RATIONAL ANIMAL SPIRITS[1]

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ABSTRACT: Financial markets consist of many traders having different "beliefs" about future earnings and prices of stocks, and therefore traders use different trading strategies, such as fundamental versus technical analysis. We discuss the concept of *Adaptive Belief Systems*, where a financial market is viewed as a nonlinear, evolutionary system between competing trading strategies. Agents choose their prediction strategy from a finite set and revise their beliefs according to evolutionary fitness measured by accumulated realized profits or wealth. Bifurcation routes to market instability and complicated, chaotic asset price fluctuations may arise when the agents sensitivity to differences in fitness, the diversity of beliefs and/or the fitness memory increase. Technical analysts may survive evolutionary competition and are not necessarily driven out of the market by fundamentalists. The evolutionary dynamics is characterized by an irregular switching between phases where fundamentalists dominate the market and prices are close to the RE-fundamental value and phases where technical analysis dominates and prices follow a self-fulfilling upward or downward trend. Technical analysts are not "irrational," but boundedly rational and may in fact earn above average net profits and achieve higher accumulated wealth than fundamentalists.

KEYWORDS: Endogenous asset price fluctuations, heterogeneous beliefs, evolutionary dynamics, bounded rationality, bifurcations, chaos.

JEL CODES: D84, E32, G12.

1. INTRODUCTION

There has been a long and still ongoing debate whether prices of risky financial assets are completely determined by economic fundamentals or whether traders' expectations or beliefs affect asset prices significantly. For example, Keynes already argued that: *"Investment based on genuine long-term expectation is so difficult as to be scarcely practible. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and given equal intelligence, he may make more disastrous mistakes"* (Keynes (1936), p. 157). In Keynes view, stock prices are not completely determined by economic fundamentals, but investors "animal spirits" constitute an additional source of stock price fluctuations. More recent collections

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of articles arguing that "market psychology" plays a significant role in the pricing of risky assets can e.g. be found in Shiller (1989) and Thaler (1994). Among financial practitioners, there also seems to be a commonly shared view that the large volatility so frequently observed in financial markets these days can not be attributed completely to economic fundamentals, but that the "psychological state of the market" may lead to sudden, large changes in stock prices triggered by news about small changes in fundamentals.

In modern financial market theory however, it has often been emphasized that the view that "market psychology" affects asset prices is incompatible with the efficient market hypothesis (EMH) and the rational expectations hypothesis (REH), see e.g. Fama (1970) or Malkiel (1987). In particular, market psychology, technical analysis and noise traders are viewed as "irrational" behaviour and therefore incompatible with a theory where markets consist of rational traders optimizing their short or long term utility such as accumulated profits or wealth. In a perfectly rational world every trader has, given all his or her available information, correct expectations and trading can only occur because of asymmetric information.

In this paper we discuss the concept of an *Adaptive Belief System* (ABS), that is, a financial market consisting of many heterogeneous agents with competing trading strategies based upon different beliefs about future prices. For some agents beliefs may be completely determined by economic fundamentals; others use simple technical trading strategies and may have (temporary) upward or downward biased beliefs. Traders update their beliefs or trading strategies over time according to an evolutionary "fitness" measure, such as accumulated past profits or wealth. The nonlinear evolutionary interaction in such a heterogeneous world may lead to asset price fluctuations similar to those observed in real markets. In particular, when traders are very sensitive to differences in fitness of trading strategies or when the diversity in beliefs is high, market instability and complicated asset price fluctuations may arise, characterized by an irregular switching between phases where prices are close to their fundamental value and optimistic or pessimistic phases where prices deviate in an upward or downward trend from their fundamental value. These complicated asset price fluctuations are driven by (boundedly) rational evolutionary selection of trading strategies (*"rational animal spirits"*). It is important to note that in such a heterogeneous agents world, technical analysis is not irrational, but may in fact earn higher profits during certain periods or even lead to above average accumulated wealth.

Rather than challenging rational expectations theory or trying to find an alternative to rational expectations equilibrium (REE), our theory *builds on* the extensive work in REE theory by expressing beliefs in terms of *deviations from a structural REE model*. These deviations can be viewed as each REE trader's *beliefs* about how the deviations from REE by the *rest* of the trading community might show up in equilibrium prices. Indeed truly rational traders must take into account the be-

havior of other "non-rational" traders in the trading community. Our theory leads to a decomposition of excess returns of the risky asset, consisting of an "REE" part which is a Martingale Difference Sequence (MDS) plus an "endogenous dynamical" part which is contributed by our theory. This "extra part" is rather like "endogenous uncertainty" in the sense of Kurz's book (1997) which develops a theory of rational belief equilibria (RBE). Brock, Lakonishok and LeBaron (1992) have documented that some popular parameterizations of the EMH such as GARCH models tend to under predict (over predict) returns (volatility) following buy signals and tend to over predict returns following sell signals of certain trading strategies. Significance is measured by bootstrapping the null distribution of trading strategy statistics under the null model under scrutiny. Our theoretical decomposition of returns may shed light on economic forces causing such empirical findings.

Let us now relate the present paper to some of our other recent work on ABS. Brock and Hommes (henceforth BH) (1997a) introduced an evolutionary dynamics in a demand-supply cobweb model with heterogenous agents, where sophisticated predictors such as rational expectation are more costly to obtain than simple rule of thumb predictors such as adaptive expectations. They showed that a *rational route to randomness,* i.e. a bifurcation route to complicated, chaotic equilibrium price fluctuations, occurs when traders become more sensitive to differences in fitness (past net realized profits) associated to the different expectations schemes. Brock (1997) and BH (1997b, 1998) introduce the same evolutionary dynamics into the present discounted value asset pricing model, and investigate rational routes to randomness in a heterogeneous agent asset pricing model.

The present paper emphasizes two novel aspects of ABS. In previous work, for analytic tractability attention has been mainly focussed on the case where fitness is given by net realized profits in the most recently observed period. Here we will focus on the case where fitness is given by accumulated past realized profits and show that a similar bifurcation route to complicated asset price fluctuations occurs when the memory of the fitness measure increases. Secondly, we present an example of a "Large Type Limit" (LTL). The LTL concept was sketched in Brock (1997) and Brock and de Fontnouvelle (1999) and will be more rigorously developed in BH (1999). In the ABS examples studied in BH (1997b, 1998) the number of different trader types is small, e.g. two, three or four, including types such as fundamentalists, trend followers, contrarians or upward or downward biased traders. In real marktets, one would expect *many* different types of traders. A Large Type Limit (LTL) is a deterministic "approximation" of a market with many different traders types, where initial beliefs are drawn from some random distribution. The LTL is a type of ensemble limit rather like thermodynamic limits in statistical mechanics. It is motivated by observing that equilibrium equations in our models tend to have a form which is a function of expressions that look like sample moments. This leads to taking a type of limit so that these sample moments converge to pop-

ulation moments. For particular distributions of characteristics, one may obtain closed form solutions for these population moments and thus obtain closed form expressions for objects that appeared intractable. Here we will study a simple, but typical example of an LTL, and investigate bifurcation routes to market instability and complicated dynamics.

The LTL theory developed here may be used to form a bridge between analytical results and the literature on artifical evolutionary stock market with computer simulation of asset trading, which has become very popular in the last few years (see e.g. Arthur et al. (1997), LeBaron et al. (1999) and LeBaron's review (1999)). Unfortunately there are no analytical results available in the simulation literature. We believe our notion of Large Type Limit may be useful as a theoretical approach to this more computationally oriented literature.

The plan of the paper is as follows. In Section 2 we present the present discounted value asset pricing model with heterogeneous agents. Section 3 focusses on the two type case, with fundamentalists versus trend chasers. Two key parameters of the model are the memory in fitness, i.e. the rate of decay of past profits in the traders' fitness functions, and the "intensity of choice" to switch trading strategies, i.e. the traders' sensitivity to differences in profits. We show that market instability and complicated asset price fluctuations arise when the intensity of choice becomes high and/or memory in the fitness measure increases. In Section 4 we go to the other extreme and treat Adaptive Belief Systems with a large number of belief types. Here we introduce the notion of belief characteristic space and the notion of Large Type Limit (LTL), and investigate bifurcation routes to complicated price fluctuations for the LTL. Finally, Section 5 concludes.

