PRICE-QUANTITY ADJUSTMENT IN A KEYNESIAN ECONOMY[1]

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In this paper a price and quantity adjustment process in continuous time is considered for an economy with production factors and final goods. We assume that each final good is produced by a constant returns to scale production technology with only factor goods as inputs. The price and quantity adjustments take place on the markets for factor goods only. During the process, the prices in the final good markets adjust instantaneously by setting the price of a final good equal to the production costs. Production adjusts instantaneously to meet the demand and keeping the output markets in equilibrium.

The adjustment process consists of three consecutive parts. First, in the short run part of the process the factor prices are assumed to be rigid and only quantity adjustments take place until an out of equilibrium situation is reached in which on each market either equilibrium under supply rationing prevails or excess demand and no supply rationing is observed. Next, in the mid run the factor prices are adjusted upwards in markets with excess demand, while on the other factor markets the supply rationing is adjusted to keep them in equilibrium. This process of adjusting quantity constraints in factor markets with excess supply and prices on factor markets with excess demands is shown to lead to a supply constrained equilibrium. Thirdly, in the long run the factor prices in markets with supply constraints are decreased, whereas supply constraints and prices in all other factor markets adjust to keep those markets in equilibrium. It is shown that eventually a Walrasian price system is reached.

KEYWORDS: Production economy, price rigidity, disequilibrium, adjustment process, simplicial algorithm.

JEL CODES: C62, C63, C68, D51.

1. INTRODUCTION

Most studies in general equilibrium theory focus on the existence of competitive equilibria in various settings. Interesting extensions of the standard Arrow-Debreu general equilibrium model with complete markets, include models with overlapping generations, models with uncertainty and market incompleteness, and models with asymmetric information. Issues of existence and (in)determinateness of equilibrium are fairly well understood.

It is rather striking that very little progress has been made on the modelling of economies that are out of equilibrium and the modelling of the forces that could

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drive an economy to an equilibrium state. The most frequently studied adjustment process is still the Walrasian tatonnement process as formalized by Samuelson (1941). Nevertheless, the Walrasian tatonnement process has many shortcomings.

One of the most obvious problems related to Walrasian tatonnement is its lack of convergence for many economies. The first example of such an economy has been given by Scarf (1960). The results of Sonnenschein (1972, 1973), Mantel (1974), and Debreu (1974) implicate that many economies exist for which Walrasian tatonnement does not lead the economy to a Walrasian equilibrium. Part of this lack of convergence may be overcome by making distributional assumptions on the characteristics in the economy. The results of Hildenbrand (1983) imply that sufficient heterogeneity in income leads to the Law of Demand and tatonnement stability, and those of Grandmont (1992) imply that sufficient dispersion in preferences leads to an aggregate excess demand function derived from a single Cobb-Douglas preference relation.

An alternative to making distributional assumptions on characteristics, is to assume that the price adjustment process is more sophisticated. Several processes have been shown to be convergent under fairly weak assumptions, see Smale (1976), van der Laan and Talman (1987), Kamiya (1990), and Herings (1997).

However, even when convergence takes place, it takes some time before the equilibrium price system is reached. The relevant market signals for an out-of-equilibrium process are then no longer the ones given by the notional excess demand used in the Walrasian tatonnement process. Instead, it is more reasonable to use market signals related to the effective excess demand associated with a Drèze equilibrium, see Drèze (1975). Walrasian tatonnement is then replaced by a process as described in Veendorp (1975). Again, in general such a process does not converge, see Day and Pianigiani (1991) and Böhm (1993). Extending more sophisticated price adjustment mechanisms to include effective demand signals will lead to adjustment processes displaying convergence to a Walrasian equilibrium, see Herings, van der Laan, Talman and Venniker (1997), and Herings, van der Laan and Venniker (1998).

For an economy with two types of goods, i.e. production factors and final goods, and in which each of the final goods is produced by a constant returns to scale production technology using the factor goods as inputs, Drèze (1991) also considers an adjustment process based on effective demand signals. His process distinguishes between quantity and price adjustments, where the former takes place infinitely faster than the latter. Drèze's process is motivated by five empirical regularities: (i) firms price at average cost plus a mark-up; (ii) no rationing of final demand; (iii) downwards rigidity of nominal wages; (iv) persistent unemployment; (v) CES-Leontief technology with constant returns; see Drèze and Bean (1991). Drèze shows, under some assumptions, that the process converges to an equilibrium state of the economy with supply rationing on some of the factor inputs, but no rationing on final demands.

In this paper we show that the five empirical regularities mentioned earlier can be incorporated in an adjustment process as specified in Herings, van der Laan, Talman and Venniker (1997), or even in the more general process in Herings, van der Laan and Venniker (1998), without losing the property that a Walrasian equilibrium is obtained in the long run.

Initially, the economy may be in an arbitrary state with the possibility of rationing on the supply-side. We first formulate a short-term process of quantity adjustment leading to an excess demand state at the original prices: markets are either in equilibrium with the possibility of rationing on the supply-side for factor markets, or, for the factor markets, there is excess demand even though the factor supply is maximal.

Then a mid-term adjustment process is formulated, where prices of factors in excess demand are adjusted upwards, and markets in equilibrium are kept in equilibrium by adjusting either prices or rationing constraints. The mid-term process reaches a supply constrained equilibrium: all markets are in equilibrium, rationing on the supply-side is possible in the factor markets, and prices of factors are at least as high as at the initial state, with no rationing if the price is higher.

Finally, we introduce a long-term process, a non-tatonnement process, where prices of all commodities may be adjusted downwards or upwards. All markets are kept in equilibrium and along a sequence of supply constrained equilibria, the economy is shown to reach a Walrasian equilibrium.

Although we restrict our analysis to pricing at average cost without mark-ups, positive mark-ups can be dealt with easily. The downwards rigidity of nominal wages is maintained in the short-term and mid-term process, but abandoned in the long-term process in order to reach a Walrasian equilibrium. There is persistent unemployment, disappearing only in the limit of the long-term process. Firms use a general constant returns to scale technology. The process is therefore consistent with the empirical observations of Drèze and Bean (1991).

2. THE MODEL

Given any positive integer k, we denote the set of indices $\{1, \ldots, k\}$ by I_k and the set of indices $\{0, 1, \ldots, k\}$ by I_k^0. Furthermore, $\mathbb{R}_+^k = \{x \in \mathbb{R}^k \mid x_j \geq 0, \ \forall j \in I_k\}$ and $\mathbb{R}_{++}^k = \{x \in \mathbb{R}^k \mid x_j > 0, \ \forall j \in I_k\}$. A vector of zeroes is denoted by $\underline{0}$. Its dimension will be clear from the context.

We consider an economy $\mathcal{E} = (\{X^h, (w^h, m^h), \succeq^h\}_{h \in I_H}, \{c^l\}_{l \in I_L}, \hat{s})$ with $K+L+1$ commodities. Commodity 0 serves as a numeraire commodity with price $p_0 = 1$, a commodity indexed by $k = 1, \ldots, K$ is a production factor k, and a commodity indexed by $l = 1, \ldots, L$ is a final good l. The prices of the production factors are denoted by the vector $p \in \mathbb{R}_+^K$ and the prices of the final goods by $q \in \mathbb{R}_+^L$.

We assume that each final good is produced by a constant returns to scale production technology using the factor goods as inputs. Let $c^l(p)$ be the minimum cost to produce one unit of final good l at factor prices $p \in \mathbb{R}^K_+$. If the cost function $c^l : \mathbb{R}^K_+ \to \mathbb{R}_+$ is continuously differentiable, then it follows with Euler's law that $c^l(p) = \sum_{k=1}^K p_k a^l_k(p)$, where $a^l_k(p) = \frac{\partial c^l(p)}{\partial p_k}$ is the net input quantity of production factor k to produce one unit of the final good l at factor prices p. We allow that the vector $a^l(p) \in \mathbb{R}^K_+$ contains zero elements. From the assumption of constant returns to scale it follows that $a^l(p)$ contains at least one positive element at each p. Continuous differentiability of c^l implies that $a^l(p)$ is continuous in p. So, the production activities can be represented by the nonnegative $K \times L$ technology matrix $A(p) = [a^1(p), \ldots, a^L(p)]$ describing the net input requirements at factor prices p for producing unit amounts of the outputs, being continuous in p and satisfying that each column contains at least one positive element. We assume that also each row contains at least one positive element, implying that at any price vector p each production factor is required as an input in at least one production activity, i.e. for all p we have that $\sum_{l \in I_L} a^l_k(p) > 0$, $k \in I_K$.

There are H households, indexed $h = 1, \ldots, H$. Household h initially holds an amount m^h of the numeraire commodity and a nonnegative vector $w^h \in \mathbb{R}^K_+$ of production factors. The vector $w \in \mathbb{R}^K_+$ is defined by $w = \sum_{h \in I_H} w^h$ and $m \in \mathbb{R}$ by $m = \sum_{h \in I_H} m^h$. Household h derives utility only from the numeraire commodity and the final goods. His consumption set is given by $X^h = \mathbb{R}^{L+1}_+$. For a consumption bundle $x^h \in X^h$, x^h_0 denotes the consumption of the numeraire commodity and x^h_l, $l \in I_L$, the consumption of final good l. The preferences of household h are given by a preference relation $\succeq^h$ on X^h.

Let $Z \subset \prod_{h \in I_H} \mathbb{R}^K_+$ be the collection of HK-dimensional vectors $\zeta = (\zeta^1, \ldots, \zeta^H)$ defined by

$$Z = \{\zeta \in \prod_{h \in I_H} \mathbb{R}^K_+ \mid \underline{0} \leq \zeta^h \leq w^h, \ h \in I_H\}.$$

For each $h \in I_H$, the vector $\zeta^h \in \mathbb{R}^K_+$ is a nonnegative vector less than or equal to w^h. Then a state $s = (p, \zeta) \in \mathbb{R}^K_+ \times Z$ of the economy is defined as a price system p for the production factors and for every household h a vector of quantity constraints ζ^h on the supply of the production factors. The quantity constraint ζ^h_k is the maximal amount of production factor k which household h expects to be able to sell. In this paper the focus will be on downwards rigidities of prices of production factors, in accordance with empirical regularity iii) of Drèze and Bean (1991). If such downwards rigidities exist, the initial state of the economy, denoted by $\hat{s} = (\hat{p}, \hat{\zeta}) \in \mathbb{R}^K_+ \times Z$ is of importance. With respect to the economy $\mathcal{E}$ the following assumptions are made.

A1. For every household $h \in I_H$, the consumption set X^h is given by $\mathbb{R}^{L+1}_+$.

A2. For every household $h \in I_H$, the preference relation $\succeq^h$ is complete, transitive, continuous, strongly monotonic and strictly convex.

A3. For every household $h \in I_H$, $m^h > 0$ and $w^h \geq \underline{0}$. Moreover, $w \gg \underline{0}$.

A4. For every final good $l \in I_L$, the cost function $c^l : \mathbb{R}^K_+ \to \mathbb{R}_+$ is weakly increasing, concave, continuously differentiable, and homogeneous of degree one. For every $p \in \mathbb{R}^K_+$, for every $k \in I_K$, $\sum_{l \in I_L} \frac{\partial c^l(p)}{\partial p_k} > 0$.

A5. At the initial state $\widehat{s} \in \mathbb{R}^K_+ \times Z$ it holds that $\widehat{p}^\top A(\widehat{p}) \gg \underline{0}$.

The assumption of strict convexity of the preference relations is for the sake of simplicity and allows us to work with demand functions instead of demand correspondences. It is well-known that Assumption A4 on the cost functions can be derived from assumptions on primitive concepts. A cost function c^l from $\mathbb{R}^K_+$ into $\mathbb{R}_+$ is said to be weakly increasing if $p^1 \gg p^2$ implies $c^l(p^1) > c^l(p^2)$. Assumption A5 says that at the initial state the production cost of each final good is positive.

