ON MULTIPLICITY OF COMPETITIVE EQUILIBRIA WHEN FINANCIAL MARKETS ARE INCOMPLETE[1]

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This paper has three goals:

- It gives a brief up-to-date account of the uniqueness of competitive equilibria when markets are incomplete.
- It demonstrates by means of an example that when markets are incomplete multiple competitive equilibria can be obtained even though all agents have Cobb-Douglas preferences.
- It suggests a new computational method to generate examples of multiple competitive equilibria.

KEYWORDS: Incomplete financial markets, multiplicity of competitive equilibria, computation of competitive equilibria.

JEL CODES: D52, D58.

1. INTRODUCTION

Financial market models are extremely useful, both for providing good theoretical intuition as well as for making practical financial decisions. The perhaps most fundamental insight of financial market models is that no financial instrument can be studied in isolation. It is always the contribution to a *portfolio* of assets which is the most important characteristic of any single financial instrument. Moreover, on many financial markets a huge number of traders takes decisions independently from each other, decisions which are clearly motivated by self-interest. And neither on product markets nor on labor markets are prices as flexible as on financial markets. Hence, the notion of a *competitive equilibrium* clearing a system of interdependent markets suggests itself as a natural theoretical foundation underlying financial market models. Indeed many of the most important financial market models like the CAPM, the Black and Scholes option pricing model and the Modigliani-Miller model, for example, can be phrased in terms of competitive equilibria. Results like the CAPM beta-pricing formula or the Black and Scholes option pricing formula refer to some competitive equilibrium.

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Unfortunately, these pricing formulae depend on the particular equilibrium chosen. Hence if there are multiple competitive equilibria these results are indeterminate in some important aspect. In the CAPM, for example, the beta-coefficients which determine the assets' risk premiums differ across multiple equilibria (cf. Bottazzi, Hens and Löffler (1998)). Hence, to provide a sound theoretical foundation the *uniqueness* of its competitive equilibria is essential for financial market models.

Recently the theoretical literature on complete financial markets has made some progress in providing acceptable restrictions leading to a unique competitive equilibrium. In this literature the following two approaches can be distinguished. Whereas the older literature referred to restrictions on the traders' heterogeneity of beliefs combined with specific functional forms of their risk aversion functions (like assuming identical beliefs and identical HARA-utility functions) the new literature allows for any heterogeneity of beliefs and is able to achieve unique equilibria from assuming that each trader's relative risk aversion is sufficiently small.

Only recently there have been some successful attempts to actually *use* the heterogeneity of agents' characteristics in order to obtain unique financial market equilibria. The general idea that aggregation can create useful structure goes back to the seminal paper by Hildenbrand (1983). In this spirit, but based on the ingenious parametrization of agents' characteristics chosen in Grandmont (1992), Calvet, Grandmont and Lemaire (1998) demonstrate that the aggregate excess demand function has the gross substitution property if agents' beliefs display some certain form of heterogeneity. However, this approach hinges on the unrealistic assumption of complete financial markets.

The question of unique competitive equilibria in incomplete markets has been addressed only very recently. Extending both approaches to incomplete markets leads to the introduction of further assumptions restricting the assets' payoffs and the traders' endowments. Laitenberger (1998), for example, is the only one whose results for incomplete markets are based on the heterogeneity of agents' attitudes towards risk. He demonstrates various results for incomplete markets using much stronger restrictions than for complete markets.

In this paper we demonstrate by means of an example that when markets are incomplete small relative risk aversion is not sufficient to obtain unique financial market equilibria. We make this point by explicitly computing multiple equilibria for one example economy. Whereas the computation of one competitive equilibrium for any given economy has long been studied (cf. Kehoe (1991) and more recently Herings (1996) for excellent surveys), no efficient computational methods are known by which multiple equilibria can be computed; or, at least, by which one equilibrium can be computed out of which the existence of multiple competitive equilibria can be inferred. The latter would be a successful method to detect multiplicity if an equilibrium with a negative index could be computed. By the index theorem (cf. Dierker (1972), Mas-Colell (1985) for the Arrow-Debreu model, and Hens (1991) for the general equilibrium model with incomplete markets in the one-good-case) there must then be multiple equilibria. Standard computational methods are, however, bound to compute equilibria with a positive index (cf. Mas-Colell (1985)).

The main example of our paper is constructed by a new method to compute multiple competitive equilibria. This method has been obtained by the following observation. Note that for the general question of uniqueness we are not interested in whether one given particular economy has multiple equilibria but whether within some class of economies examples with multiple equilibria can be found. The trick of our method is to take the data of two prospective competitive equilibria as given and then to compute the characteristics of an economy (within some class of economies) which generates the preassigned data as its competitive equilibria. Following this new approach we were even able to compute equilibria with a negative index.