2. ADAPTIVE BELIEFS IN THE BASIC MEAN VARIANCE FRAMEWORK

Consider an asset pricing model with one risky asset and one risk free asset. Let p_t, denote the price (ex dividend) per share of the risky asset and let $\{y_t\}$ be a stochastic dividend process of the risky asset at time t. Let there be a risk free asset which is perfectly elastically supplied at gross return R. We have

$$\mathbf{W}_{t+1} = RW_t + (\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)z_t,$$

for the dynamics of wealth where bold face type denotes random variables at date $t + 1$ and z_t denotes the number of shares of the risky asset purchased at date t. Let E_t, V_t denote conditional expectation and conditional variance based on a publically available information set such as past prices and dividends and let E_{ht} and V_{ht} denote conditional expectation and variance of investor type h. We shall sometimes call these conditional objects the "beliefs" of trader type h. Assume each investor type is a myopic mean variance maximizer so that demand for shares, z_{ht}

solves

$$\text{Max}\{E_{ht}\mathbf{W}_{t+1} - (a/2)V_{ht}(\mathbf{W}_{t+1})\}, \text{ i.e.,} \tag{1}$$
$$z_{ht} = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)/aV_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t),$$

where the parameter a measures the risk aversion. Let z_{st} denote the supply of shares per investor at date t and n_{ht} the fraction of investors of type h at date t. Then equilibrium of supply and demand implies,

$$\sum n_{ht}\{E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)/aV_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)\} = z_{st}. \tag{2}$$

Hence, in the case when there is only one type h, equilibration of supply and demand yields the pricing equation

$$Rp_t = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) - aV_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)z_{st}. \tag{3}$$

Given a precise sequence of information sets $\mathcal{F}_t$ we may use (3) to define a notion of *fundamental solution* by letting E_{ht}, V_{ht} denote conditional mean and variance upon $\mathcal{F}_t$. Now specialize (3) to the special case of zero supply of outside shares, i.e., $z_{st} = 0$, for all t, to obtain[2]

$$Rp_t = E_t(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}). \tag{4}$$

It is well known that (4) typically has infinitely many solutions but, for the standard case $R > 1$, only one *fundamental solution* satisfies the "no bubbles" condition

$$\lim(E_1 p_t/R^t) = 0,$$

where the limit is taken as $t \to \infty$, provided the series summation $E_1\{y_t/R^{t-1}\}$ is absolutely convergent. In the special case where the dividend process $\{\mathbf{y}_t\}$ is IID, i.e. $E\{\mathbf{y}_{t+1} \mid \mathcal{F}_t\} = \bar{y}$ is constant,[3] the fundamental solution has the particularly simple form

$$p_t^* = \bar{p} = \bar{y}/(R-1).$$

Now return to the case of heterogeneous beliefs. Assuming that the conditional variance is the same for all types h, i.e. $V_{ht} \equiv V_t$, the market equilibrium equation (2) can be rewritten as

$$Rp_t = \sum n_{ht}E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) - aV_t(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)z_{st},$$

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[2]For the general case of time varying supply z_{st} of outside shares one may proceed similarly by replacing $\mathbf{y}_{t+1}$ in (4) by $\mathbf{y}_{h,t+1}^{\#} = y_{t+1} - az_{st}V_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1})$ and interpret $\mathbf{y}_{h,t+1}^{\#}$ as the risk adjusted dividend; see Brock (1997). The special case $z_{st} = 0$ of zero supply of outside shares is, up to a constant, equivalent to the case of constant supply of outside shares, i.e. $z_{st} = z_s$, for all t.

[3]It is easy to generalize the development to cases where the dividend process $\{\mathbf{y}_t\}$ is not IID. See e.g. Brock and Hommes (1997b) for an example with the dividend process given by a random walk with martingale difference sequence errors.

which for the case of zero supply of outside shares, i.e. $z_{st} = 0$, simplifies to the pricing equilibrium equation[4]

$$Rp_t = \sum n_{ht} E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}). \tag{5}$$

In order to proceed, we shall make some simplifying assumptions concerning the beliefs of type h :

A1. $V_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t) \equiv V_t(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t) \equiv \sigma^2$, for all h, t.

A2. $E_{ht}(\mathbf{y}_{t+1}) \equiv E_t(\mathbf{y}_{t+1})$, for all h, t.

A3. All beliefs $E_{ht}(\mathbf{p}_{t+1})$ are of the form

$$E_{ht}(\mathbf{p}_{t+1}) = E_t \mathbf{p}_{t+1}^* + f_h(x_{t-1}, \dots, x_{t-L}), \text{ for all } h, t.$$

Assumption A1 states that beliefs about the conditional variance are constant and are the same for all types.[5] [6] Assumption A2 amounts to homogeneity of beliefs on the one step ahead predictor of earnings; as stated above here we will focus on the case of IID dividends, i.e. $E_t(\mathbf{y}_{t+1}) = \bar{y}$, for all h, t. Finally, assumption A3 restricts the beliefs $E_{ht}(\mathbf{p}_{t+1})$ to time stationary functions f_h of past *deviations* of a commonly shared view of the fundamental. We write $f_{ht} \equiv f_h(x_{t-1}, \dots, x_{t-L})$.

These assumptions allow us to improve tractability by working in the space of deviations from the benchmark fundamental. Let $x_t = p_t - p_t^*$, denote the *deviation from the fundamental* solution. Notice that working in deviations from the fundamental also allows us to consider more general dividend processes $\{y_t\}$, with p_t^* the corresponding fundamental. Using assumptions A2 and A3 in the equilibrium equation (5), and since $\sum n_{ht} = 1$ for all t and the fundamental solution p_t^* satisfies (4), the equilibrium equation can be rewritten in deviations form:

$$Rx_t = \sum n_{ht} f_h(x_{t-1}, \dots, x_{t-L}) \equiv \sum n_{ht} f_{ht}. \tag{6}$$

Turn now to the development of the dynamics of the fractions n_{ht}. Recall that all traders are assumed to have common, constant conditional variances $\sigma^2 =$

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[4]See Footnote 2.

[5]Notice that we assume that our traders only have differences of opinion on the conditional mean of the returns distribution. There is not a contradiction in allowing them to have the same opinions on conditional variances. Indeed in a diffusion context one can follow Dan Nelson's work (cf. Bollerslev, Engle and Nelson (1994)) on continuous record asymptotics and argue that one should expect easier agreement on conditional variances than conditional means.

[6]Recently, Gaunersdorfer (1999) investigated a similar heterogeneous beliefs asset pricing model with identical, but time varying conditional beliefs on variance for all types h, given by an exponential moving average. Instead of a 3-D dynamical system as in Brock and Hommes (1998), she then obtains a 5-D dynamical system, with similar bifurcation routes to complicated asset price fluctuations.

$V_{ht}(R_{t+1})$, on excess returns $R_{t+1} = p_{t+1} + y_{t+1} - Rp_t$. Let $\rho_{ht} = E_{ht}(R_{t+1})$ and consider the goal function

$$\text{Max}_z\{E_{ht}R_{t+1}z - (a/2)z^2 V_{ht}(R_{t+1})\} = \text{Max}_z\{\rho_{ht}z - (a/2)z^2\sigma^2\}. \qquad (7)$$

Note that (7) is equivalent to the objective (1) up to a constant, so the optimum choice of shares of the risky asset is the same. Denote the optimum solution of (7) by $z(\rho_{ht})$.

Before we define the "fitness function" we compute the realized excess return over period t to period $t + 1$,

$$
\begin{aligned}
R_{t+1} &= p_{t+1} + y_{t+1} - Rp_t = x_{t+1} + p_{t+1}^* + y_{t+1} - Rx_t - Rp_t^* \\
&= x_{t+1} - Rx_t + p_{t+1}^* + y_{t+1} - E_t(p_{t+1}^* + y_{t+1}) + E_t(p_{t+1}^* + y_{t+1}) - Rp_t^* \\
&\equiv x_{t+1} - Rx_t + \delta_{t+1}.
\end{aligned}
\qquad (8)
$$

Notice that, since p_t^* satisfies (4), δ_{t+1} is a Martingale Difference Sequence (MDS) w.r.t. $\mathcal{F}_t$, i.e. $E\{\delta_{t+1} \mid \mathcal{F}_t\} = 0$ for all t. Indeed we may view the decomposition (8) as separating the "explanation" of realized excess returns R_{t+1} into the contribution $x_{t+1} - Rx_t$ of the theory being exposited here and the conventional Efficient Markets Theory term δ_{t+1}.[7]

A simple form for the fitness function or the performance measure $\pi(R_{t+1}, \rho_{ht})$, is

$$
\begin{aligned}
\pi_{h,t} &= \pi(R_{t+1}, \rho_{ht}) = R_{t+1}z(\rho_{ht}) \\
&= (x_{t+1} - Rx_t + \delta_{t+1})\frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2} \\
&= (x_{t+1} - Rx_t + \delta_{t+1})\frac{f_{ht} - Rx_t}{a\sigma^2},
\end{aligned}
\qquad (9)
$$

that is, fitness is given by realized profits for trader type h.[8] Notice that in general, realized return depends upon stochastic dividends and is given by $R_{t+1} = x_{t+1} -$

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[7]Studies like Brock et al. (1992) parametrize δ_{t+1} by, for example, a GARCH or EGARCH model, and test for the presence of an "additional term" $x_{t+1} - Rx_t$ by bootstrapping the null distribution of objects like trading strategies under the null hypothesis that $x_{t+1} - Rx_t = 0$ for all t. Brock et al. (1992) reject the null hypothesis that $x_{t+1} - Rx_t = 0$. Hence, it appears that extra structure is needed to "explain" excess returns data. This kind of finding motivates our theoretical work.