Because of the assumption of constant returns to scale, the price $q_l(s)$ of final good l in state $s = (p, \zeta)$ of the economy follows from the price system p and is given by

$$q_l(s) = c^l(p). \tag{1}$$

The supply of production factors by the households in state s of the economy is given by

$$z^{\mathrm{S}}(s) = \sum_{h \in I_H} \zeta^h, \tag{2}$$

since ζ^h is the amount household h is allowed to sell. In a supply constrained equilibrium, ζ^h coincides with the amount sold by household h. The income of household h at state s is given by

$$r^h(s) = m^h + p^\top \zeta^h > 0. \tag{3}$$

At any state a household collects his income from his initial holdings of the numeraire commodity and the expected value of the sale of factors. At state s, the demand $x^h(s) = (x^h_0(s), x^h_1(s), \ldots, x^h_L(s))^\top \in \mathbb{R}^{L+1}$ of household h is defined as the best element x^h for $\succeq^h$ on X^h under the budget constraint

$$x^h_0 + \sum_{l \in I_L} q_l(s) x^h_l \leq r^h(s). \tag{4}$$

Because of the assumptions made on preferences, this element is unique and lies on the budget hyperplane. The total demand at state s is given by $x(s) =$

$(x_0(s), x_1(s), \ldots, x_L(s))^\top = \sum_{h \in I_H} x^h(s)$ with $x_0(s)$ the total demand for the numeraire commodity and $x_l(s)$ the total demand for final good l, $l = 1, \ldots, L$. In the following we denote the total demand for final goods at state s by $d(s) = (x_1(s), \ldots, x_L(s))^\top$. The total input of the production sector needed to produce this demand $d(s)$ for final goods at state s is given by

$$z^{\mathrm{D}}(s) = A(p)d(s). \tag{5}$$

We define a supply constrained equilibrium as follows.

DEFINITION 2.1: A *supply constrained equilibrium (SCE)* for the economy $\mathcal{E}$ is a state $s^* \in \mathbb{R}_+^K \times Z$ such that $z^{\mathrm{D}}(s^*) = z^{\mathrm{S}}(s^*)$.

At an equilibrium state $s^* = (p^*, \zeta^*)$ the expectations ζ^{*h} of each household h about its supply possibilities are fulfilled and the income of household h is given by $r^h(s^*)$. Since by equation (5) the total production of final goods equals the total demand for final goods and hence at an equilibrium state all commodity markets are in equilibrium, it follows from Walras' law that at an equilibrium state s^* also the total demand for the numeraire commodity, $x_0(s^*)$, equals the total initial holdings, m, of the numeraire commodity.

At a supply constrained equilibrium, households are not necessarily able to sell their entire initial factor endowment. On the contrary, typically it holds that $\zeta^{*h} < w^h$ for some households. The economy is then in a situation of underemployment at an equilibrium state. There is some but not a full utilization of all production factors. The underemployment rate of factor k is given by $(w_k - z_k^{\mathrm{D}}(s^*))/w_k$. If for some household h the indifference surfaces associated to any strictly positive x^h do not intersect the boundary of $\mathbb{R}_+^{L+1}$, then there is a positive total demand for all final goods and the underemployment rate of factor k is less than one because of Assumption A4.

A Walrasian equilibrium arises when at a supply constrained equilibrium the underemployment rate of all production factors is equal to zero. An exception is made for the case where the price of some factor is zero. Since households do not derive utility from production factors, underemployment of a production factor with zero price forms no real constraint.

DEFINITION 2.2: A *Walrasian equilibrium (WE)* for the economy $\mathcal{E}$ is an SCE s^* such that, for all $k \in I_K$, $p_k^*(w_k - \sum_{h \in I_H} \zeta_k^{*h}) = 0$.

Given the initial state $\hat{s}$, we denote the set of price vectors greater than or equal to $\hat{p}$ by $\hat{P}$, i.e.

$$\hat{P} = \{p \in \mathbb{R}_+^K \mid p \geq \hat{p}\}.$$

The following definitions are helpful when describing the adjustment process.

DEFINITION 2.3: A *$\widehat{p}$-state* for the economy $\mathcal{E}$ is a state $s = (p, \zeta) \in \mathbb{R}^K_+ \times Z$ with $p = \widehat{p}$.

DEFINITION 2.4: A *$\widehat{P}$-state* for the economy $\mathcal{E}$ is a state $s = (p, \zeta) \in \mathbb{R}^K_+ \times Z$ with $p \in \widehat{P}$.

Our price and quantity adjustment process can be decomposed into three parts: a short-term, a mid-term, and a long-term process. Starting from the initial state $\widehat{s}$, the short-term process yields a path of $\widehat{p}$-states until a $\widehat{p}$-state is reached at which the quantity constraints are such that total factor demand exceeds total factor supply, and total factor demand only strictly exceeds total factor supply if the factor is used at full capacity. We call such a $\widehat{p}$-state an excess demand $\widehat{p}$-state.

DEFINITION 2.5: An *excess demand $\widehat{p}$-state (ED $\widehat{p}$-state)* for the economy $\mathcal{E}$ is a $\widehat{p}$-state $\overline{s} \in \mathbb{R}^K_+ \times Z$ such that $z^{\mathrm{D}}(\overline{s}) \geq z^{\mathrm{S}}(\overline{s})$ and $z^{\mathrm{D}}_k(\overline{s}) > z^{\mathrm{S}}_k(\overline{s})$ implies $\overline{\zeta}^h_k = w^h_k$, $h \in I_H$.

Analogously, we define an excess demand $\widehat{P}$-state.

DEFINITION 2.6: An *excess demand $\widehat{P}$-state (ED $\widehat{P}$-state)* for the economy $\mathcal{E}$ is a $\widehat{P}$-state $\overline{s} \in \mathbb{R}^K_+ \times Z$ such that $z^{\mathrm{D}}(\overline{s}) \geq z^{\mathrm{S}}(\overline{s})$ and $z^{\mathrm{D}}_k(\overline{s}) > z^{\mathrm{S}}_k(\overline{s})$ implies $\overline{\zeta}^h_k = w^h_k$, $h \in I_H$.

In the mid-term process the prices of factors that are in positive excess demand will rise, while quantity adjustments will keep the other markets in equilibrium. Continuing from the excess demand $\widehat{p}$-state found by the short-term process, the mid-term process yields a path of excess demand $\widehat{P}$-states, which ends at a supply constrained $\widehat{P}$-equilibrium.

DEFINITION 2.7: A *supply constrained $\widehat{P}$-equilibrium (SC $\widehat{P}$-equilibrium)* for the economy $\mathcal{E}$ is an SCE $s^* \in \mathbb{R}^K_+ \times Z$ such that $p^* \geq \widehat{p}$ and $p^*_k > \widehat{p}_k$ implies $\zeta^{*h}_k = w^h_k$ for all h.

All concepts defined so far are static in nature. In Drèze (1991), a dynamic process is described that starts at an arbitrary initial state $\widehat{s}$ and that leads, under certain conditions, via an ED $\widehat{p}$-state to an SC $\widehat{P}$-equilibrium. We will give another process that satisfies the five empirical regularities of Drèze and Bean (1991) and that, for an economy satisfying A1-A5, leads via short-term adjustments of $\widehat{p}$-states to an ED $\widehat{p}$-state, via mid-term adjustments of ED $\widehat{P}$-states to an SC $\widehat{P}$-

equilibrium, and via long-term adjustments of SCEs to a WE.

3. THE DRÈZE PROCESS

In Drèze (1991), a discrete price and quantity adjustment process is proposed for the model of the previous section. Its main characteristics are that quantities move faster than prices and that prices of production factors are downwards rigid, reflecting the fundamentals of Keynesian economics.

To be more precise, let ε_k, $k \in I_K$, be small positive numbers. For numbers ξ_k^h satisfying $0 < \xi_k^h < \frac{\varepsilon_k}{2}$ and such that w_k^h is an integer multiple of ξ_k^h for all h and k, and (small) positive numbers δ_k, the quantity constraint ζ_k^h in the market for factor k is adjusted by discrete steps of size ξ_k^h and the price p_k is increased by discrete steps of size δ_k.[2] Starting from an initial state $\widehat{s}$, this adjustment continues until an approximate SC $\widehat{P}$-equilibrium is reached. An approximate SC $\widehat{P}$-equilibrium is a state of the economy such that in the market for each factor k the absolute value of the difference between supply and demand is at most ε_k and the price is at least equal to $\widehat{p}_k$. As argued by Drèze (1991), such an approximate equilibrium seems to be tolerable, since then the differences between demand and supply can be absorbed by small adjustments in productivity, product quality or inventories. Furthermore, along the process, only price increments may occur and hence the empirical regularity iii) of Drèze and Bean (1991) is satisfied along the path of states generated by the process.

In the Drèze process, quantities are adjusted as long as in some factor market either excess supply prevails or excess demand for factor k exceeds ε_k and at least one household is rationed on supply in that market. The process continues until a situation is reached where all factor markets are in excess demand and for any factor k it holds that either excess demand is at most equal to ε_k or excess demand is more than ε_k and all households are unconstrained in that market. When the first situation holds for all markets, the economy has attained an approximate SC $\widehat{P}$-equilibrium and the process terminates. Otherwise, the process switches to price adjustments. Because of price adjustment, the economy may return to a state where quantity adjustments prevail.

Given some initial state $\widehat{s}$ at which any number $\widehat{\zeta}_k^h$ is a nonnegative integer multiple of ξ_k^h, the Drèze process generates a sequence of states $s(t)$, $t = 0, 1, \dots$, starting at $s(0) = \widehat{s}$. The state $s(t)$ belongs to one out of two possible regimes. Either

(i) there exists a factor k such that
either (a) $z_k^{\mathrm{D}}(s(t)) - z_k^{\mathrm{S}}(s(t)) < 0$, or (b) $z_k^{\mathrm{D}}(s(t)) - z_k^{\mathrm{S}}(s(t)) > \varepsilon_k$ and $z_k^{\mathrm{S}}(s(t)) < w_k$,
or

(ii) for all $k \in I_K$, both (a) and (b) do not hold.

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[2]This is a slight generalization of Drèze (1991), who takes ξ_k^h independent of h.

In regime (i) there is scope for quantity adjustment to equilibrate the markets by tightening the rationing constraints in case (a) of excess supply, and, if possible, weakening the rationing constraints in case (b) of excess demand. In regime (ii) there is no scope for quantity adjustment and prices will adjust. If regime (i) holds true, then an arbitrary index k for which (a) or (b) holds is taken. In case (a) $z_k^{\mathrm{S}}(s(t)) > 0$, so there exists at least one household h such that $\zeta_k^h(t) = \kappa \xi_k^h$ with κ a positive integer. Then an arbitrary household h satisfying this condition is taken and a quantity adjustment takes place by setting $\zeta_k^h(t+1) = \zeta_k^h(t) - \xi_k^h$, while all other quantities and prices at stage $t+1$ are set equal to those in t. In case (b) $z_k^{\mathrm{S}}(s(t)) < w_k$ and there exists at least one household h such that $\zeta_k^h(t) = w_k^h - \kappa \xi_k^h$ with κ a positive integer. An arbitrary household h satisfying this condition is taken and a quantity adjustment takes place by setting $\zeta_k^h(t+1) = \zeta_k^h(t) + \xi_k^h$, while all other quantities and prices at stage $t+1$ are set equal to those in t. The process proceeds in this way until a stage t is reached in which regime (ii) occurs. Suppose there exists an index k' such that

$$z_{k'}^{\mathrm{D}}(s(t)) - z_{k'}^{\mathrm{S}}(s(t)) > \varepsilon_{k'}. \tag{6}$$

Then $z_{k'}^{\mathrm{S}}(s(t)) = w_{k'}$ (because otherwise case (b) of (i) is true) and hence there is excess demand for factor k' without rationing of the households, while for each factor k for which condition (6) is not true the excess demand is nonnegative but at most ε_k. The price of an arbitrary factor k for which condition (6) holds is raised by setting $p_k(t+1) = p_k(t) + \delta_k$, while all other quantities and prices at stage $t+1$ are set equal to those in t. The process continues by adjusting quantities in case regime (i) occurs again, and by adjusting prices if regime (ii) holds true and condition (6) holds for at least one k. The process terminates if a state $s(t)$ is reached in which (ii) is true and $0 \leq z_k^{\mathrm{D}}(s(t)) - z_k^{\mathrm{S}}(s(t)) \leq \varepsilon_k$ for all k and hence $s(t)$ is an approximate SC $\widehat{P}$-equilibrium.