The remainder of the paper is organized as follows. In the next section we present a simple model of an exchange economy with incomplete financial markets, define its competitive equilibria and derive some invariance properties on the number of competitive equilibria. Thereafter we briefly review some recent results on the uniqueness of competitive equilibria when financial markets are incomplete. This lays the foundation for understanding why even with Cobb-Douglas preferences standard arguments known from the complete markets case for obtaining uniqueness break down when markets are incomplete. We present the data of the example showing the multiplicity of equilibria in exchange economies with Cobb-Douglas preferences when financial markets are incomplete and finally we present our new method to compute multiple equilibria.

2. THE MODEL

The model chosen is the simplest version of a general equilibrium model with incomplete financial markets. We refer to Magill and Quinzii (1996) and Hens (1998) for an account of the full variety of the general model. For the purpose of this paper it will however be legitimate to restrict attention to exchange economies with a finite horizon in which a single consumption good is available in a finite number of states of the world. These states represent symmetric uncertainty among a finite number of traders. The basic intuition is that without these simplifying assumptions the equilibrium set would only become larger thus weakening our result which shows that with Cobb-Douglas preferences the equilibrium set may have at least three elements. Nevertheless this very simple general equilibrium model is still general enough to include cornerstones of finance like the static CAPM (cf. Geanakoplos and Shubik (1989) for displaying the CAPM as a special case of this model). Within this very simple model the 'fundamental theorem of asset pricing' can be proved which provides the foundation for the important no-arbitrage reasonings as applied in the Black and Scholes and the Modigliani-Miller model.

The details of the model are as follows:

There are two periods $t = 0, 1$. In the second period a finite number of states, $s = 1, \dots, S$, may occur. Uncertainty is assumed to be symmetric across traders.

In each state $s = 1, \dots, S$ there is a spot market on which a single consumption good is traded. The consumption good should be interpreted as a Hicks-Leontief 'composite commodity' called 'expenditure for commodities in state s'. The composite commodity is infinitesimally divisible, hence the commodity space of the model is $\mathbb{R}^S$, the Euclidean space of dimension S.

Traders $i = 1, \dots, I$ are described by their consumption sets $X^i \subset \mathbb{R}^S$, their utility functions $U^i : X^i \to \mathbb{R}$ and their endowments $\omega^i \in X^i$. We will evoke the standard assumptions $X^i = \mathbb{R}^S_+$, U^i is continuous, strictly monotonic, strictly quasi-concave for every $i = 1, \dots, I$. In period $t = 0$ agents can trade, without any short sales restrictions, $j = 1, \dots, J$ assets delivering $A^j_s \in \mathbb{R}$ units of the single consumption good in state $s = 1, \dots, S$. Asset trades have to be self-financing, i.e. agents face the budget restriction $q \cdot \theta^i \leq 0$, where $q \in \mathbb{R}^J$ denotes the vector of asset prices and $\theta^i \in \mathbb{R}^J$ is agent i's portfolio choice. The characteristics of the asset markets can be summarized by the matrix of asset payoffs $(A^j_s)^{j=1,\dots,J}_{s=1,\dots,S} = A \in \mathbb{R}^{S \times J}$. Thus the financial market model is given by

$$\text{GEI} = [\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)^I_{i=1}, A].$$

A portfolio of assets $\theta \in \mathbb{R}^J$ results in a vector of period one payoffs $(A\theta) \in \mathbb{R}^S$. Financial markets are called 'complete' if any vector of income transfers $y \in \mathbb{R}^S$ can be achieved on the asset markets by choice of some portfolio $\theta \in \mathbb{R}^J$, i.e. if the column span of the payoff matrix A, $\langle A \rangle$ is equal to $\mathbb{R}^S$. Otherwise financial markets are 'incomplete'. The latter case is commonly assumed to be the more realistic one because of transaction costs, bounded rationality, moral hazard, and adverse selection (cf. Magill and Quinzii (1996, Chapter 1)).

Note that the model chosen in our paper does not explicitly include consumption in the first period. However, as Geanakoplos and Polemarchakis (1986) have pointed out, on restricting the asset payoffs such that $A^1_1 = 1$, while $A^j_1 = 0$, $j = 2, \cdots, J$, and $A^1_s = 0$, $s = 2, \cdots, S$, i.e. for

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \bar{A} \end{pmatrix},$$

consumption in $s = 1$ can be interpreted as first period consumption.

Denoting by $x^i \in \mathbb{R}^S_+$ agent i's planned consumption, her second period budget constraints are $x^i \leq \omega^i + A\theta^i$. Note that these constraints do not include spot market prices. This is because of strict monotonicity of U^i in each state spot market prices would be strictly positive. Hence, since assets pay off in units of the consumption good, budget constraints are invariant with respect to spot market prices.