[8]Another possibility for the performance measure is e.g. $\pi_{h,t} = \pi(R_{t+1}, \rho_{ht}) = R_{t+1}z(\rho_{ht}) - (a/2)[z(\rho_{ht})]^2\sigma^2$, where the second term captures risk adjustment; see Gaunersdorfer (1999) on the risk adjusted case. The non-riskadjusted case considered here may be regarded as a fitness function that is slightly inconsistent with the traders being myopic mean-variance maximizers of wealth, in the sense that in their optimal portfolio decisions traders take risk into account, whereas the updating of beliefs is determined by non-risk adjusted realized profits. On the other hand, from a practical viewpoint accumulated realized profits or wealth may be the most relevant performance measure real market practitioners care about.

$Rx_t + \delta_{t+1}$. In Section 3 we will mainly focus on the *deterministic skeleton*, i.e. the nonlinear asset pricing dynamics, with $\delta_{t+1} = 0$ for all t and constant dividend $\bar{y}$ per time period. However, Section 3 also contains some numerical simulations with a stochastic dividend process $y_t = \bar{y} + \varepsilon_t$, where $\bar{\varepsilon}_t$ is IID, with a uniform distribution on a small interval $[-\varepsilon, +\varepsilon]$, to investigate the effect of noise upon the asset pricing dynamics. Notice that in the IID dividend case we simply have $\delta_{t+1} = \varepsilon_{t+1}$.

More generally, one can introduce *memory* into the fitness measure, by considering a weighted average of past realized profits, as follows,

$$U_{h,t} = \pi_{h,t} + wU_{h,t-1}, \tag{10}$$

where the parameter w represents the "memory strength". In the special case $w = 0$, fitness coincides with realized profit in the most recently past period; at the other extreme case $w = 1$, fitness coincides with accumulated wealth. In the intermediate case $0 < w < 1$, fitness is an exponentially moving average of past realized profits.

The updated fractions $n_{h,t}$ will be given by the discrete choice probability

$$n_{h,t} = \exp[\beta U_{h,t-2}]/Z_t, \quad Z_t \equiv \sum \exp[\beta U_{h,t-2}], \tag{11}$$

where the parameter β is the *intensity of choice* measuring how fast agents switch between different prediction strategies; see Brock and Hommes (1997a) for a motivation of using the discrete choice set up for predictor selection. In the extreme case when the intensity of choice is infinite, the entire mass of traders uses the strategy that has highest fitness. At the other extreme, when the intensity of choice is zero, the mass of traders distributes itself evenly across the set of available strategies and all fractions will be constant and equal. The higher the intensity of choice β, the more "rational" traders are in the sense that they become more sensitive to the fitness of the different trading strategies.

The *Adaptive Belief System* is thus given by market equilibrium (6) and updating of fractions (11), with fitness given by (9) and (10). The timing of updating of beliefs is important. We can only allow the fractions $n_{h,t}$ on the RHS of (6) to depend upon observable deviations x_{t-1} at date $t-1$ and further back in the past. Therefore, on the RHS of (11) fitness U and return R are dated at time $t-2$, to ensure that past realized profits $R_{t+1-j}z(\rho_{h,t-j})$, $j \geq 1$, are indeed *observable* quantities that can be used in predictor selection.

3. FUNDAMENTALISTS VERSUS TREND CHASERS

In this section we investigate a simple example of an ABS with two different trader types. The first type are fundamentalists, believing that at each date t the asset price p_t will be at its RE fundamental value p_t^*, or equivalently at each date

their expected deviation from the fundamental equals zero, that is, $f_{1t} \equiv 0$. The second trader type consists of "trend chasers," expecting

$$f_{2t} = gx_{t-1}.$$

Trend following may be seen as an extremely simple example of technical analysis, where traders believe that prices will deviate from the RE fundamental, following a constant growth rate. Brock and Hommes (1998) investigate the same 2-type example as well as other simple examples with three or four different trader types, focussing on the special case where memory is only one lag, i.e. $w = 0$, so that the performance measure becomes realized profit in the previous period. They show that, when average per period costs for fundamentalists are larger than average per period costs for trend chasers, an increase in the intensity of choice β leads to a *rational route to randomness*, that is, a bifurcation route to chaotic asset price fluctuations. More precisely, an increase of the intensity of choice β causes the fundamental steady state to become unstable in a *pitchfork* bifurcation in which two additional stable non-fundamental steady states are created. As the intensity of choice increases further the two non-fundamental steady states become unstable in a Hopf-bifurcation and periodic, quasi-periodic and even chaotic asset price fluctuations arise for large values of the intensity of choice. All these results have been shown for the case where memory is only one single period, i.e. for $w = 0$. Here we investigate the effect of the memory parameter w, and its role in generating market instability. It will be convenient to work with the difference in fractions

$$m_t \equiv n_{1t} - n_{2t} = \text{Tanh}(\frac{\beta}{2}[U_{1,t-2} - U_{2,t-2} - C]), \tag{12}$$

where the parameter C represents the per period average costs of fundamentalists. This cost C may be positive because "training" costs must be borne to obtain enough "understanding" of how markets work in order to believe that they should price according to the EMH fundamental. Straightforward computation, using (6), (12), (9) and (10), shows that in the case of fundamentalists versus trend chasers the dynamics is described by

$$Rx_t = \frac{1 - m_t}{2} gx_{t-1} \tag{13}$$

$$m_t = \text{Tanh}(\frac{\beta}{2}[wu_{t-1} - \frac{gx_{t-3}}{a\sigma^2}(x_{t-1} - Rx_{t-2} + \delta_{t-1}) - C]) \tag{14}$$

$$u_t = wu_{t-1} - \frac{gx_{t-3}}{a\sigma^2}(x_{t-1} - Rx_{t-2} + \delta_{t-1}) - C, \tag{15}$$

where $u_t = U_{1,t} - U_{2,t}$ is the difference in fitness of the two trader types and δ_{t-1} is the stochastic component in the realized excess returns, which in the case of an IID dividend process $y_{t-1} = \bar{y} + \varepsilon_{t-1}$ coincides with the stochastic term ε_{t-1}. Substituting (14) into (13) leads to a system of difference equations which is of

third order in x_t and of first order in u_t, or equivalently a four dimensional first order system. We are now ready to explore existence and stability of steady states of (13)-(15):

LEMMA 3.1: *(Existence and stability of steady states for (13)-(15)). Let $m^{\mathrm{eq}} = \mathrm{Tanh}(-\frac{\beta C}{2(1-w)})$, $m^* = 1 - 2R/g$ and x^* be the positive solution (if it exists) of $\mathrm{Tanh}(\frac{\beta}{2(1-w)}[\frac{(R-1)g}{a\sigma^2}(x^*)^2 - C]) = m^*$.*

(a) *For $0 < g < R$, $E_1 = (0, m^{\mathrm{eq}})$ is the unique, globally stable steady state.*

(b) *For $g > 2R$, there exist three steady states $E_1 = (0, m^{\mathrm{eq}})$, $E_2 = (x^*, m^*)$ and $E_3 = (-x^*, m^*)$; the steady state $E_1 = (0, m^{\mathrm{eq}})$ is an unstable saddle point.*

(c) *For $R < g < 2R$ there are two possibilities:*
   (i) if $m^* < m^{\mathrm{eq}}$ then E_1 is the unique, (locally) stable steady state.
   (ii) if $m^* > m^{\mathrm{eq}}$ then there are three steady states E_1, E_2 and E_3; the steady state $E_1 = (0, m^{\mathrm{eq}})$ is an unstable saddle point.

PROOF: From (13) we get that a steady state (x^*, m^*, u^*) has to satisfy

$$Rx^* = \frac{1 - m^*}{2}gx^*,$$

implying $x^* = 0$ or $R = g(1 - m^*)/2$. Solving the latter for m^* yields

$$m^* = 1 - \frac{2R}{g}. \tag{16}$$

According to (14) and (15), for the deterministic skeleton (i.e. with $\delta_{t-1} = 0$), we also have $m^* = \mathrm{Tanh}(\frac{\beta}{2}u)$. Solving u^*, as a function of x^*, from (15) then yields

$$m^* = \mathrm{Tanh}(\frac{\beta}{2(1-w)}[\frac{(R-1)g}{a\sigma^2}(x^*)^2 - C]). \tag{17}$$

The fundamental steady state is then given by $(0, m^{\mathrm{eq}}) = (0, \mathrm{Tanh}(-\frac{\beta C}{2(1-w)}))$ and the non-fundamental steady states are (x^*, m^*) and $(-x^*, m^*)$, where m^* is given by (16) and x^* is the postive solution (if it exists) of (17). For $0 < g < R$, $m^* < -1$ so that the fundamental steady state is the unique steady state. For $g > 2R$, $0 < m^* < 1$ and (17) has one positive solution x^* and one negative solution $-x^*$ so that two non-fundamental steady states exist. Finally, for $R < g < 2R$, (17) has two solutions if and only if $m^* = 1 - 2R/g > m^{\mathrm{eq}}$. Finally, the stability results of the fundamental steady state follows easily from (13). The eigenvalues of the fundamental steady state are $\lambda_1 = \frac{(1-m^{\mathrm{eq}})g}{2R} = (1 - \mathrm{Tanh}(-\frac{\beta C}{2(1-w)}))g/(2R)$ and $\lambda_2 = 0$.