Drèze shows that this process reaches such a state within a finite number of steps if the demand functions of the households for final goods satisfy the conditions of Non Inferiority and Marginal Propensity to Consume. Non Inferiority rules out inferior goods and Marginal Propensity to Consume states that the marginal propensity to spend is less than, and bounded away from, unity. That is, higher income leads to strictly more consumption of the numeraire commodity.

4. THE PRICE AND QUANTITY ADJUSTMENT PROCESS

The price and quantity adjustment process of this paper is based on the same empirical regularities as the Drèze process. It consists of a short-term, a mid-term and a long-term process.

In the short-term process, prices are kept fixed and only quantities are adjusted until each factor market is in equilibrium under supply rationing or is in excess demand without supply rationing. When the first case is true for all factor markets, the process has reached an SC $\widehat{P}$-equilibrium. When the second case holds for at least one factor market an ED $\widehat{p}$-state is reached. This part of the path is referred to as the short-term adjustment process and corresponds to states in regime (i) of the Drèze process.

After reaching an ED $\widehat{p}$-state, mid-term adjustments take place in which the prices of factors in excess demand are adjusted upwards, and factor markets in equilibrium are kept in equilibrium by either quantity or price adjustments. This corresponds to regime (ii) of the previous section. Although mid-term and short-term adjustments may alternate and the rationing regimes in the markets may differ between different periods of mid-term adjustments, we will show that the process eventually reaches an SC $\widehat{P}$-equilibrium, a concept that is empirically well supported.

Proceeding from such an SCE found by the short-term and mid-term adjustments, a long-term adjustment process takes place, which is shown to reach a Walrasian equilibrium along a path of SCEs. As argued by Drèze (1991), there are two main roads to attain such an equilibrium, namely downwards price flexibility and fiscal expansion. In fact, these two ways are two sides of the same coin. Increasing the numeraire commodity balances of the households by a fiscal expansion policy triggers a relative downward adjustment of the prices. Therefore, in the long-term process we concentrate here on the instrument of downward price adjustments, which can be modelled by introducing a variable reflecting the price level of the factor prices. Indeed, in the long-term process, prices of factors with supply rationing are decreased, whereas rationing schemes and prices of factors without rationing are adjusted to keep all markets equilibrated.

It is possible that the short-term process ends at an SC $\widehat{P}$-equilibrium. In general this will happen if the prices of all factors are high initially. Then there is no mid-term process. It is also possible that the mid-term process ends at a WE. This situation will occur if the prices of all factors are low initially. In this case there is no long-term process. Generically, it cannot happen that the short-term process generates a WE. In fact, this can only be the case if $\widehat{p}$ is a Walrasian equilibrium price system, which is, generically, not the case. The formal descriptions of the short-term, mid-term and long-term processes are given in Sections 6, 7 and 8.

It is not realistic that all factor prices are downwards rigid, even in the short run. It is certainly possible to extend the price and quantity adjustment process of this paper in several directions. One option is to introduce a group of factors, whose markets are always cleared by price adjustments and never by quantity adjustments, see also Herings and Drèze (1998). For those factors, prices could adjust downwards, even in the short run.

5. THE REDUCED TOTAL EXCESS DEMAND FUNCTION

To facilitate the exposition, we introduce the so-called reduced total excess demand function in this section. Along the short-term process all prices are kept fixed, while quantities may adjust, i.e. for all k it holds that $p_k = \widehat{p}_k$ and $\zeta_k^h \leq w_k^h$, $h \in I_H$. In the mid-term process for all k the following complementarity conditions prevail: $\zeta_k^h < w_k^h$ for at least one h implies $p_k = \widehat{p}_k$, and $p_k > \widehat{p}_k$ implies $\zeta_k^h = w_k^h$, $h \in I_H$. The long-term process is characterized by a similar complementarity condition involving also the price level of the factor prices. Owing to these complementarities between prices and quantities, it is possible to describe the price and quantity adjustment for a factor good by a single variable.

The set of variables defining the prices and rationing schemes is taken to be the set

$$Y = Y_0 \cup \left((0,1] \times \{y \in \mathbb{R}_+^K \,|\, \min\{y_1, \ldots, y_K\} \leq 1\}\right),$$

where $Y_0 = \{0\} \times \mathbb{R}_+^K$. To any $y = (y_0, y_1, \ldots, y_K)^\top \in Y$ we relate a unique price system $p(y) \in \mathbb{R}_+^K$ and a unique rationing scheme $\zeta^h(y) \in \mathbb{R}_+^K$, $h \in I_H$, satisfying $\zeta^h(y) \leq w^h$. To do so, let $g_k^h : [0,1] \to [0,1]$ be continuous, nondecreasing functions satisfying $g_k^h(0) = 0$, $g_k^h(1/2) = \widehat{\zeta}_k^h / w_k^h$ and $g_k^h(1) = 1$.[3] For $y \in Y$, we define

$$p_k(y) = (1 - y_0)\widehat{p}_k + \max\{0, y_k - 1\}, \quad k \in I_K, \tag{7}$$

$$\zeta_k^h(y) = g_k^h(\min\{1, y_k\}) w_k^h, \quad k \in I_K,\ h \in I_H. \tag{8}$$

The variable y_0 reflects the long-term price level and is restricted to vary between zero and one. For $k \in I_K$, the variable y_k determines the price and rationing on factor market k. Observe that $y_k = 0$ implies $\zeta_k^h(y) = 0$, $0 \leq y_k \leq 1$ implies $0 \leq \zeta_k^h(y) \leq w_k^h$, and $y_k \geq 1$ implies $\zeta_k^h(y) = w_k^h$. Furthermore, $p_k(y) = (1 - y_0)\widehat{p}_k$ if $y_k \leq 1$ and $p_k(y) = (1 - y_0)\widehat{p}_k + y_k - 1 > (1 - y_0)\widehat{p}_k$ if $y_k > 1$. In particular, it holds that $p_k(y) \geq \widehat{p}_k$ if $y_0 = 0$. So, any $y \in Y_0$ induces a state $s(y) = (p(y), \zeta(y))$ in $\mathbb{R}_+^K \times Z$ satisfying the short-term or mid-term complementarity conditions between prices and quantities mentioned above. In particular, the vector $v \in Y_0$ with $v_k = \frac{1}{2}$, $k \in I_K$, induces the initially given state $\widehat{s}$. The short-term and mid-term processes generate vectors y that belong to Y_0.

In the following, let the set $\widehat{Y}$ be given by

$$\widehat{Y} = \{y \in Y \mid q(s(y)) \gg \underline{0}\},$$

so any $y \in \widehat{Y}$ induces a price vector $p(y)$ such that for each final good the production costs are positive. We will show that the price and quantity adjustment process will never generate prices at which some output is a free commodity, so that we may restrict attention to the set $\widehat{Y}$. Clearly, this holds for the short-term and mid-term

[3] If $w_k^h = 0$, then the value of $g_k^h(1/2)$ can be chosen arbitrarily between zero and one.

processes because of Assumption A5 and the fact that $p(y) \geq \hat{p}$ if $y \in Y_0$. For the long-term process, this follows from part (vi) of Lemma 5.1 below.

We define the demand $\hat{x}^h : \hat{Y} \to \mathbb{R}^{L+1}$ of household h by $\hat{x}^h(y) = x^h(s(y))$, the best element for $\succeq^h$ on X^h under his budget constraint (4) taking into account the prices $q(s(y))$ for the final goods and income $r^h(s(y)) = m^h + p(y)^\top \zeta^h(y)$. In this way we obtain total demand for money and final goods, $\hat{x}(y)$, total demand for final goods, $\hat{d}(y)$, total input needed to produce this demand, $\hat{z}^{\mathrm{D}}(y)$, and total factor supply, $\hat{z}^{\mathrm{S}}(y)$. Total excess demand for factors at state $s(y)$ equals

$$\hat{z}(y) = \hat{z}^{\mathrm{D}}(y) - \hat{z}^{\mathrm{S}}(y), \quad y \in \hat{Y}. \tag{9}$$

The function $\hat{z} : \hat{Y} \to \mathbb{R}^K$ is called the reduced total excess demand function. The price and quantity adjustment process can be described in terms of y and the reduced total excess demand function $\hat{z}$ only. Notice that, for $y^* \in \hat{Y}$, $s(y^*)$ is a supply constrained equilibrium if and only if $\hat{z}(y^*) = \underline{0}$, and a Walrasian equilibrium when in addition $y_0^* = 1$ or $y_k^* \geq 1$ for all $k \in I_K$.

We give some properties of the reduced total excess demand function $\hat{z}$, among which its continuity, a version of Walras' law, and its boundary behaviour, i.e. the behaviour of $\hat{z}$ when $y_k = 0$ or $y_k \to \infty$ for some $k \in I_K$.

LEMMA 5.1: *Let the economy $\mathcal{E}$ satisfy the Assumptions A1-A5. Then*

(i) *$\hat{z}$ is continuous on $\hat{Y}$,*

(ii) $p(y)^\top \hat{z}(y) = m - \hat{x}_0(y)$, *for all $y \in \hat{Y}$,*

(iii) *for $k \in I_K$, $y_k = 0$, implies $\hat{z}_k(y) \geq 0$,*

(iv) *there exists a real number $M > 0$ such that for all $y \in Y_0$,* $\max\{y_1, \ldots, y_K\} \geq$ *M implies $\hat{z}_k(y) < 0$ for some $k \in I_K$,*

(v) *there exists a real number $M > 0$ such that for all $y \in \hat{Y} \setminus Y_0$,* $\max\{y_1, \ldots, y_K\}$ *$\geq M$ implies $\hat{z}(y) \neq \underline{0}$,*

(vi) *if $(y^n)_{n \in \mathbb{N}}$ is a sequence of points in $\hat{Y}$ such that $\hat{z}(y^n) = \underline{0}$, $n \in \mathbb{N}$, and $y^n \to \overline{y}$, then $q(s(\overline{y})) \gg \underline{0}$.*

PROOF:

(i) For every $y \in \hat{Y}$, by definition, $q(s(y)) \gg \underline{0}$, and $r^h(s(y)) > 0$, $h \in I_H$. From Theorem 4.2.5 in Herings (1996) and the continuity of the function r^h in y, it follows that $\hat{x}^h$, $h \in I_H$, is a continuous function on $\hat{Y}$. From this the continuity of $\hat{z}$ follows trivially.