Now we are in a position to define the central concept of financial market models.

DEFINITION 2.1: A *competitive equilibrium* is an allocation $(\overset{\star}{x}, \overset{\star}{\theta})$ and a vector of asset prices $\overset{\star}{q}$ such that

$$(1) \qquad (\overset{\star}{x}{}^i, \overset{\star}{\theta}{}^i) \in \arg \max_{\substack{\theta \in \mathbb{R}^J \\ x \in \mathbb{R}^S_+}} U^i(x) \;\; s.t. \;\; \overset{\star}{q} \cdot \theta \leq 0 \;\wedge x \leq \omega^i + A\theta,$$

$$(2) \qquad \sum_{i=1}^{I} \overset{\star}{x}{}^i = \sum_{i=1}^{I} \omega^i,$$

$$(3) \qquad \sum_{i=1}^{I} \overset{\star}{\theta}{}^i = 0.$$

The assumptions on U^i assure that $\overset{\star}{x}{}^i$ in (1) is unique.

Since the utility functions U^i are strictly monotonic, consumers maximizing their utility always exhaust tomorrow's budget constraints, i.e. we can substitute the inequalities by equalities. Therefore, knowing $\overset{\star}{\theta}{}^i$, $\overset{\star}{x}{}^i$ can be computed by

$$\overset{\star}{x}{}^i = \omega^i + A\, \overset{\star}{\theta}{}^i.$$

This equation allows to transform the maximization problems of the consumers, to restrict attention to portfolio demands and to neglect consumption plans. We get the following definition.

DEFINITION 2.2: A *financial market equilibrium* is a portfolio allocation $\overset{\star}{\theta}$ and a vector of asset prices $\overset{\star}{q}$ such that

$$(1) \qquad \overset{\star}{\theta}{}^i \in \arg \max_{\theta \in \mathbb{R}^J} U^i(\omega^i + A\theta) \;\; s.t. \;\; \overset{\star}{q} \cdot \theta \leq 0 \;\wedge \omega^i + A\theta \geq 0,$$

$$(2) \qquad \sum_{i=1}^{I} \overset{\star}{\theta}{}^i = 0.$$

Asset demand is homogeneous of degree zero in asset prices. Therefore, considering the number of equilibria we have to introduce a price normalization. It is a well-known result that after such a normalization the number of equilibria is generically finite. So, speaking of a unique equilibrium means the existence of exactly one equilibrium price vector $\overset{\star}{q}$ after normalization. The maximizing property of equilibria requires asset prices to be arbitrage-free, i.e. there cannot be a positive revenue tomorrow without investment today. Denote the set of these arbitrage free asset prices by

$$Q = \left\{ q \in \mathbb{R}^J \mid \nexists\, \theta \in \mathbb{R}^J \text{ with } q \cdot \theta \leq 0 \wedge A\theta > 0 \right\}.$$

For simplification the following assumptions on the asset matrix A are made.

ASSUMPTION A

(i) There are no redundant assets, i.e. rank $A = J$.

(ii) There always exists a desirable portfolio, i.e. there is $\bar{\theta} \in \mathbb{R}^J$ *such that* $A\bar{\theta} > 0$.

A(i) implies that there is a one-to-one correspondence between consumption and portfolio plans, because every consumption plan is financed by exactly one portfolio plan. Therefore, $\overset{\star}{\theta}{}^i$ is the only maximizer in (1) of Definition 2.2. A(ii) assures that today's budget constraint is satisfied with equality.

We are now able to prove that changing the asset matrix of a GEI-economy does not change the number of equilibria, as long as the asset span $\langle A \rangle$ remains the same.

THEOREM 2.3: *Consider two GEI-economies* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A)]$ *and* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, B)]$ *such that* $\langle A \rangle = \langle B \rangle$. *Then – after price normalization – both economies have the same number of financial market equilibria and therefore of competitive equilibria.*

PROOF:[2] Let $E(A)$ and $E(B)$ be the normalized equilibrium price sets. We have to show

$$\mid E(A) \mid = \mid E(B) \mid .$$

If $\overset{\star}{q} \in E(A)$, there are $\overset{\star}{\theta}{}^i \in \mathbb{R}^J, i = 1, ..., I$, such that $(\overset{\star}{\theta} = (\overset{\star}{\theta}{}^i)_{i=1}^I, \overset{\star}{q})$ is financial market equilibrium for $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A)]$. Then it is straightforward to verify that $(\hat{\theta} = (\hat{\theta}^i)_{i=1}^I, \hat{q})$ defined by

$$\hat{\theta}^i = (B^T B)^{-1} B^T A \, \overset{\star}{\theta}{}^i \; (i = 1, \ldots, I), \tag{1}$$

$$\hat{q} = B^T A (A^T A)^{-1} \, \overset{\star}{q} \tag{2}$$

is a financial market equilibrium in $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, B]$.