Obviously $\lambda_1 > 1$, if and only if $m^* > m^{\mathrm{eq}}$. *Q.E.D.*

Hence, when the trend chasers extrapolate only weakly ($0 < g < R$), then the fundamental steady state E_1 is globally stable. If costs $C = 0$ the steady state fractions of fundamentalists and trendchasers are equal for any β, since the difference in realized profits is zero at $x = 0$. Now if $C > 0$, we see that the mass on fundamentalists decreases to zero as the intensity of choice β increases to $+\infty$ or when the memory becomes infinite (i.e. the weight w approaches $+1$). This makes economic sense. There's no point in paying any cost in a steady state for a trading strategy that yields no extra profit in that steady state. As intensity of choice β increases or the weight of net realized profits increases, the mass on the most profitable strategy in net terms, increases. When the trendchasers extrapolate very strongly ($g > 2R$) there are always two additional non-fundamental steady states E_2 and E_3, one above and one below the fundamental steady state, even when there are no information costs. Finally, the case of strongly extrapolating trendchasers ($R < g < 2R$) and positive information costs for fundamentalists, i.e. $C > 0$, is the most interesting. For $\beta = 0$, $m^{\mathrm{eq}} = 0 > m^*$, whereas for large β or for w close to 1, $m^{\mathrm{eq}} \approx -1 < m^*$. Hence, as the intensity of choice increases or when memory becomes infinite, a *pitchfork* bifurcation occurs for some $\beta = \beta^*$ or for some $w = w^*$, in which the fundamental steady state E_1 becomes unstable and two additional (stable) steady states $E_2 = (p^*, m^*)$ and $E_3 = (-p^*, m^*)$ are created. Additional memory thus leads to the same primary bifurcation as an increase of the intensity of choice, and the creation of non-fundamental steady states. In particular, there exist critical values β^* and w^*, such that for $\beta > \beta^*$ and/or for $w^* < w \leq 1$, the fundamental steady state is an unstable saddle point.

BH (1998) show that, in the special case when memory is only one single period, i.e. for $w = 0$, as the intensity of choice β increases beyond the primary bifurcation value β^*, a secondary bifurcation occurs, namely a Hopf bifurcation in which the non-fundamental steady states become unstable and two (coexisting) attracting invariant circles are created on which periodic or quasi-periodic asset price fluctuations occur. Moreover, as the intensity of choice is further increased, even chaotic asset price fluctuations arise. Here we explore what happens when memory strength w increases beyond its primary bifurcation value w^*. By introducing memory into the fitness function, the dimension of the dynamical system has increased from 3 to 4. An analytical treatment of the local stability of the non-fundamental steady states, therefore becomes more delicate, and we use numerical simulations to investigate the bifurcation scenario. In all our numerical simulations we observed that, as the memory parameter increases the secondary bifurcation is again a Hopf-bifurcation of the non-fundamental steady states, in which two (coexisting) invariant circles, one around each of the two non-fundamental steady states, are created.

Figure 1 shows plots of the attractors in the (x_t, m_t) plane without noise (1a,c,e) and with noise (1b,d,f) added to the dividend process and to the system equation (13). Immediately after the secondary bifurcation, periodic or quasi-periodic fluctuations occur. In Figure 1a, the orbit converges to one of the two attracting invariant "circles" created after the Hopf bifurcation of the non-fundamental steady states. As w increases (Figure 1c), the invariant circle grows and almost has the shape of a "rectangle". For even larger w-values ($w \geq 0.95$, including $w = 1$ as in Figure 1e) in all our numerical simulations orbits converge to the fundamental steady state. It is important to note however that for $w > w^*$ the fundamental steady state is a saddle point, and therefore it must be locally unstable. Adding small noise to the dividend process and system equation (13) will therefore drive asset prices away from the fundamental steady state again, as illustrated in Figure 1f. Figure 2 shows the time series, both with and without noise, corresponding to the attractors in Figure 1. Prices are characterized by a switching between an unstable phase of an upward (or downward) trend and a stable phase with prices close to their fundamental value. In the noise free case, this switching seems to be fairly regular. In the presence of small noise however, the switching becomes higly irregular and unpredictable. Notice also that in the case when fitness equals accumulated net wealth, i.e. $w = 1$, the length of the period where one of the two groups dominates the market tends to become longer.

At this point, the reader is urged to compare Figure 1 to Figure 1 and the subsequent discussion in BH (1998, pp. 1250-1251), for the case where fitness is realized profit in the last period, i.e. $w = 0$ and β large. These figures are very similar. BH (1998) argue that in the case $w = 0$ and β large the system is close to having a homoclinic orbit, with time paths diverging from the fundamental steady state along the unstable manifold in the horizontal direction, and returning close to the steady state along the stable manifold in the vertical direction. We conjecture that, for β fixed, as the memory parameter w approaches $+1$ the system also gets close to having a homoclinic orbit. A proof of this conjecture about the global dynamical behaviour when memory strength increases seems to be hard, since for $w > 0$ the system is 4-D; in any case a proof is beyond the scope of this paper. However, the result should not come as a complete surprise, because in the case $w = 0$ and β large along the periodic and chaotic equilibrium time paths, average profits of the trend chasers will be higher than average profits of the fundamentalists due to fundamentalists' positive information costs.[9] Therefore, in