(ii) Since preferences are strongly monotonic, for each household $h \in I_H$, the budget constraint will be satisfied with equality,

$$\widehat{x}_0^h(y) + \sum_{l \in I_L} q_l(s(y))\widehat{x}_l^h(y) = r^h(s(y)) = m^h + p(y)^\top \zeta^h(y), \quad y \in \widehat{Y}.$$

Summing up over all households and substituting $q(s(y))^\top = p(y)^\top A(p(y))$, we get

$$\widehat{x}_0(y) + p(y)^\top A(p(y))\widehat{d}(y) = m + p(y)^\top \widehat{z}^{\mathrm{S}}(y).$$

Since $A(p(y))\widehat{d}(y) = \widehat{z}^{\mathrm{D}}(y)$ this reduces to

$$p(y)^\top \widehat{z}(y) = m - \widehat{x}_0(y).$$

(iii) If $y_k = 0$, then $\zeta_k^h(y) = 0$, $h \in I_H$, and hence $\widehat{z}_k(y) = \widehat{z}_k^{\mathrm{D}}(y) \geq 0$.

(iv) Suppose the statement is false. Then, for every $M \in \mathbb{N}$, there exists $y^M \in Y_0$ such that $\max\{y_1^M, \ldots, y_K^M\} = M$ and $\widehat{z}(y^M) \geq \underline{0}$. Then it follows from (ii) that $\widehat{x}_0(y^M)$ is bounded by m and that $\widehat{z}_k(y^M) \leq \frac{m}{\widehat{p}_k}$, $k \in I_K$, so

$$\widehat{z}_k^{\mathrm{D}}(y^M) \leq \widehat{z}_k^{\mathrm{S}}(y^M) + \frac{m}{\widehat{p}_k} \leq w_k + \frac{m}{\widehat{p}_k}, \ k \in I_K.$$

Hence, the demand for factors is bounded, and therefore the production of final goods is bounded, and so the demand $\widehat{d}(y^M)$ is bounded. Define $p^M = p(y^M)$ and $\widetilde{p}^M = \frac{p^M}{\|p^M\|_\infty}$. Consider a convergent subsequence of $(\widetilde{p}^M, \zeta(y^M))_{M \in \mathbb{N}}$ with limit, say, $(\overline{p}, \overline{\zeta})$. Let $k' \in I_K$ be such that $\overline{p}_{k'} = 1$. Let $h' \in I_H$ be a household with $w_{k'}^{h'} > 0$. Since $y_{k'}^M \to \infty$ it holds that $\overline{\zeta}_{k'}^{h'} = w_{k'}^{h'}$. The budget constraint of household h' is given by

$$x_0^{h'} + \sum_{l \in I_L} c^l(p^M)x_l^{h'} = m^{h'} + p^{M\top}\zeta^{h'}(y^M).$$

Dividing this constraint by $\|p^M\|_\infty$ results in

$$\frac{1}{\|p^M\|_\infty}x_0^{h'} + \sum_{l \in I_L} c^l(\widetilde{p}^M)x_l^{h'} = \frac{1}{\|p^M\|_\infty}m^{h'} + \widetilde{p}^{M\top}\zeta^{h'}(y^M).$$

It follows that the sequence of normalized incomes of household h'

$$\left(\frac{1}{\|p^M\|_\infty}m^{h'} + \widetilde{p}^{M\top}\zeta^{h'}(y^M)\right)_{M \in \mathbb{N}}$$

is convergent with limit $\overline{p}^\top \overline{\zeta}^{h'} \geq w_{k'}^{h'} > 0$. The sequence of normalized numeraire and final goods prices

$$\left(\frac{1}{\|p^M\|_\infty}, c(\widetilde{p}^M)\right)_{M \in \mathbb{N}}$$

converges to $(0, c(\overline{p})) > \underline{0}$, where the inequality follows since the price $c^l(\overline{p})$ of a final good l using factor k' as an input exceeds $\overline{p}_{k'}a^l_{k'}(\overline{p}) = a^l_{k'}(\overline{p}) > 0$, because $\overline{p}_{k'} = 1$. Since the price of the numeraire commodity in $(0, c(\overline{p}))$ is zero, and income of household h' is positive, $\|\widehat{d}^{h'}(y^M)\|_\infty \to \infty$, a contradiction to the boundedness of $\widehat{d}(y^M)$.

(v) Suppose the statement is false. Then, for every $M \in \mathbb{N}$, there exists $y^M \in \widehat{Y} \setminus Y_0$ such that $\max\{y^M_1, \ldots, y^M_K\} = M$ and $\widehat{z}(y) = \underline{0}$. Then,

$$\widehat{z}^{\mathrm{D}}_k(y^M) = \widehat{z}^{\mathrm{S}}_k(y^M) \leq w_k, \ k \in I_K.$$

The demand for factors is bounded, and therefore the production of final goods is bounded, and so the demand $\widehat{d}(y^M)$ is bounded. Notice that $\widehat{x}_0(y^M)$ is bounded by m. The derivation of a contradiction is now identical to case (iv), which proves the statement.

(vi) Suppose the statement is false. Then, there is a sequence $(y^n)_{n\in\mathbb{N}}$ in $\widehat{Y}$ such that $\widehat{z}(y^n) = \underline{0}$, $n \in \mathbb{N}$, $y^n \to \overline{y}$, and there is $k' \in I_K$ such that $q_{k'}(s(\overline{y})) = 0$. The sequence of incomes $(m^h + p(y^n)^\top \zeta^h(y^n))_{n\in\mathbb{N}}$ of a household h is convergent with limit $m^h + p(\overline{y})^\top \zeta^h(\overline{y}) > 0$. The sequence of numeraire and final goods prices $(1, q(s(y^n)))_{n\in\mathbb{N}}$ is convergent with limit $(1, q(s(\overline{y})))$. Since $q_{k'}(s(\overline{y})) = 0$ and the income of household h is positive, $\|x^h(s(y^n))\|_\infty \to \infty$, a contradiction to $\widehat{z}(y^n) = \underline{0}$, $n \in \mathbb{N}$. *Q.E.D.*

The statements (iv) and (v) of Lemma 5.1 will be used to show that the adjustment process does not leave the set $\{y \in \widehat{Y} \mid 0 \leq y_k \leq M, \ k \in I_K\}$ for an arbitrarily chosen number M satisfying both statements. Furthermore, statement (vi) claims that for any convergent sequence of points in $\widehat{Y}$ such that all factor markets are in equilibrium, it holds that also the limit point is in $\widehat{Y}$, i.e. if we have a convergent sequence of points inducing positive production costs for the final goods and satisfying that all factor markets are in equilibrium, then the production costs induced by the limit point are also positive. This property is useful in showing that the reduced excess demand is well-defined at any limit point of the long-term process. Before giving any further proofs, we give the formal definitions of the short-term, mid-term and long-term process in the Sections 6, 7, and 8. For ease of notation, let Y^{s} be the subset of Y given by

$$Y^{\mathrm{s}} = \{y \in Y_0 \mid 0 \leq y_k \leq 1, \ k \in I_K\}.$$

The upper boundary of Y^{s} is given by $\overline{Y^{\mathrm{s}}} = \{y \in Y^{\mathrm{s}} \mid y_k = 1 \text{ for at least one } k \in I_K\}$. Furthermore, let Y^{m} and Y^{l} be subsets of Y given by

$$Y^{\mathrm{m}} = \{y \in Y_0 \mid \exists k \in I_K, \ y_k \geq 1\}$$
$$Y^{\mathrm{l}} = \{y \in Y \mid \min\{y_1, \ldots, y_k\} \leq 1\}.$$

6. THE SHORT-TERM PROCESS

In the short-term process only quantities are adjusted, whereas prices stay put. These adjustments correspond to adjustments of the vector y in the subset Y^{s} of $\widehat{Y}$ defined in the previous section. The initial state $\widehat{s}$ is represented by $v \in Y^{\mathrm{s}}$, given by $v_k = \frac{1}{2}$, $k \in I_K$. Recall that $v_0 = 0$. The quantity adjustments are governed by the excess demands on the factor markets. For any point y reached by the short-term process it holds that $y \in Y^{\mathrm{s}}$ and there exists $\gamma \in [0,1]$ such that for every $k \in I_K$,

$$\begin{array}{ll} y_k = \frac{1}{2}\gamma, & \text{if } \widehat{z}_k(y) < 0, \\ \frac{1}{2}\gamma \leq y_k \leq 1 - \frac{1}{2}\gamma, & \text{if } \widehat{z}_k(y) = 0, \\ y_k = 1 - \frac{1}{2}\gamma, & \text{if } \widehat{z}_k(y) > 0. \end{array} \tag{10}$$

The dynamics corresponding to (10) are as follows. The initial state v satisfies the properties given above for $\gamma = 1$. It is the starting point of the path generated by the adjustment process. The adjustment process starts by decreasing γ, i.e. decreasing y_k if there is excess supply in the market for commodity k, and increasing y_k if there is excess demand in the market for commodity k. The process continues like this, until one of the markets gets equilibrated, say the market for factor k^1. Then y_{k^1} adjusts such that the market for factor k^1 stays in equilibrium, whereas y_k for any factor k in excess supply is kept minimal (equal to $\frac{1}{2}\gamma$) and for any factor k in excess demand is kept maximal (equal to $1 - \frac{1}{2}\gamma$). This induces supply rationing in those markets which is, when compared to the initial state, relatively maximal in case of excess supply and relatively minimal in case of excess demand. If there was excess supply in the market for commodity k^1 before it got equilibrated, then y_{k^1} is increased away from the minimum value, $\frac{1}{2}\gamma$, and if there was excess demand in the market for commodity k^1 before equilibration, then y_{k^1} is decreased away from the maximum value, $1 - \frac{1}{2}\gamma$. The process continues until a market for a commodity $k^2 \neq k^1$ gets equilibrated, after which also y_{k^2} adjusts such that the market for factor k^2 stays in equilibrium. It may happen that in order to keep the market for factor k^1 in equilibrium, y_{k^1} becomes equal to the minimum value $\frac{1}{2}\gamma$ or to the maximum value $1 - \frac{1}{2}\gamma$. In the first case, y_{k^1} is kept equal to $\frac{1}{2}\gamma$ and excess supply results in this market. In the latter case y_{k^1} is kept equal to $1 - \frac{1}{2}\gamma$ and excess demand results in the market for commodity k^1.

In general, the short-term process proceeds by allowing the variable y_k to vary between the bounds $\frac{1}{2}\gamma$ and $1 - \frac{1}{2}\gamma$ when the market for factor k is in equilibrium and keeping y_k on this lower (upper) bound when that market is in excess supply (demand). The short-term process terminates as soon as a point $y^* \in Y^{\mathrm{s}}$ is reached in which all markets are in equilibrium or γ has become equal to zero. In the former case the point y^* induces an SC $\widehat{P}$-equilibrium. In the latter case it follows from (10) that $y_k^* = 1$ for any commodity k in positive excess demand, $0 \leq y_k^* \leq 1$ for any commodity k in equilibrium, and $y_k^* = 0$ for any commodity k in positive excess supply. However, by property (iii) of Lemma 5.1 we have that $\widehat{z}_k(y^*) \geq 0$ when $y_k^* = 0$. So, if $\gamma = 0$ it holds for any k that either $\widehat{z}_k(y^*) > 0$ and $y_k^* = 1$, so $\widehat{\zeta}_k^h(y^*) = w_k^h$, $h \in I_H$, or $\widehat{z}_k(y^*) = 0$ and $0 \leq y_k^* \leq 1$. Therefore, if $\widehat{z}_k(y^*) = 0$ for all k, then y^* induces an SC $\widehat{P}$-equilibrium at prices $\widehat{p}$. If not, we have that $y_k^* = 1$ for at least one index k and so $y^* \in \overline{Y}^{\text{s}}$ and y^* induces an ED $\widehat{p}$-state. Therefore, the short-term process ends either at an SC $\widehat{P}$-equilibrium with price vector $\widehat{p}$ or at an ED $\widehat{p}$-state.

The properties of the short-term adjustment process are closely related to the ideas underlying the Walrasian tatonnement process, but now with adjustments of quantities instead of prices. In fact, at a point y reached by the process we have that $y_{k'} = \min_{k \in I_K} y_k$ if $\widehat{z}_{k'}(y) < 0$, so that $y_{k'}$ has been maximally decreased from the initial value and therefore supply rationing is maximally thightened from the initial supply rationing if there is excess supply of commodity k'. On the other hand, $y_{k'} = \max_{k \in I_K} y_k$ if $\widehat{z}_{k'}(y) > 0$, so that $y_{k'}$ has been maximally increased from the initial value and therefore supply rationing has been maximally weakened from the initial supply rationing if there is excess demand for commodity k'.

When the short-term process ends at a point y^* inducing an SC $\widehat{P}$-equilibrium, the process continues with long-term adjustments to find a Walrasian equilibrium. Observe that in this case it typically holds that $\max_{k \in I_K} y_k^* < 1$. It is possible to give robust examples where the short-term process terminates at an SC $\widehat{P}$-equilibrium; it suffices to take $\widehat{p}$ sufficiently high. In case the short-term process ends at a point y^* inducing an ED $\widehat{p}$-state it holds that $\max_{k \in I_K} y_k^* = 1$, i.e. $y^* \in \overline{Y}^{\text{s}}$. In this case the mid-term process is needed to take the economy to an SC $\widehat{P}$-equilibrium. Generically, the short-term process will not generate a WE.

To follow the short-term path numerically, we formulate an algorithm that generates points satisfying (10) for an approximation of the reduced total excess demand function in Section 11. Since the equations in (10) are not solvable explicitly in general, it is clear that we have to resort to an approximation in one way or another anyhow. The approximation is constructed in such a way that its inaccuracy can be made arbitrarily small. It will be shown in Section 11 that the points $y \in Y^{\text{s}}$ satisfying (10) for the approximation form a piecewise linear path. For the moment, just note as a heuristic that the dimension of the space $Y^{\text{s}} \times [0,1]$ of the parameters (y, γ) is equal to $K+1$, whereas there are K independent equations in (10) that have to hold with equality, leaving one degree of freedom. For a generic differentiable economy, one expects the solution set of (10) to be a 1-dimensional piecewise differentiable manifold with boundary. For such proofs in comparable settings, we refer to Herings (1996, 1997).

7. THE MID-TERM PROCESS

The properties of the mid-term adjustment process can be formulated as follows. A point y reached by the mid-term process belongs to the set Y^{m} and there exists

$\alpha \geq 1$ such that for every $k \in I_K$,

$$\begin{array}{ll} y_k = \alpha, & \text{if } \widehat{z}_k(y) > 0, \\ 0 \leq y_k \leq \alpha, & \text{if } \widehat{z}_k(y) = 0. \end{array} \tag{11}$$

Notice that a point $y^* \in \overline{Y}^{\mathrm{s}}$ inducing an ED $\widehat{p}$-state satisfies these properties for $\alpha = 1$.

The mid-term process initially proceeds from an ED $\widehat{p}$-state induced by a point $y^* \in \overline{Y}^{\mathrm{s}}$ found by the short-term process by increasing α from one. Then y_k is increased from one and therefore p_k from $\widehat{p}_k$ if there is excess demand in the market for factor k. The process no longer operates by making quantity adjustments only. The price of a factor k in excess demand is kept maximal by keeping y_k equal to α and the variable y_k corresponding to the market of a factor k in equilibrium is adjusted to keep this market in equilibrium. As soon as the market of a factor k with $y_k = \alpha$ becomes in equilibrium, y_k gets decreased away from α. On the other hand, if for the market of a factor k in equilibrium, y_k becomes equal to α, then y_k is not increased above α but is kept equal to α and excess demand may occur on this market. Again these properties are closely related to the ideas behind the Walrasian tatonnement process. Furthermore, the adjustment in the market for a factor k switches from quantity adjustment to price adjustment if the value of the variable y_k switches from below 1 to above 1 and reversely if the value goes from above 1 to below 1. In the latter case the price p_k has fallen back on its initial value $\widehat{p}_k$. This expresses the hypotheses that quantities adjust faster than prices and that factor prices are downwards rigid.

Any state $s(y)$ induced by a point y reached by the mid-term process is an ED $\widehat{P}$-state. So, starting from the ED $\widehat{p}$-state induced by a point y^* found by the short-term process, the mid-term process follows a path of ED $\widehat{P}$-states. Because of property (iv) of Lemma 5.1, there exists some $M > 0$ such that $\widehat{z}_k(y) < 0$ for some $k \in I_K$ if $\|y\|_\infty \geq M$. Since $\widehat{z}(y) \geq \underline{0}$ for any point y on the mid-term path, this implies that $\|y\|_\infty < M$ for any such y. From this it follows that the mid-term path either comes back to a point y^{**} in $\overline{Y}^{\mathrm{s}}$ at which $\alpha = 1$, and hence $y_k^{**} = 1$ for all markets k in excess demand and $y_k^{**} \leq 1$ for all markets in equilibrium, or ends at a point $\overline{y}$ satisfying $\widehat{z}(\overline{y}) = \underline{0}$. In the latter case all markets are in equilibrium and it holds that $p(\overline{y}) \geq \widehat{p}$ and $p_k(\overline{y}) > \widehat{p}_k$ implies $\zeta_k^h(\overline{y}) = w_k^h$ for all h. So, supply constraints are only active in markets that face a binding downwards price rigidity and hence an SC $\widehat{P}$-equilibrium has been found. In the former case a second ED $\widehat{p}$-state has been found and the process returns inside Y^{s} and continues by increasing γ from zero with short-term adjustments until again a point in Y^{s} is reached which induces an SC $\widehat{P}$-equilibrium with all prices equal to the initially given prices or a new point in $\overline{Y}^{\mathrm{s}}$ is found inducing an ED $\widehat{p}$-state. In the latter case the process continues with mid-term adjustments in Y^{m}, and so on. The combination of short-term and mid-term adjustment processes eventually leads to an SC $\widehat{P}$-equilibrium.

An SC $\widehat{P}$-equilibrium $\bar{y}$ is a WE if $\min\{\bar{y}_1, \ldots, \bar{y}_K\} \geq 1$. So, it may occur that the mid-term process ends at a WE, for instance if all factor prices are low enough initially.

Again, as a heuristic, the dimension of the space $Y^{\mathrm{m}} \times [1, \infty)$ of the parameters (y, α) is $K + 1$, whereas all K equations in (11) that have to hold with equality are independent, leaving one degree of freedom. Therefore, one should expect the set of solutions to (11) to be a 1-dimensional piecewise differentiable manifold with boundary for a generic differentiable economy.

8. THE LONG-TERM PROCESS

Both the Drèze process and the sequence of short-term and mid-term processes described in the previous sections converge to an SC $\widehat{P}$-equilibrium, which in some cases might actually be a WE. To reach a WE in general, factor prices should also be adjusted downwards in the long run. The long-term process generates a path of points $y \in \widehat{Y} \cap Y^1$ satisfying

$$\widehat{z}(y) = \underline{0}. \tag{12}$$

Along the path generated by the long-term process, prices and quantities are adjusted such that all factor markets are equilibrated. According to equations (7) and (8), any y satisfying (12) induces an SCE where, for $k \in I_K$,

$$\begin{array}{lll} p_k(y) = (1 - y_0)\widehat{p}_k & \text{and } \zeta_k^h(y) = g_k^h(y_k)w_k^h,\ h \in I_H, & \text{if } y_k \leq 1, \\ p_k(y) > (1 - y_0)\widehat{p}_k & \text{and } \zeta_k^h(y) = w_k^h,\ h \in I_H, & \text{if } y_k > 1, \end{array} \tag{13}$$

i.e. the price of a factor k subject to supply rationing is decreased to $(1 - y_0)\widehat{p}_k$ and the price of a factor without rationing is greater than or equal to this lower bound. These properties are again closely related to the ideas underlying the Walrasian tatonnement process. Notice that at any point y on the path of the long term process at least one factor price does not exceed the initial price since $y \in Y^1$ and therefore $\min\{y_1, \ldots, y_K\} \leq 1$.

Starting from a point $\bar{y}$ generated by either the short-term or mid-term process and inducing an SC $\widehat{P}$-equilibrium, the parameter y_0 is initially increased, implying that the prices of factors subject to supply rationing are all decreased proportionally. Simultaneously, the parameters y_k, $k \in I_K$, adjust to keep all markets in equilibrium. Rationing schemes adjust in factor markets for which $y_k < 1$, whereas prices adjust in factor markets for which $y_k > 1$. When y_k switches from below 1 to above 1, the adjustment regime in that market switches from quantity adjustment to price adjustment, and reversely.

Property (vi) of Lemma 5.1 guarantees that the long-term process does not leave $\widehat{Y}$. Because of property (v), there exists some $M > 0$ such that $\widehat{z}(y) \neq \underline{0}$

if $\|y\|_\infty \geq M$, and hence the path of the long term process remains in $\widehat{Y}$. The long-term process terminates at a point $\widetilde{y} \in \widehat{Y}$ in the boundary of $\widehat{Y}$ satisfying $\widetilde{y}_0 = 1$, or $\min\{\widetilde{y}_1, \ldots, \widetilde{y}_K\} = 1$, or $\widetilde{y}_0 = 0$. If $\widetilde{y}_0 = 1$, then a WE obtains. Indeed, $(1 - \widetilde{y}_0)\widehat{p}_k = 0$ for all k, and therefore, for any $k \in I_K$, either $p_k(\widetilde{y}) > (1 - \widetilde{y}_0)\widehat{p}_k = 0$ and $\zeta_k^h(\widetilde{y}) = w_k^h$, $h \in I_H$, or $p_k(\widetilde{y}) = (1 - \widetilde{y}_0)\widehat{p}_k = 0$. In the former case there is no rationing in factor market k. In the latter case rationing is not binding, because the factor price is equal to zero. If $\min\{\widetilde{y}_1, \ldots, \widetilde{y}_K\} = 1$, also a WE obtains, since then $\zeta^h(\widetilde{y}) = w^h$, $h \in I_H$. In case $\widetilde{y}_0 = 0$ it holds that $\widetilde{y}$ belongs to Y_0, so an SC $\widehat{P}$-equilibrium is induced again. The process switches either to mid-term adjustments generating points in Y^{m} inducing a path of ED $\widehat{P}$-states, or to short-term adjustments generating points in Y^{s} inducing a path of $\widehat{p}$-states. Again, as a heuristic, the dimension of the parameter space $\widehat{Y} \cap Y^{\mathrm{l}}$ is $K + 1$, whereas all K equations in (12) are independent, leaving one degree of freedom. For a generic differentiable economy, one expects the set of solutions to (12) to be a 1-dimensional piecewise differentiable manifold with boundary.

For a generic economy, the adjustment process can be summarized as follows. The combination of short-term, mid-term and long-term processes eventually converges to a WE. There is an odd number of SC $\widehat{P}$-equilibria and an odd number of WEs. Recall that an SC $\widehat{P}$-equilibrium might also be a WE, in which case the equilibrium counts for both. Consider the first SC $\widehat{P}$-equilibrium reached by either the short-term or the mid-term process. It is either also a WE or a starting point of a path described by the long-term process (12) having either another SC $\widehat{P}$-equilibrium or a WE as its other end point. In the former case, the second SC $\widehat{P}$-equilibrium is a starting point of a path from the short-term and the mid-term process, leading to a new SC $\widehat{P}$-equilibrium. This third SC $\widehat{P}$-equilibrium is then again either also a WE or the starting point of a second path described by the long-term process (12) having yet another SC $\widehat{P}$-equilibrium or a WE as another end point, and so on. Because all paths are bounded, eventually an SC $\widehat{P}$-equilibrium is found that either is a WE or leads to a WE. The path starting from v leads to one WE and contains an odd number of SC $\widehat{P}$-equilibria. Any other connected set of points in Y satisfying (10), (11) or (12) is either a loop containing no WE and an even number of SC $\widehat{P}$-equilibria or a path having two different WE as its end points and containing no other WE and an even number of SC $\widehat{P}$-equilibria.