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[9]A natural question is whether the positive per period information costs for fundamentalists is the main reason for obtaining these results. BH (1998) also investigate a number of examples with three or four different trader types, *without* any information costs for fundamentalists, for the finite memory case $w = 0$. In these examples, along the cycles and chaotic equilibrium paths average profits of non-fundamental traders are not necessarily smaller than average profits of fundamentalists. Whereas the precise role of information costs in the evolutionary competition with large memory remains to be investigated, these examples suggest that similar results may
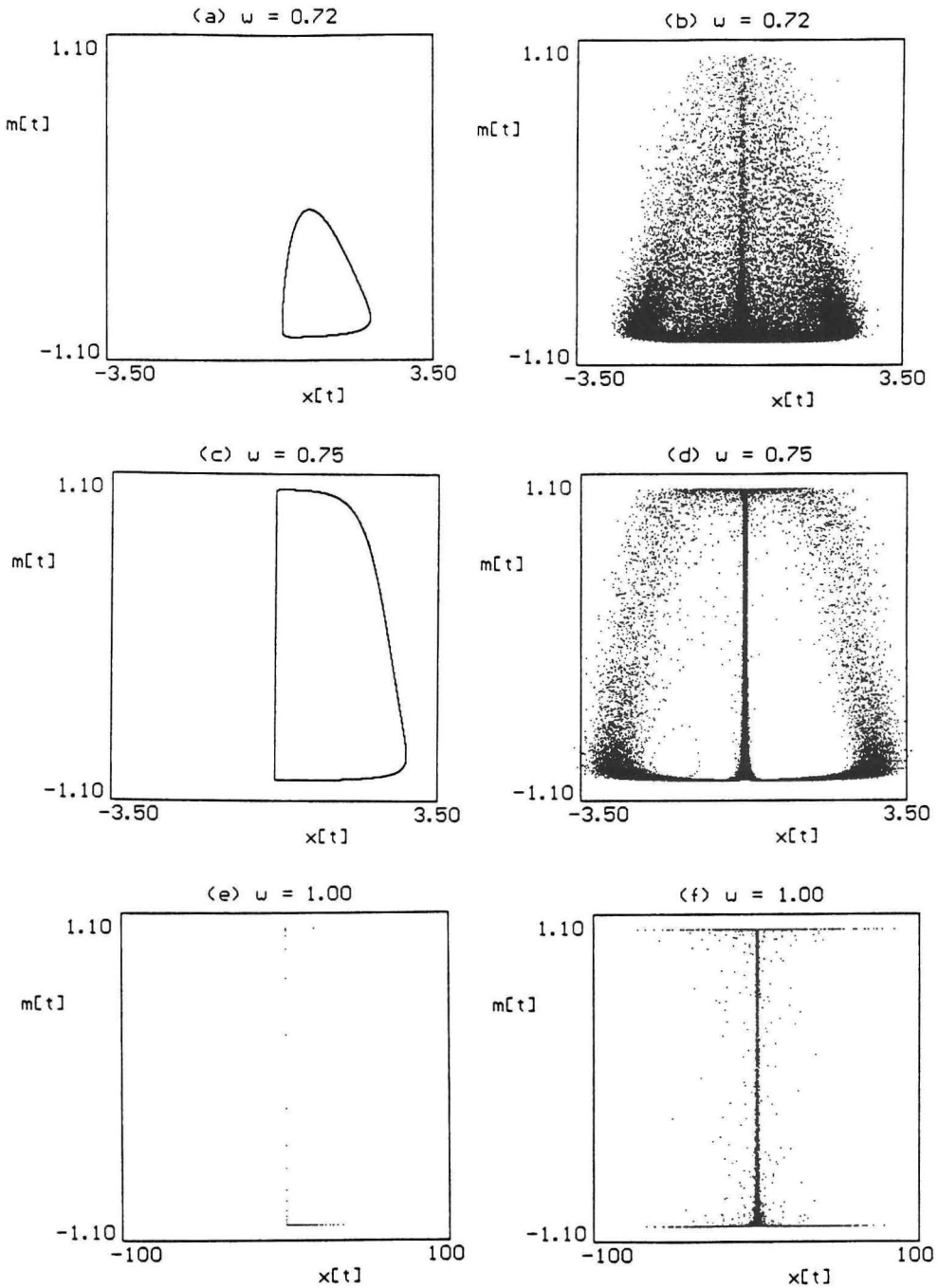
FIGURE 1. Attractors for different values of the memory parameter w, with all other parameters fixed at $\beta = 1$, $C = 1$, $g = 1.2$, $R = 1.1$ and $a = \sigma^2 = 1$. $(a - b)w = 0.72$, $(c - d)w = 0.75$, and $(e - f)w = 1$. In (b), (d) and (f) dividend noise δ_t (added to the dividend process) and system noise (added to equation (13)) have been added, both drawn from a uniform distribution on the interval $[-0.05, 0.05]$ in (b) and (d) and the interval $[-1, 1]$ in (f).
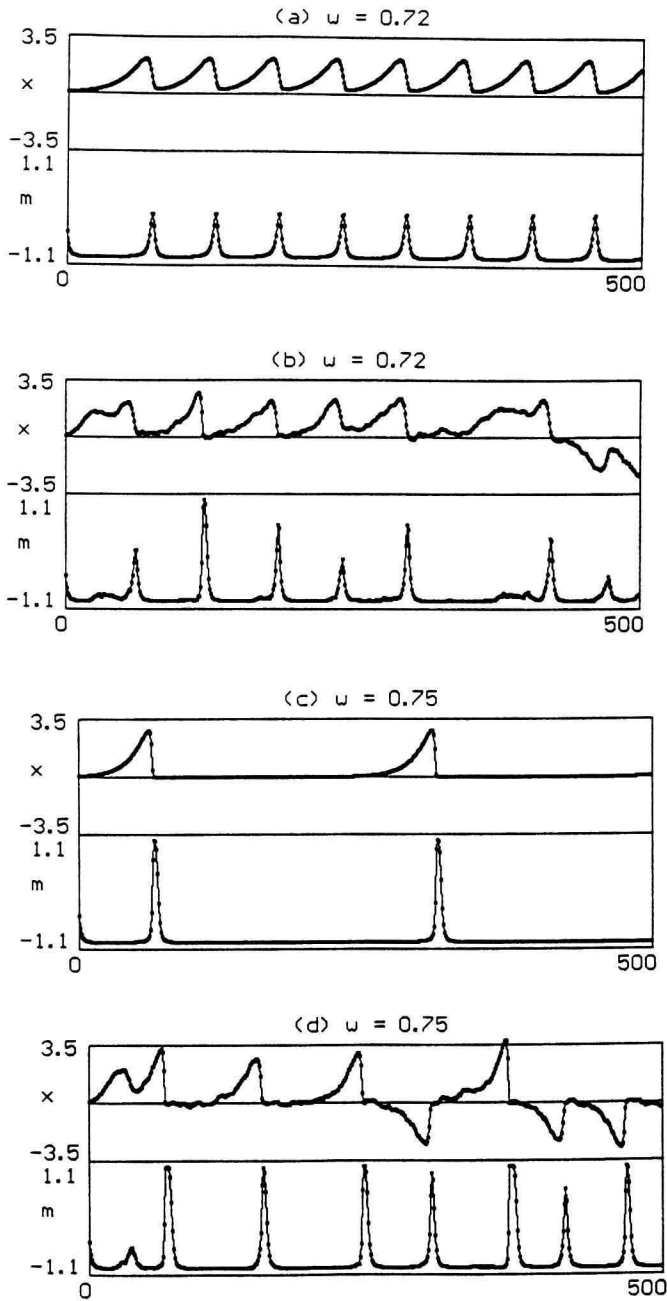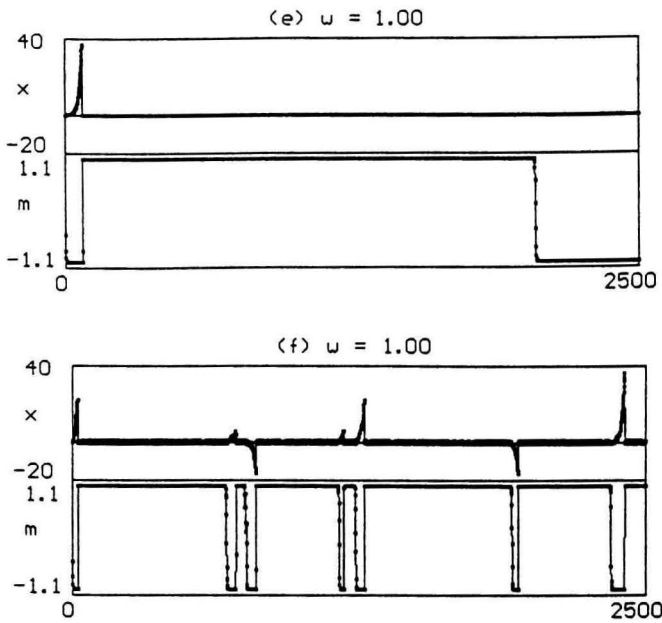
FIGURE 2: a-d.

FIGURE 2: e and f. Time series corresponding to the attractors in Figure 1, with and without noise. The time series for $w = 1$ in (e-f) correspond to the case where fitness is given by accumulated net wealth.

the case $w = 0$ and β large, accumulated net wealth of the trend chasers becomes larger than accumulated net wealth of the fundamentalists. This suggests that also in the case when β is finite and fixed and memory w gets sufficiently close to 1, fundamentalists may not be able to drive out trend chasers. Instead, even with large memory the (noisy) evolutionary dynamics is characterized by an irregular switching between periods where the market is dominated by trend followers and prices diverge from the fundamental and periods where differences of accumulated profits of fundamentalists and trend followers become large enough to push prices back close to the fundamental. This suggests that, as memory increases, the system gets close to having a homoclinic orbit associated to the fundamental saddle point steady state.

In summary, we conclude that the bifurcation scenario w.r.t. the memory parameter w (with β fixed) is essentially the same as the rational route to randomness w.r.t. the intensity of choice β (with w fixed at 0). Increasing the sensitivity of traders to yesterday's profits has a similar effect as increasing memory of past generated profits. Memory thus appears to be another possibly destabilizing force in the evolutionary adaptive belief system with accumulated realized profits as the fitness measure. When fitness equals accumulated wealth, random "news" about fundamentals may trigger temporary speculative bubbles.

A word of caution about the generality of these results needs to be said however. In recent work in progress, Brock and Hommes (1999) show that if the fitness measure is given by risk adjusted profits (see Footnote 8), when costs for all trading rules are zero and memory is infinite, all bounded orbits in the evolutionary dynamics must converge to the RE fundamental steady state. In the risk adjusted case, more memory is thus a *stabilizing* force. In the non risk adjusted case, the fact that in their evaluation of the performance of the different trading strategies traders do not take into account the risk they have taken in accumulating their wealth, thus plays an important role for memory to become a *destabilizing* force and the survival of non-fundamental traders in the evolutionary competition. In general, the role of memory in generating stability or instability is thus ambiguous.

4. LARGE TYPE LIMITS

In the previous section we focused on a very simple example of an adaptive belief system, with only two trader types. Brock and Hommes (1998) present a detailed analysis of bifurcation routes to chaotic asset price fluctuations, in a number of other examples with two, three or four different types of traders. In real markets however, one would expect a much larger degree of heterogeneity, with a large number of different beliefs or trading strategies. In this section we discuss the

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be true in cases without any information costs.

notion of Large Type Limit (LTL), which is in fact a (deterministic) dynamical system approximating a market with many different trader types. The notion of LTL for the asset pricing model was discussed already in Brock (1997). Here we present a simple, concrete example of an LTL; a more detailed treatment will be given in Brock and Hommes (1999). Brock and de Fontnouvelle (1999) investigate LTLs in the overlapping generations model.

In order to motivate the concept of Large Type Limit (LTL) consider the equilibrium equation

$$Rx_t = \sum n_{ht} f_{ht} = \sum \exp(\beta U_{ht}) f_{ht} / \sum \exp(\beta U_{ht}). \tag{18}$$

Now suppose there are parameter vectors θ, ϕ, and functions $f(\cdot)$, and $U(\cdot)$, such that for each h, and each t, we have

$$f_{ht} = f(\theta_h, x_{t-1}, \ldots), \qquad U_{ht} = U(\phi_h, x_{t-1}, \ldots).$$

Furthermore, suppose that there is a joint distribution function $G(\cdot, \cdot)$, for example, multivariate normal from which θ_h, ϕ_h are drawn for each h. Divide both numerator and denominator of (18) by H and observe that (18) may be rewritten as

$$Rx_t = \widehat{E}\{\exp(\beta U(\tilde{\phi}, x_{t-1}, \ldots)) f(\tilde{\theta}, x_{t-1}, \ldots)\} / \widehat{E}\{\exp(\beta U(\tilde{\phi}, x_{t-1}, \ldots))\}, \tag{19}$$

where $\widehat{E}$ denotes the analog estimator $(1/H)\sum$, i.e.,

$$\widehat{E}\{.\} \equiv (1/H)\sum\{.\}.$$

Note that we have written θ and ϕ with tildes to encourage the reader to view them as random variables. We now have the notation and framework set up to motivate and to introduce the notion of LTL.

Arthur et al. (1997) and LeBaron et al. (1999) (cf. LeBaron's review (1999)) have been conducting evolutionary adaptive simulations of "artificial stock markets" that parallels the literature on "artificial life." We wish to formulate an analytical apparatus that will enable us to obtain some analytic insight into the behavior of these simulation experiments. The simulation experiments can be viewed as a "computer model" of a real world stock market that goes much further than pencil and paper analytics in helping us understand what goes on in real markets.

The "artificial stock market" of Arthur et al. (1997) and LeBaron et al. (1999) operates by starting out with a population of H "rules" (which are rather like our f_{ht} functions). These rules compete against each other over time and replicate according to their relative fitnesses (rather like our equilibrium dynamical system (19)). There is another layer of complexity in the artificial stock markets that is not present in our equilibrium dynamical system (19). It is "sex" and "mutation". In Arthur et al. (1997) and LeBaron et al. (1999) rules are coded as bitstrings and a version of Holland's classifier system and genetic algorithm is used to produce

new rule types, while the lowest performing rule types are discarded from the "ecology" to make room for the new ones. Our framework can make a very crude approximation to these dynamic simulations with evolutionary forces of mutation and sexual reproduction if we assume these forces operate in a slow scale of time relative to the replication time scale, "t," operating the equilibrium dynamical system (19). Here is the crudest but most tractable approximation. Draw a system (19) of size H "species" of "rules" f_h, and run it for T periods, then draw another system (19) of size H independently, and run it for T more periods. Repeat this process. We can use LTL theory to study the behavior of this sequence of dynamical systems for large T. We build this ensemble by first describing a "trial." Draw H "samples" (θ_h, ϕ_h) from the joint distribution $G(\cdot, \cdot)$ at date 0, one for each $h = 1, 2, \ldots, H$ and fix them for all future times. This gives us, via the equilibrium equation, equation (19), one dynamical system which we call a "trial" dynamical system. Build an ensemble of dynamical systems by considering a collection of N trial dynamical systems. We are now in a position to pose and possibly answer questions about the "average" behavior of this ensemble for large H and large N. We are particularly interested in the behavior for large H. Indeed under modest regularity conditions on the sampling process the sample moments appearing on the RHS of (19) converge in probability to the population moments. If the regularity conditions are strengthened on the sampling process, the quality of the convergence is strengthened. In fact we can obtain *exponentially fast* convergence of each sample moment to its population counterpart under regularity conditions and IID sampling (cf. Ellis (1985), Chapter 2). Since the RHS of (19) is a smooth function (C^∞) of quantities that converge exponentially we might expect exponential convergence of the RHS of (19) as $H \to \infty$. In any event, this motivates analysis of the population counterpart of (19) in a first cut at a complete probabilistic analysis.

We conduct the first cut analysis of (19) here by replacing, from now on, all sample moments by their population counterparts and analysing the resulting deterministic dynamical system

$$Rx_t = E\{\exp(\beta U(\tilde{\phi}, x_{t-1}, \ldots)) f(\tilde{\theta}, x_{t-1}, \ldots)\} / E\{\exp(\beta U(\tilde{\phi}, x_{t-1}, \ldots))\}. \quad (20)$$

This raises the problem of calculating the population moments appearing on the RHS of (20) for specific applications. We illustrate the power of LTL theory here by calculating a representative example. We calculate an example of an LTL for the case where fitness is realized profits in the most recently observed period (i.e. the weight $w = 0$) and where parameters are distributed multivariate normal. Recall from (9) that actual realized profits are given by

$$\pi_{h,t-2} = R_{t-1} z(\rho_{h,t-2}) = \left(\frac{f_{ht} - Rx_{t-2}}{a\sigma^2} \right) (x_{t-1} - Rx_{t-2} + \delta_{t-1})$$

where $\{\delta_t\}$ is the Martingale Difference Sequence component in the decomposition

of the excess returns process $\{R_t\}$. In the notion of (22)-(24) below, we obtain,

$$Rx_t = E\{f_t(\theta)\exp[\eta(\rho_{t-1} + \delta_{t-1})f_{t-2}(\theta)]\}/E\{\exp[\eta(\rho_{t-1} + \delta_{t-1})f_{t-2}(\theta)]. \quad (21)$$

If we set $\delta_{t-1} = 0$ we get what we shall call "the skeleton" of the stochastic dynamical system (21). We wish to uncover stabilizing and destabilizing economic forces by studying the skeleton.

We specialize U and f in (20) to,

$$U_{ht} = (x_{t-1} - Rx_{t-2})[(f_{ht} - Rx_{t-2})/(a\sigma^2)], \quad (22)$$
$$f_{ht} = f_t(\theta_h) = \theta_{0h} + \theta_{1h}x_{t-1} + \cdots + \theta_{Lh}x_{t-L},$$

thus, (20) becomes

$$Rx_t = E\{\exp[\eta\rho_{t-1}f_{t-2}(\theta)]f_t(\theta)\}/E\{\exp[\eta\rho_{t-1}f_{t-2}(\theta)]\}, \quad (23)$$

where, $\eta \equiv \beta/(a\sigma^2)$,

$$f_t(\theta) = \theta_0 + \theta_1 x_{t-1} + \cdots + \theta_L x_{t-L}, \quad (24)$$

$\rho_{t-1} = x_{t-1} - Rx_{t-2}$, $\tau = \eta\rho_{t-1}$, $s_0 = \tau$, $s_1 = \tau x_{t-2-1}, \ldots, s_L = \tau x_{t-2-L}$, and put $\exp(\eta\rho_{t-1}f_{t-2}(\theta)) = \exp(\sum s_i\theta_i)$.

In order to calculate a closed form expression for the RHS of (23) we use moment generating function formulae from normal distribution theory. Note that

$$E\{\exp(\mathbf{n})\} = \exp(E(\mathbf{n}) + (1/2)\text{Var}(\mathbf{n}))$$

for normal random variable $\mathbf{n} = \sum s_i\theta_i$. Note also that $E\{\exp(\sum s_i\theta_i)\theta_k\}$ can be obtained by simply differentiating $E\{\exp(\sum s_i\theta_i)\}$ w.r.t s_k.

Assume that the θ's are uncorrelated (i.e. independent for this multivariate normal case) for simplicity. It is straightforward to extend the method to correlated θ's. Thus, we see that it is easy to use moment generating function formulae for multivariate normals to obtain the following closed form expression for equation (23),

$$Rx_t = m_0 + \eta\rho_{t-1}\sigma_0{}^2 + a_{1t}x_{t-1} + \cdots + a_{Lt}x_{t-L}, \text{ where,} \quad (25)$$
$$a_{kt} = m_k + \eta\rho_{t-1}\sigma_k{}^2 x_{t-2-k}, \quad \eta = \beta/(a\sigma^2), \quad (26)$$

with the mean $m_k = E\theta_k$ and $\sigma_k{}^2$ the variance of θ_k. Notice that, if the maximum number of lags used in the (linear) predictors is L, then the large type limit (LTL) becomes a (nonlinear) difference equation in x of order $L + 2$, or equivalently an $L + 2$ dimensional dynamical system.

The simplest special case of (25) that still possesses dynamics arises in the pure bias case when $f_t(\theta) = \theta_0$. When there is no intrinsic mean bias, i.e. $m_0 = E\theta_0 = 0$, the simplest LTL becomes

$$Rx_t = m_0 + \eta\rho_{t-1}\sigma_0{}^2 = m_0 + \eta\sigma_0{}^2(x_{t-1} - Rx_{t-2}) = \eta\sigma_0{}^2(x_{t-1} - Rx_{t-2}). \quad (27)$$

The natural bifurcation parameter for this system is $\alpha = \eta\sigma_0{}^2$. For this second order system, it is easy to see that instability occurs, with complex eigenvalues leaving the unit circle, as α increases beyond the bifurcation point $\alpha_c = 1$. Hence an increase in choice intensity β, a decrease in risk aversion "a", a decrease in conditional variance σ^2, or an increase in diversity of beliefs $\sigma_0{}^2$ can push α beyond α_c and set off instability of the fundamental steady state $x = 0$ in (27).

We conclude that in the pure bias case, such as (27), instability tends to appear when $\eta \equiv \beta/(a\sigma^2)$ increases and when the variance σ_0^2 of the θ_0-distribution increases. Therefore one expects an increase in heterogeneity dispersion to increase chances of instability in these pure bias cases.