9. AN ILLUSTRATIVE EXAMPLE

We consider an economy with a numeraire commodity, one final good and two production factors. The technology to produce the final good is given by $x_1 = 2\sqrt{z_1 z_2}$, with x_1 the output and $z_k \geq 0$ the input of production factor k, $k = 1, 2$. Cost minimizing behaviour of the production sector gives $a(p) = (\frac{1}{2}\sqrt{p_2/p_1},\ \frac{1}{2}\sqrt{p_1/p_2})^\top$ as the vector of inputs needed to produce one unit of the output. The cost function is given by $c(p) = \sqrt{p_1 p_2}$. It follows that $q(s) = \sqrt{p_1 p_2}$. We assume that there is one representative agent, with utility function defined by $u(x_0, x_1) = x_0 x_1$. The initial endowment is given by $w = (1,2)^\top$ and $m = 4$. This specification of the economy is closely related to the model of Malinvaud (1977), in which there is only one factor. The Walrasian equilibrium prices are given by $p^* = (2,1)^\top$ and $q^* = \sqrt{2}$. In the calculations that follow, it is useful to keep in mind that

$$z^{\mathrm{D}}(p,\zeta) = \left(\frac{p_1\zeta_1 + p_2\zeta_2 + 4}{4p_1}, \frac{p_1\zeta_1 + p_2\zeta_2 + 4}{4p_2} \right)^\top .$$

We now take $\widehat{s} = (\widehat{p}, \widehat{\zeta})$ with $\widehat{p} = (1\frac{1}{2}, 1\frac{1}{2})^\top$ and $\widehat{\zeta} = (\frac{1}{2}, 1)^\top$ as the initial state of the economy. The functions g_k, $k = 1,2$, are given by $g_k(e_k) = e_k$, $e_k \in [0,1]$, where e_k indicates the extend to which factor k is employed. Indeed, both factors are employed at a rate of 50 % at the starting point. Straightforward calculations show that the income of the representative household at state $\widehat{s}$ is given by $r(\widehat{s}) = 6\frac{1}{4}$ and that $q(\widehat{s}) = 1\frac{1}{2}$, giving a demand for the final good equal to $d(\widehat{s}) = 2\frac{1}{12}$. Factor demand is $z^{\mathrm{D}}(\widehat{s}) = (1\frac{1}{24}, 1\frac{1}{24})^\top$, which exceeds the supply of factors $z^{\mathrm{S}}(\widehat{s}) = \widehat{\zeta} = (\frac{1}{2}, 1)^\top$. Since factors are not used at full capacity, supply of factors will instantaneously adjust to meet the demand.

In the system (10), the parameter γ decreases from 1 to $\frac{14}{15}$, y_k increases from $\frac{1}{2}$ to $\widehat{y}_k = \frac{8}{15}$, $k = 1,2$, inducing $p(\widehat{y}) = \widehat{p}$ and $\zeta(\widehat{y}) = (\frac{8}{15}, \frac{16}{15})^\top$. At $\widehat{y}$, the market for factor 2 is equilibrated, while the market for factor 1 is still in excess demand. Quantities in the market for factor 2 are adjusted now to keep that market in equilibrium. On this part of the path it holds that

$$z_2^{\mathrm{D}}(p,\zeta) = \zeta_2 \text{ and } p = \widehat{p}.$$

Supply of factor 1 continues to increase. So, in (10), γ decreases from $\frac{14}{15}$ and simultaneously y_2 decreases away from $1 - \frac{1}{2}\gamma$ in such a way that the market for factor 2 remains in equilibrium. The increase in supply of factor 1 leads to more income, more demand for the final good, and therefore more demand for both factors. So, this part of the adjustment process corresponds to the Keynesian multiplier effect. Supply of factor 2 remains increasing to equilibrate that market. The process generates the path of points y given by $y_0 = 0$, $y_2 = \frac{4}{9} + \frac{1}{6}y_1$ moving from $\widehat{y} = (0, \frac{8}{15}, \frac{8}{15})^\top$ to $\overline{y} = (0, 1, \frac{11}{18})^\top \in \overline{Y}^{\mathrm{s}}$ with $p(\overline{y}) = \widehat{p}$ and $\zeta(\overline{y}) = (1, \frac{11}{9})^\top$. Equivalently, the process generates states (p,ζ) satisfying $p = \widehat{p}$, $\zeta_2 = \frac{8}{9} + \frac{1}{3}\zeta_1$, $\frac{8}{15} \le \zeta_1 \le 1$. At the ED $\widehat{p}$-state $s(\overline{y})$, factor 1 is used at its maximum capacity, but there is still excess demand for it. The mid-term process comes into play to increase its price.

The adjustment process turns from short-term to mid-term adjustments by increasing α from one in system (11), keeping $y_1 = \alpha$, and allowing y_2 to vary between

0 and α to keep market 2 in equilibrium. As long as $y_2 \leq 1$, it holds that

$$z_2^{\mathrm{D}}(p,\zeta) = \zeta_2,\ \zeta_1 = 1 \text{ and } p_2 = \widehat{p}_2.$$

The increase in p_1 leads to more income for the agent, leading again to higher demand for output, and consequently higher factor demands. Supply rationing in the market for factor 2 will diminish further. Straightforward calculations show that the path is given by $y_0 = 0$, $y_2 = \frac{1}{2} + \frac{1}{9}y_1$ until at $\alpha = 1\frac{1}{2}$ the point $y^* = (0, 1\frac{1}{2}, \frac{2}{3})^{\top}$ is reached, inducing an SC $\widehat{P}$-equilibrium state with $p(y^*) = (2, 1\frac{1}{2})^{\top}$ and $\zeta(y^*) = (1, 1\frac{1}{3})^{\top}$. Equivalently, the process generates states (p,ζ) satisfying $\zeta_1 = 1$, $p_2 = \widehat{p}_2$, $\zeta_2 = \frac{8}{9} + \frac{2}{9}p_1$, $1\frac{1}{2} \leq p_1 \leq 2$, until at $p_1 = 2$ it holds that $z_1(p,\zeta) = 0$. At this SC $\widehat{P}$-equilibrium, the price of the first factor is above its initial price and there is no supply rationing in the market for this factor. The price for the second factor is still equal to its initial price, and this factor is underutilized. Its employment rate did increase, however, from 50 % initially to 67 % now.

Finally, consider the long-term process. To clear all markets without rationing, the price of the second factor will have to decrease. The price of the first factor adjusts to keep that market in equilibrium. On this part of the path it holds that

$$z^{\mathrm{D}}(p,\zeta) = \zeta \text{ and } \zeta_1 = 1.$$

Decreasing p_2 leads to more demand for factor 2. Due to the Cobb-Douglas specification of technology, this is an income-neutral adjustment for the agent. However, decreased input prices lead to a decrease of the final good price, and an increased final good demand. Straightforward calculations show that the path is given by $y_1 = 1\frac{1}{2} + 1\frac{1}{2}y_0$, $y_2 = \frac{2}{3(1-y_0)}$, $0 \leq y_0 \leq \frac{1}{3}$, until the point $y^{**} = (\frac{1}{3}, 2, 1)^{\top}$ is reached, inducing a WE with $p(y^{**}) = (2,1)^{\top}$ and $\zeta(y^{**}) = (1,2)^{\top}$. Equivalently, the process generates states (p,ζ) satisfying $p_1 = 2$, $\zeta_1 = 1$, $\zeta_2 = \frac{2}{p_2}$, $\frac{3}{2} \geq p_2 \geq 1$.

It is interesting to consider the development of the national product during the course of the adjustment process. There are two ways to define the national product. One is to consider the demand for the final product and the alternative is to consider production actually possible with the factors supplied. During the short-term and mid-term processes these definitions are different, because of the mismatch in factor supply and factor demand. The most appropriate definition would be to consider the minimum of the two, since this would correspond to trade in the final good actually realized if trade had to take place during either the short-term or the mid-term process. Due to the specifics of the example, this is equivalent to taking the second definition. In this example, output can be verified to increase monotonically during the entire adjustment process. It starts at 1.41 at the initial state. When the second factor market equilibrates at $s(\widehat{y})$, output has risen to 1.51, and it attains a value of 2.21 at the ED $\widehat{p}$-state $s(\overline{y})$. The mid-term adjustment process leads to the SC $\widehat{P}$-equilibrium state $s(y^*)$ for which the output is 2.31. Finally, the Walrasian output level is 2.83.

10. CONVERGENCE RESULTS

In the example of the previous section, it is possible to solve for the path of the adjustment process analytically, due to the Cobb-Douglas specification of preferences and technology. In this section we give an algorithm that generates an approximation of the path for any economy satisfying A1-A5. The algorithm will solve (10), (11) and (12) for a piecewise linear approximation of the reduced total excess demand function $\widehat{z}$. The technique of simplicial approximation for the study of adjustment processes under price rigidities has already been introduced by van der Laan (1982) and has been applied to the adjustment from constrained equilibrium to Walrasian equilibrium in Herings (1996), Herings, van der Laan, Talman and Venniker (1997) and Herings, van der Laan and Venniker (1998). Alternatively, we could invoke differentiability assumptions, and prove that for a generic economy (10), (11) and (12) yield a 1-dimensional solution path that connects the initial state $\widehat{s}$ to a uniquely determined WE, see Herings (1996, 1997) for similar proofs in this setting. We have opted for the piecewise linear approximation, since the proofs are less tedious and provide us with an implementable algorithm.

Let $\widetilde{Y}$ denote the subset of points in Y that satisfy (10), (11) or (12). Not all the points in the set $\widetilde{Y}$ will actually be reached by the adjustment process. This will only hold for the points that are connected to the starting point $v = (0, \frac{1}{2}, \ldots, \frac{1}{2})^{\top}$ inducing the initially given state $\widehat{s}$. The component of v in $\widetilde{Y}$, i.e. the maximally connected subset of $\widetilde{Y}$ containing v, is denoted by $\widetilde{Y}^v$.

DEFINITION 10.1: The *price and quantity adjustment process* for the economy $\mathcal{E}$ is given by $\widetilde{Y}^v$.

The price and quantity adjustment process is defined by considering explicitly the points generated by it. Since we have given a topological definition of the process, we also give a topological definition of convergence.

DEFINITION 10.2: The price and quantity adjustment process for the economy $\mathcal{E}$ is *convergent* if $\widetilde{Y}^v$ is an arc having the starting point v and a WE as its boundary points.

An arc is a set homeomorphic to the unit interval $[0,1]$. Any arc described as the zero points of a system of continuously differentiable functions satisfying certain regularity properties is the solution of a system of differential equations, see for instance Garcia and Zangwill (1981). This is the approach adopted by Smale (1976) and Kamiya (1990) in their price adjustment processes for pure exchange economies without price rigidities. Such a system of differential equations will depend on the excess demand function and its first derivatives, exactly the amount of information indicated by Saari and Simon (1978). However, in our process there are three reasons for nondifferentiabilities. The first one is due to the fact that along the path of the process a supply constraint for a household may change from binding to nonbinding and vice versa. The second one is obtained when for a factor a change occurs in the short-term or mid-term complementarity conditions (10) and (11). The third one occurs when a change is made from quantity adjustment to price adjustment or vice versa, i.e. for some $k \in I_K$ the value of the variable y_k crosses 1 or the value of the variable y_0 changes form zero to positive or the reverse. These problems can be solved by taking a sequence of systems of differential equations. We avoid these problems by using the technique of simplicial approximation. In fact, this technique only needs continuity of the underlying function.

Although each point y belonging to $\widetilde{Y}^v$ is in $\widehat{Y}$, the simplicial technique to trace this path approximately may generate points not in $\widehat{Y}$. However, if $y \in Y \setminus \widehat{Y}$, then $q_l(s(y)) = 0$ for at least one $l \in I_L$, so $\widehat{z}(y)$ is not defined. To solve this problem, we compactify the consumption sets. The compactification should be such that it does not affect the adjustment process. To take care not to influence the short-term and mid-term process, observe that property (ii) of Lemma 5.1 guarantees that total factor demand, and therefore total demand for final goods is bounded for all $y \in Y_0$ by, say, $M^1 > 0$. Next, with respect to the long-term process, we define $w_{\max} = \max\{w_1, \ldots, w_K\}$. Using the same arguments as Debreu (1959), it can be derived that maximal production possible by using at most $w_{\max}$ of each factor, is bounded by some $M^2 > 0$. If total demand for some final good is greater than or equal to M^2, then the required inputs of at least one factor must exceed $w_{\max}$, contradicting factor market equilibrium. So, any bound M greater than or equal to $\max\{M^1, M^2\}$ can be taken to compactify the consumption sets. The resulting reduced total excess demand function, for simplicity denoted again by $\widehat{z}$ can be shown to be continuous on Y by standard arguments and the set $\widetilde{Y}$ and therefore the price and quantity adjustment process is unaffected by this compactification.

For $t \in \mathbb{N}$, $0 \leq t \leq K+1$, a t-dimensional simplex or t-simplex is defined as the convex hull of $t+1$ affinely independent vectors $y^1, \ldots, y^{t+1}$ in $\mathbb{R}^{K+1}$, called the vertices of the simplex, and is denoted by $\sigma(y^1, \ldots, y^{t+1})$ or shortly by σ. A k-face of a t-simplex σ is the convex hull of $k+1$ vertices of σ, $0 \leq k \leq t$. A $(t-1)$-face of a t-simplex σ is called a facet of σ. A triangulation Σ of the set $[0,1] \times \mathbb{R}^K_+$ is a locally finite collection of $(K+1)$-simplices such that $[0,1] \times \mathbb{R}^K_+$ is the union of all simplices in Σ and the intersection of any two simplices in Σ is either empty or a common face of both. The mesh size of a triangulation Σ is defined by $\mathrm{mesh}(\Sigma) = \sup\{\|x-y\|_\infty \mid x, y \in \sigma, \sigma \in \Sigma\}$.

Given a proper triangulation[4] Σ of $[0,1] \times \mathbb{R}^K_+$, a function $\widehat{Z} : Y \rightarrow \mathbb{R}^K$ is called the piecewise linear approximation of the reduced total excess demand function $\widehat{z}$ on Y with respect to Σ if for each vertex $y^j \in Y$ of any simplex $\sigma(y^1, \ldots, y^{K+2}) \in \Sigma$,

[4]Properness is a technical condition on the triangulation, which will be explained in the next section.

$\widehat{Z}(y^j) = \widehat{z}(y^j)$ and $\widehat{Z}$ is affine on each face in Y of a simplex in Σ. In the next section we describe an algorithm that generates a one-one, piecewise linear, continuous function $\pi : [0,1] \to Y$ such that $\pi(t)$, $t \in [0,1]$, satisfies (10), (11) or (12) with respect to $\widehat{Z}$. Moreover, it holds that $\pi(0) = v$, and that $\pi(1) = y^* \in Y$ yields a Walrasian equilibrium with respect to $\widehat{Z}$, in the sense that $\widehat{Z}(y^*) = 0$ and either $y_0^* = 1$ or $\min\{y_1^*, \ldots, y_K^*\} \geq 1$. We refer to π as the approximate price and quantity adjustment process.

THEOREM 10.3: *Let the economy $\mathcal{E}$ satisfy Assumptions A1-A5. Let Σ be a proper triangulation of $[0,1] \times \mathbb{R}_+^K$. Then, for every $y \in \pi([0,1]) \cap Y^{\mathrm{s}}$ there exists a real number $\gamma \in [0,1]$ such that for every $k \in I_K$*

$$\begin{array}{ll} y_k = \frac{1}{2}\gamma, & \text{if } \widehat{Z}_k(y) < 0, \\ \frac{1}{2}\gamma \leq y_k \leq 1 - \frac{1}{2}\gamma, & \text{if } \widehat{Z}_k(y) = 0, \\ y_k = 1 - \frac{1}{2}\gamma, & \text{if } \widehat{Z}_k(y) > 0, \end{array}$$

for every $y \in \pi([0,1]) \cap Y^{\mathrm{m}}$ there exists a real number $\alpha \geq 1$ such that for every $k \in I_K$

$$\begin{array}{ll} y_k = \alpha, & \text{if } \widehat{Z}_k(y) > 0, \\ 0 \leq y_k \leq \alpha, & \text{if } \widehat{Z}_k(y) = 0, \end{array}$$

and for every $y \in \pi([0,1]) \cap (Y^1 \setminus Y_0)$ it holds that $\widehat{Z}(y) = \underline{0}$.

PROOF: Analogous to the proof of Theorem 1 in Herings, van der Laan and Venniker (1998). *Q.E.D.*

The next theorem shows the accuracy of the approximation along the piecewise linear path.

THEOREM 10.4: *Let the economy $\mathcal{E}$ satisfy Assumptions A1-A5. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every proper triangulation Σ of $[0,1] \times \mathbb{R}_+^K$ satisfying mesh$(\Sigma) < \delta$, it holds that*

1. *for every $y \in \pi((0,1]) \cap Y^{\mathrm{s}}$, for every $k \in I_K$,*

$$\begin{array}{ll} \widehat{z}_k(y) < \varepsilon, & \text{if } y_k = \frac{1}{2}\gamma, \\ -\varepsilon < \widehat{z}_k(y) < \varepsilon, & \text{if } \frac{1}{2}\gamma < y_k < 1 - \frac{1}{2}\gamma, \\ \widehat{z}_k(y) > -\varepsilon, & \text{if } y_k = 1 - \frac{1}{2}\gamma, \end{array}$$

2. *for every $y \in \pi([0,1]) \cap Y^{\mathrm{m}}$, for every $k \in I_K$,*

$$\begin{array}{ll} \widehat{z}_k(y) > --\varepsilon, & \text{if } y_k = \alpha, \\ -\varepsilon < \widehat{z}_k(y) < \varepsilon, & \text{if } 0 \leq y_k < \alpha, \end{array}$$

3. for every $y \in \pi([0,1]) \cap (Y^1 \setminus Y_0)$, $\|\hat{z}(y)\|_\infty < \varepsilon$.

PROOF: Analogous to the proof of Theorem 2 in Herings, van der Laan and Venniker (1998). *Q.E.D.*

Although the assumptions made on the model are not sufficient to guarantee that $\widetilde{Y}^v$ is convergent, the piecewise linear path traced by the algorithm always ends with a point y^* yielding a Walrasian equilibrium with respect to $\widehat{Z}$. The last part of Theorem 10.4 shows that y^* is an approximate Walrasian equilibrium, i.e. at y^* the excess demand $\hat{z}(y^*)$ can assured to be arbitrarily close to zero by taking the mesh size of the triangulation small enough. Using this it can be proved that $\widetilde{Y}^v$ contains a Walrasian equilibrium. The next theorem says that $\widetilde{Y}^v$ always connects the starting point with a WE, even when $\widetilde{Y}^v$ is not convergent.

THEOREM 10.5: *Let the economy* $\mathcal{E}$ *satisfy Assumptions A1-A5. Then* $\widetilde{Y}^v$ *contains a WE.*

PROOF: Analogous to the proof of Theorem 5.4.1 in Herings (1996). *Q.E.D.*

The final result shows that the whole path of the approximate adjustment process gets arbitrarily close to $\widetilde{Y}^v$. The reverse is not necessarily true, but holds if $\widetilde{Y}^v$ is convergent. For a nonempty compact set S of $\mathbb{R}^n$ we define the distance function $d_S: \mathbb{R}^n \to \mathbb{R}$ by $d_S(\overline{y}) = \min_{y \in S} \|y - \overline{y}\|_2$, $\overline{y} \in \mathbb{R}^n$. Notice that both the sets $\widetilde{Y}^v$ and $\pi([0,1])$ are compact.

THEOREM 10.6: *Let the economy* $\mathcal{E}$ *satisfy Assumptions A1-A5. Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every proper triangulation* Σ *of* $[0,1] \times \mathbb{R}^K_+$ *satisfying* $mesh(\Sigma) < \delta$, *it holds that*
(i) for every $t \in [0,1]$, $d_{\widetilde{Y}^v}(\pi(t)) < \varepsilon$,
(ii) for every $y \in \widetilde{Y}^v$, $d_{\pi([0,1])}(y) < \varepsilon$ *if the price and quantity adjustment process is convergent.*

PROOF: Analogous to the proof of Theorem 3 in Herings, van der Laan and Venniker (1998). *Q.E.D.*

11. TECHNICAL DETAILS OF THE ALGORITHM

Let $R = \{r \in \mathbb{R}^K \mid r_k \in \{-1, 0, +1\}, k \in I_K\}$ be the set of K-dimensional sign vectors. For any $r \in R$, define the sets $I^-(r)$, $I^0(r)$, and $I^+(r)$ by $I^-(r) = \{k \in I_K \mid r_k = -1\}$, $I^0(r) = \{k \in I_K \mid r_k = 0\}$, $I^+(r) = \{k \in I_K \mid r_k = +1\}$. Let $i^-(r)$,

$i^0(r)$, and $i^+(r)$ denote the number of elements in these respective sets. For any $r \neq \underline{0}$, define the $(i^0(r)+1)$-dimensional set $Y^s(r)$ by

$$\begin{aligned} Y^s(r) = \{y \in Y^s \mid \exists \gamma \in [0,1] \text{ such that } & \forall k \in I^-(r) : y_k = \frac{1}{2}\gamma, \\ & \forall k \in I^0(r) : \frac{1}{2}\gamma \leq y_k \leq 1 - \frac{1}{2}\gamma, \\ & \forall k \in I^+(r) : y_k = 1 - \frac{1}{2}\gamma \ \}, \end{aligned}$$

for any $r \neq \underline{0}$ with $i^+(r) \geq 1$, define the $(i^0(r)+1)$-dimensional set $Y^m(r)$ by

$$\begin{aligned} Y^m(r) = \{y \in Y^m \mid \exists \alpha \geq 1 \text{ such that } & \forall k \in I^+(r) : y_k = \alpha, \\ & \forall k \in I^0(r) : y_k \leq \alpha, \\ & \forall k \in I^-(r) : y_k = 0 \ \}, \end{aligned}$$

and for any $r \neq \underline{0}$ with $i^+(r) = 0$, define the $(i^0(r)+1)$-dimensional set $Y^l(r)$ by

$$Y^l(r) = \{y \in Y^l \mid \forall k \in I^-(r) : y_k = 0\}.$$

For $r \neq \underline{0}$, the $(i^0(r)+1)$-dimensional set $Y(r)$ is defined by $Y(r) = Y^s(r) \cup Y^m(r) \cup Y^l(r)$, where any of the sets $Y^\kappa(r)$, $\kappa \in \{\text{s, m, l}\}$, is assumed to be empty if it is not defined above. For ease of notation, we denote $Y(\underline{0}) = [0,1] \times \mathbb{R}^K_+$.