Next consider a more general example with linear predictors with 3 lags, i.e. $f_t(\theta) = \theta_0 + \theta_1 x_{t-1} + \theta_2 x_{t-2} + \theta_3 x_{t-3}$. The LTL in (25)-(26) then reduces to the 5-D dynamical system

$$Rx_t = m_0 + m_1 x_{t-1} + m_2 x_{t-2} + m_3 x_{t-3}$$
$$+ \eta(x_{t-1} - Rx_{t-2})(\sigma_0{}^2 + \sigma_1{}^2 x_{t-1}x_{t-3} + \sigma_2{}^2 x_{t-2}x_{t-4} + \sigma_3{}^2 x_{t-3}x_{t-5}),(28)$$

where $\eta = \beta/(a\sigma^2)$. Steady states of (28) are the fundamental steady state $x = 0$ and solutions x^* (if they exist) of

$$(x^*)^2 = (\frac{\overline{\mu} - R}{\eta(R-1)} - \sigma_b^2)/\overline{\sigma}^2, \tag{29}$$

where $\overline{\mu} = m_1 + m_2 + m_3$ and $\overline{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. From this expression it is clear that two non-fundamental steady states, one positive and one negative, exist when the sum $\overline{\mu}$ of the means m_i of the three trend parameters g_i is large enough. Indeed, increasing one of the means m_i leads to a pitchfork bifurcation of the fundamental steady state and the creation of two non-fundamental steady states in the LTL. In fact, a further increase of one of the means m_i leads to a bifurcation route to complicated price fluctuations in the LTL similar to the bifurcation route for the two-type case discussed in Section 3, with a Hopf bifurcation of the non-fundamental steady state leading to periodic, quasi-periodic and even chaotic fluctuations in the LTL. However, here we will focus on a different bifurcation route, in an example where the fundamental steady state is the unique steady state of the LTL.

Figure 3 shows some (projections of) attractors of the 5-D LTL in the $x_t - x_{t-1}$ plane, for increasing values of the parameter $\eta = \beta/(a\sigma^2)$, with the other parameters fixed at $m_0 = 0$, $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.1$, $\sigma_b^2 = 1$, $\sigma_1^2 = 0.1$ and $\sigma_2^2 = \sigma_3^2 = 0.05$. The reader may easily check that for this choice of the parameters (29) has no real solutions, so that the fundamental steady state is the unique steady state of the LTL. Increasing the parameter η is equivalent to increasing the intensity of choice β, decreasing the risk aversion parameter a, or decreasing the belief on the variance σ^2 of the returns process. Figure 3 suggest that the fundamental steady state loses stability through a Hopf-bifurcation, in which an attracting invariant

circle is created, with (quasi-)periodic dynamics. As the parameter η increases further, the invariant circle breaks up into a strange attractor with complicated asset price fluctuations as illustrated in the time series in Figure 4. For η-values larger than 1.5, the LTL becomes globally unstable and prices diverge to infinity.

It is important to note that this bifurcation scenario for the LTL, is similar to the bifurcation scenario in some of the examples with only three or four trader types, analyzed in BH (1998). In these few type examples, the primary Hopf bifurcation and the consecutive breaking of the invariant circle into a strange attractor are due to *diversity in bias*, that is, when types with positive and negative bias-parameter b in their linear forecasting rules $f_{ht} = gx_{t-1} + b$, coexist. Opposite biases among the different types generate this bifurcation scenario in the few type examples. For the LTL, this particular bifurcation scenario also seems to be related to diversity in bias parameters among the many different trader types, as measured by the variance σ_b^2 of the bias. From (29) it is clear that when this diversity is large, the fundamental steady state will be the unique steady state. In all our numerical simulations of the LTL (28) with large diversity in bias, we observed a similar bifurcation scenario as illustrated in Figure 3.

More generally, one may ask what is the relationship between bifurcation routes to complicated asset price fluctuations in the LTL and in a typical sample with many traders with initial beliefs drawn from the corresponding distribution. A more detailed analysis of this relationship will be given in BH (1999), but here we wish to make a brief remark. It can be argued that regularity conditions on a "population linearization" of (20) can be located so that if for the LTL the primary bifurcation is one of the generic co-dimension one bifurcations[10] as η increases through a critical value η_c, then the probability that the primary bifurcation of the "sample linearization" of (19) is the same co-dimension one bifurcation goes to unity as $H \to \infty$. In fact, we expect the complete bifurcation route to complicated dynamics in the LTL to be similar to the bifurcation scenario in a typical sample with many traders.

5. CONCLUDING REMARKS

The theory we have started to build is difficult and controversial. When one wishes to build an extension of classical finance models to include evolutionary choice over a space of trading strategies one must expect controversy. Our paper is motivated by the kinds of evidence of departures of rational expectations equilibrium in real markets reviewed e.g. in Shiller (1989) and Thaler (1994), and

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[10]The co-dimension k of a bifurcation is the minimum number of parameters needed so that the bifurcation occurs in generic k-parameter families of dynamical systems. Co-dimension one bifurcations are thus the most common, since they already occur generically when varying one single parameter; see Kuznetsov (1995) for an introduction to bifurcation theory.
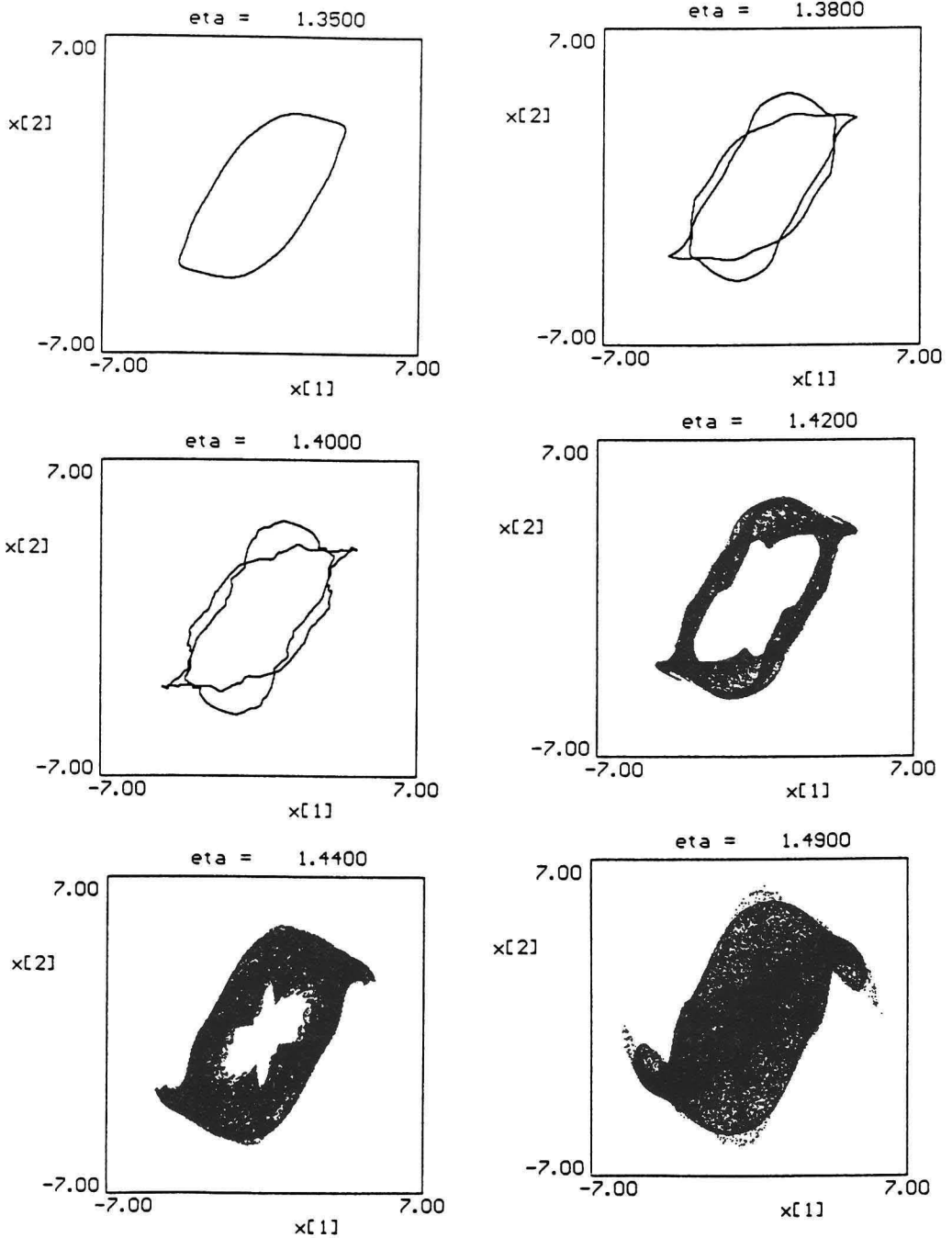
FIGURE 3. Projections of attractors of the 5-D LTL, for increasing values of the parameter $\eta = \beta/(a\sigma^2)$. After the primary Hopf-bifurcation of the fundamental steady state, as η increases, the attracting invariant circle with (quasi-) periodic dynamics breaks into a a strange attractor with chaotic asset price fluctuations.
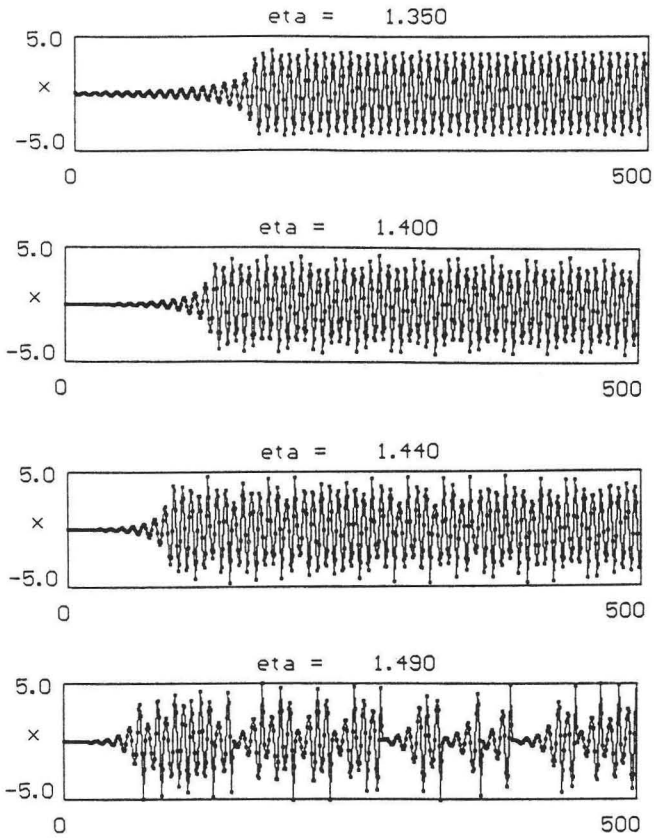
FIGURE 4. Asset price fluctuations in the LTL for different values of the parameter η.