Let Σ be a triangulation of $Y(\underline{0})$ such that for any $r \neq \underline{0}$ and for any $\kappa \in \{\text{s, m, l}\}$ the restriction of Σ to any nonempty set $Y^\kappa(r)$ induces a triangulation $\Sigma^\kappa(r)$ in $(i^0(r)+1)$-simplices of $Y^\kappa(r)$. Such a triangulation of $Y(\underline{0})$ is said to be a proper triangulation and has the property that for any $r \neq \underline{0}$, the collection of simplices $\Sigma(r) = \Sigma^s(r) \cup \Sigma^m(r) \cup \Sigma^l(r)$ yields a triangulation of $Y(r)$. Again for ease of notation, we denote $\Sigma(\underline{0}) = \Sigma$, so that for any $r \in R$, $\Sigma(r)$ triangulates $Y(r)$. For the remainder of this section some proper triangulation of $Y(\underline{0})$ is assumed to be given, for example we can take the K'-triangulation as proposed by Todd (1978).

Let $r \in R$ be a sign vector with $i^0(r) = t$, for some $t \in I^0_K$, and let $\sigma(y^0, \ldots, y^{t+1})$ be a $(t+1)$-dimensional simplex in $\Sigma(r)$. Consider solutions

$$(\lambda_0, \ldots, \lambda_{t+1}, (\mu_k)_{k \in I^-(r) \cup I^+(r)}) \in \mathbb{R}^{K+2}$$

of the following system of $K+1$ equations:

$$\sum_{j \in I^0_{t+1}} \lambda_j \begin{pmatrix} 1 \\ \hat{z}(y^j) \end{pmatrix} - \sum_{k \in I^-(r) \cup I^+(r)} \mu_k \begin{pmatrix} 0 \\ r_k e(k) \end{pmatrix} = \begin{pmatrix} 1 \\ \underline{0} \end{pmatrix}, \tag{14}$$

where $e(k)$ denotes the k-th K-dimensional unit vector. If $\lambda_j \geq 0$, $j \in I^0_{t+1}$, and $\mu_k \geq 0$, $k \in I^-(r) \cup I^+(r)$, then $(\lambda_0, \ldots, \lambda_{t+1}, (\mu_k)_{k \in I^-(r) \cup I^+(r)})$ is called an admissible solution to (14). An admissible solution of this $(K+1)$-system is called nondegenerate if at most one of the $K+2$ variables is equal to zero. The usual assumption made in the literature that there exist no degenerate admissible solutions is not very convincing for the problem under consideration. Following Eaves (1971), Todd (1976), Wright (1981), Herings, Talman and Yang (1996) and others, nondegeneracy can be handled by applying lexicographic pivoting. A row vector of $\mathbb{R}^m$ is said to be lexicographically positive if it is nonzero and its first nonzero component is positive. An $m \times m$-matrix A is said to be lexicopositive if each row is lexicographically positive. For $r \in R$ and a t-simplex $\tau(y^1, \ldots, y^{t+1}) \in \Sigma(r)$ with $t = i^0(r)$, let the $(K+1) \times (K+1)$-matrix $A_{r,\tau}$ be defined by

$$A_{r,\tau} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \widehat{z}(y^1) & \cdots & \widehat{z}(y^{t+1}) & -r_{k^1}e(k^1) & \cdots & -r_{k^{K-t}}e(k^{K-t}) \end{pmatrix},$$

where $\{k^1, \ldots, k^{K-t}\} = I^-(r) \cup I^+(r)$. Now we have the following definition.

DEFINITION 11.1: Let τ be a t-simplex in Y and let r be a sign vector with $i^0(r) = t$. Then τ is *r-complete* if τ is a facet of a simplex in $\Sigma(r)$ and $A_{r,\tau}^{-1}$ exists and is lexicopositive.

Given an r-complete t-simplex $\tau(y^1, \ldots, y^{t+1})$ in $Y(r)$, system (14) has an admissible solution with $\lambda_0 = 0$ and $(\lambda_1, \ldots, \lambda_{t+1}, \mu_{k^1}, \ldots, \mu_{k^{K-t}})$ equal to the first column of the matrix $A_{r,\tau}^{-1}$, for any $(t+1)$-simplex $\sigma(y^0, y^1, \ldots, y^{t+1})$ in $\Sigma(r)$ having τ as a facet. Due to the properties of a triangulation, there is only one such simplex if τ lies in the boundary of $Y(r)$ and there are precisely two such simplices otherwise. The basic idea of the algorithm is, starting in v, to generate for varying vectors $r \in R$ a unique sequence of adjacent r-complete facets of simplices in $\Sigma(r)$.

DEFINITION 11.2: An r^1-complete simplex τ^1 and an r^2-complete simplex τ^2 are *adjacent complete simplices* if $r^1 = r^2 = r$ and τ^1 and τ^2 are both facets of a simplex σ of $\Sigma(r)$ in Y, or if τ^1 is a facet of τ^2 and τ^2 is a simplex of $\Sigma(r^1)$, or if τ^2 is a facet of τ^1 and τ^1 is a simplex of $\Sigma(r^2)$.

The next lemmas show that the sequence of adjacent complete simplices is uniquely determined. Lemma 11.3 determines the unique complete starting simplex, Lemma 11.4 determines movements in the interior of a region $Y(r)$ and Lemma 11.5 considers the case where a complete facet in the boundary of a region $Y(r)$ is generated.

LEMMA 11.3: *Let the sign vector r be such that $r_k = -1$ if $\widehat{z}_k(v) \leq 0$ and $r_k = +1$ if $\widehat{z}_k(v) > 0$. Then the 0-simplex $\tau = \tau(\{v\})$ is an r-complete simplex and is not an $\overline{r}$-complete simplex for any other sign vector $\overline{r} \in R$.*

PROOF: Consider the matrix $A_{r,\tau}$ given by

$$A_{r,\tau} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \widehat{z}(v) & -r_1 e(1) & \cdots & -r_K e(K) \end{pmatrix}.$$

This matrix is not singular and its inverse is given by

$$A_{r,\tau}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a & -r_1 e(1) & \cdots & -r_K e(K) \end{pmatrix},$$

where the vector a is given by $a_k = r_k \widehat{z}_k(v)$, $k \in I_K$. Since $a_k > 0$ if $\widehat{z}(v) \neq 0$ and $-r_k > 0$ if $\widehat{z}_k(v) = 0$, the matrix $A_{r,\tau}^{-1}$ is lexicopositive. Moreover, there is no other sign vector $\bar{r} \in R$ for which the matrix $A_{\bar{r},\tau}^{-1}$ is also lexicopositive. *Q.E.D.*

LEMMA 11.4: *Let σ be a $(t+1)$-simplex of $\Sigma(r)$ in Y where r is a sign vector with $i^0(r) = t$. If σ has an r-complete facet τ, then exactly one of the following cases holds:*

1. *The simplex σ is an $\tilde{r}$-complete simplex for precisely one sign vector $\tilde{r}$ and no facets of σ other than τ are r-complete.*
2. *The simplex σ has exactly one other r-complete facet $\tilde{\tau}$ and σ is not an $\tilde{r}$-complete simplex.*

PROOF: Analogous to the proof of Lemma 3.5 in Herings, Talman and Yang (1996). *Q.E.D.*

LEMMA 11.5: *Let τ be an r-complete t-simplex being a facet of a simplex of $\Sigma(r)$ in Y where r is a sign vector with $i^0(r) = t$. Suppose that τ is a subset of $Y(\tilde{r})$ where $i^0(\tilde{r}) = t - 1$. Then exactly one of the following cases holds:*

1. *The simplex τ is an $\bar{r}$-complete simplex for precisely one sign vector $\bar{r} \neq r$ and τ has no $\tilde{r}$-complete facets;*
2. *Precisely one facet of τ is $\tilde{r}$-complete and τ is not an $\bar{r}$-complete simplex for any sign vector $\bar{r} \neq r$.*

PROOF: Analogous to the proof of Lemma 3.6 in Herings, Talman and Yang (1996). *Q.E.D.*

The next theorem follows from the results stated above.

THEOREM 11.6: *Let the economy $\mathcal{E}$ satisfy Assumptions A1-A5. Let Σ be a proper triangulation of $Y(\underline{0})$ and let τ be an r-complete simplex for some $r \in R$. If $\tau = \{v\}$ or if both $r = \underline{0}$ and $\tau \subset \{y \in Y^{\mathrm{m}} \mid \min\{y_1, \ldots, y_K\} \geq 1\}$ or if both $r = \underline{0}$ and $\tau \subset \{y \in Y^1 \mid \min\{y_1, \ldots, y_K\} = 1\}$ or if both $r \leq \underline{0}$ and $\tau \subset \{y \in Y^1 \mid y_0 = 1\}$, then there exists exactly one adjacent complete simplex to τ. Otherwise, τ has exactly two adjacent complete simplices. Moreover, for some $N > 0$, there exists a finite sequence of simplices $\tau^0, \ldots, \tau^N$, $\tau^N \neq \tau^0$, such that $\tau^0 = \{v\}$, τ^N has just one adjacent complete simplex, and τ^{n-1} and τ^n are adjacent complete simplices for $n = 1, \ldots, N$.*

The sequence of simplices mentioned in the theorem can be generated by an algorithm that starts in τ^0 and makes alternating lexicographic pivot steps in system (14) and replacement steps in the triangulation in order to determine the next complete simplex. The final simplex generated by the algorithm, τ^N, induces a Walrasian equilibrium with respect to the piecewise linear approximation $\widehat{Z}$ and thus according to the last part of Theorem 10.4 an approximate Walrasian equilibrium. We conclude with the steps of the algorithm.

STEPS OF THE ALGORITHM

STEP 0. Take $v = (0, \frac{1}{2}, \ldots, \frac{1}{2})^\top$, r as in Lemma 11.3, $\tau = \{v\}$, and let σ be the unique simplex in $\Sigma(r)$ having τ as a facet.

STEP 1. Make a lexicographic pivot step in (14) with the vector $(1, \hat{z}(y)^\top)^\top$ where y is the vertex of σ opposite to τ.

STEP 2. If $(1, \hat{z}(y^i)^\top)^\top$, for some $i \in I^0_{t+1}$, is the leaving column, go to Step 3 with τ equal to the facet of σ opposite to y^i. Otherwise, $(0, r_h e(h)^\top)^\top$ is the leaving column for some $h \in I^-(r) \cup I^+(r)$ and go to Step 4.

STEP 3. Stop if either both $r = \underline{0}$ and $\min\{y_1, \ldots, y_K\} = 1$ for every $y \in \tau$ or both $r \leq \underline{0}$ and $y_0 = 1$ for every $y \in \tau$. If $\tau \in \Sigma(\bar{r})$ for some $\bar{r} \neq r$, then go to Step 5. Otherwise there is precisely one simplex $\bar{\sigma} \in \Sigma(r)$, $\bar{\sigma} \neq \sigma$, sharing τ with σ. Set $\sigma = \bar{\sigma}$ and return to Step 1.

STEP 4. Set $r_h = 0$. Stop if both $r = \underline{0}$ and $\min\{y_1, \ldots, y_K\} \geq 1$ for all $y \in \tau$. Otherwise there is precisely one simplex $\bar{\sigma} \in \Sigma(r)$ containing τ as a facet. Set $\sigma = \bar{\sigma}$ and return to Step 1.

STEP 5. Let h be the unique index for which $\bar{r}_h \neq r_h$. Set $r = \bar{r}$ and make a lexicographic pivot step with $(0, -r_h e(h)^\top)^\top$. Go to Step 2.

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