also observed in experimental markets, e.g. in Smith et al. (1988). An attractive feature of our theory is that the models are formulated in deviations from a rational expectations fundamental solution. REE is thus nested as a special linear subcase within our nonlinear evolutionary adaptive belief system. As a result, our theory leads to a decomposition of excess returns into a conventional REE martingale difference sequence part and an *endogenous evolutive dynamics part*. This decomposition makes the theory suited for empirical testing. Our theory may be viewed as an attempt to "back off" from REE in a bounded rationality sense, as in Sargent (1993), but *nests REE* in such a way that *REE-econometric technology* such as methods based upon orthogonality conditions can be readily adapted to test the "significance" of the extra "free parameters" that our theory adds to REE theory. In related recent work, for example Baak (1999) implemented statistical tests for the "significance" of the "extra parameters" embodied in the departures from REE for US cattle data. For stock market data, trading volume will play an important role in this testing because the heterogeneity of beliefs in our theory is associated with a typically larger level of trading volume than theories such as REE which impose more homogeneity on beliefs. De Fontnouvelle (1999) develops a theory which extends Brock and Hommes (1997a) to noisy REE settings which should be helpful in the extension we propose.

There is an alternative way of looking at our formulation. Kleidon's recent review of literature on market "pathologies" such as crashes and blowoffs (Kleidon (1993)) stresses that there can be situations where traders are uncertain *about the behavior of other traders* and this can lead to momentary departures from rational expectations equilibrium. He reviews experimental work on bubbles and crashes which shows that markets with more "experienced traders" converge onto the fundamental but there may be blowoffs and crashes in the first one or two trials. He also reviews work that uncovers conditions that frustrate convergence to equilibrium. The main point in this part of Kleidon's review is that models which do "...not assume irrational behavior but that relax the assumption that prices necessarily fully aggregate individual information...allow for deviations from a fully revealing REE, with consequent booms and crashes," are very useful for understanding the sources of market "pathologies" like the Crashes of October 1987, 1989 (Kleidon and Mehra (1997, p. 657)). Confidence in such models is strengthened by their consistence with the experimental results. These models focus the attention on concepts such as "external news" and "internal news."

We think a theory is needed that nests both views in a way such that econometric methods are suggested to test the null of REE against the alternative. Pro-REE economists loosely argue that the situations in which these extra free parameters are significant may not be frequent. Anti-REE types argue that such departures may be frequent for anti-REE types. We wish to contribute to the task of building a theory in which the data can speak to this issue. In spite of the evidence reviewed

by Kleidon, the controversy on REE still rages.

Another important question concerning our work is whether the types of departures from REE and the types of nonlinearities that we focus upon here might not be "washed out" by aggregation across a rich enough heterogeneity of trader types.[11] After all in demand theory recent work of Hildenbrand (1994) and Grandmont (1992) suggests plausible assumptions on heterogeneity of demanders so that aggregate demand ends up "downward sloping" so that the usual Walrasian adjustment dynamics are stable. Might not a similar force "wash out" the potentially destabilizing forces caused by the type of trader heterogeneity discussed here? Notice however, that in our theory the heterogeneity actually present at any given time is "endogenous" in the sense that it is evolutionarily "selected" from a "population" of heterogeneity. This is a different situation than the one discussed by Hildenbrand and Grandmont. Indeed in the LTL there are a continuum of potential types and the amount of mass put upon each type evolves according to the trading performance of that type relative to the other types. This evolutionary selection adds another layer of dynamics onto the market dynamics which results in the equilibrium dynamics which we study. This evolutionary selection dynamic can counter act the "washing out" effect of aggregation because mass is attracted to parts of the heterogeneity space that performs well.

Turn now to another force which can check the "washing out" effect of aggregation. Our theory may be extended using the statistical mechanics approach to financial economics discussed in Brock (1993). In that work financial models are developed where expectational complementarities cause a breakdown of the law of large numbers because they induce strong dependence in the statistical sense. Laws of large numbers and central limit theorems require weak dependence. The framework developed in Brock (1993) is still econometrically tractable even though the law of large numbers and the central limit theorem breaks down. This is so because recent results on the Curie-Weiss model of statistical physics are extended in Brock (1993) to develop tractable limit results in the context of financial markets models. We plan to develop a related extension of our theory by adding expectational complementarities along the lines of Brock (1993). This extension hopes to uncover useful sufficient conditions on expectational complementarities so that aggregation does not "wash them out."

Finally, there is a serious issue of econometric identification from data of effects that look like "contagion, bubbles, herding, etc." but which are plausibly just sensible market pricing of fundamentals. A good place to see the heart of this identification problem is the literature on the recent financial crises. Good places to start are Calvo (1998), Calvo and Mendoza (1998), Krugman (1998), and Garber et al. (1998).

Note that Garber argued that much of these extreme financial movements are

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[11]We thank Dave Furth for pointing out this aggregation problem.

primarily fundamentally driven in an REE setting. I.e., the bottom line is that REE theory can do a good job of explaining these recent financial movements once one takes into account the impact on the underlying fundamental of real world factors such as the interaction between potential insolvency chains with induced moral hazard in the regulatory framework of lending institutions such as banks. This nexus can act as a magnifier of shocks to fundamentals in the underlying REE pricing process so that REE returns look like they contain social interaction effects such as contagion when such effects are absent. The relative size of volume movements over time may be useful in assisting econometric identification here.

Since financial crises are cited as an example of market induced instabilities like those studied in our work, let us look more closely at the recent crises. Calvo (1998) states "Recent financial crises in emerging markets have shared the following characteristics:
1. have been preceded by large capital outflows,
2. evolved through a complicated interaction among the domestic financial and non financial sectors, international investors and banks, and sovereign governments,
3. few people were able to predict them,
4. led to a sharp growth slowdown, if not sheer output collapse."
In point 2, Calvo is referring to phenomena such as moral hazard in banks, sovereign government actions, uncertainty of default, bad incentives inherent in existing regulatory structures, and the like. These are all components of the "baseline fundamental" in our theory.

Hence, the only addition of our theory is the possible magnification or distortion of the equilibrium market returns process from the implicit REE market returns process in our theory. So the econometric identification issue is this. How can one use observable data on returns, volatility of returns, trading volume, etc. in the asset markets as well as data on observable components of the underlying fundamental where many components of the underlying fundamental are not observed by the scientist but are observed by the market participants to adduce evidence for departures from REE pricing as developed by our theory or any theory of departures from REE pricing?

This is not an easy problem, but it must be attacked in our view if economics is going to make any progress in narrowing the width of disagreements amongst scientists on the nature of the dynamics of crises like the recent financial crises. See Krugman (1998) for a recent commentary on the depth of this disagreement. At the risk of repeating ourselves let us conclude this paper with a short comment on what audiences we are trying to reach and what we are trying to do here.

Audiences for this work include not only academics but also policy makers. There is a lot of interest amongst policy makers in financial social interactions measurement and research. After all whenever the topic of regulation to assure "orderly" markets comes up, one is talking about possible problems caused by

social interactions in the financial markets. Terms such as "bubbles," "excessive speculation," "market panics," "herding," etc., reflect policy maker and public concern in this area. It is very difficult to *measure and quantify* such effects not only in theory, but also in empirical work. The work discussed here may be seen as an attempt to build a theory in which "excessive speculation" can be measured. In future work, we hope to contribute to "measuring" in both financial data and experimental markets whether the endogenous speculative part of the theory is significant.

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