Starting with an equilibrium according to B we could reason analogously. Therefore, since the matrix products in (1) and (2) are invertible, the sets $E(A)$ and $E(B)$ must have the same number of elements. *Q.E.D.*

REMARK: Since the number of equilibria is invariant with respect to different representations of the same asset span, for the purpose of showing uniqueness we

[2]For those who know the concept of no-arbitrage equilibria (cf. Magill and Quinzii (1996)) the content of this theorem should be clear, because obviously the asset matrix does not influence the no-arbitrage equilibria as long as the matrix span is the same.

can always select one such representation which is most suitable for our argument. Given some representation by an asset matrix $A \in \mathbb{R}^{S\times J}$ useful properties like gross substitution or monotonicity need to be demonstrated only for some matrix $B \in \mathbb{R}^{S\times J}$ which represents the same asset span as A, i.e. $B = AM$ for some invertible $J \times J$ matrix M. Especially, this implies that given Assumption A (ii) when examining uniqueness of financial market equilibria Theorem 2.3 allows us to restrict ourselves to asset matrices with one non-negative asset.

Moreover, in the case of complete markets, for example, without loss of generality we can restrict attention to 'Arrow securities' , i.e. to securities which pay off one unit of the commodity in exactly one state, so that the asset matrix A is the $S \times S$ identity matrix. Choosing the identity matrix in the case of complete markets it is straightforward that financial market equilibria are equivalent to Walrasian equilibria in the Arrow-Debreu model.

Under the standard assumptions on utility functions mentioned above the *existence* of competitive equilibria can be established given some restrictions on the joint distribution of endowments and the asset market structure $(A, (\omega^i)_{i=1}^I)$ are satisfied (cf. Gottardi and Hens (1996)). The latter guarantee continuity and boundary behavior of excess demand. A sufficient condition for the satisfaction of these restrictions are interior endowments, i.e. the assumption that $\omega^i \gg 0, i = 1,\dots, I$. In order not to run into difficulties due to boundary endowments, throughout this paper we therefore assume interiority of endowments. In addition to the assumptions needed for existence much stronger assumptions are necessary for uniqueness. To these assumptions we turn in the next section.

3. UNIQUENESS OF FINANCIAL MARKET EQUILIBRIA

The uniqueness of competitive equilibria has been the subject of intensive research within the Arrow-Debreu model. For a recent account of this area of research see Mas-Colell, Whinston, and Green (1995, Chapter 17.F). Only recently a systematic study of this question has begun for incomplete financial market models (cf. Detemple and Gottardi (1998) and Voß (1997)).

In the early stages of general equilibrium theory it was believed that uniqueness of competitive equilibria should follow from agents' optimization behavior. It has, however, become clear that only very little structure, like continuity, homogeneity, and Walras Law, is actually implied by utility maximization on a complete set of markets. In fact without posing further restrictions on the individual characteristics the equilibrium price set is essentially arbitrary (cf. Mas-Colell (1977)). Recently, it has been shown that similar 'no structure results' can be obtained with incomplete financial markets (cf. Bottazzi and Hens (1996), Gottardi and Hens (1998)).

In order to achieve unique competitive equilibria besides the standard continuity, monotonicity and convexity assumptions further restrictions on the distribution of agents' characteristics have to be imposed. A huge variety of approaches has been suggested to serve this purpose. To present the most important ones in a systematic way, it is instructive to display them around the 'Weak Axiom of Revealed Preferences'. From a mathematical point of view, to achieve the uniqueness of competitive equilibria amounts to assure that a certain nonlinear system of equations – the interdependent system of market clearing conditions – has a unique solution. Under certain differentiability assumptions on the agents' utility functions it can be demonstrated that for a generic set of endowments this system of equations, say $\theta(q) = 0$, has a finite number of normalized solutions (cf. Hens (1991)), which we denote by $E = \{\overset{\star}{q} \in Q | \theta(\overset{\star}{q}) = 0\}$.

The differentiability assumptions are the following ones.

ASSUMPTION U *For all $i \in I$ the utility functions $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ satisfy the following properties:*

(i) U^i is continuous on $\mathbb{R}^S_+$ and C^∞ on $\mathbb{R}^S_{++}$.

(ii) $\left\{\tilde{x} \in \mathbb{R}^S_+ \mid U^i(\tilde{x}) \geq U^i(x)\right\} \subseteq \mathbb{R}^S_{++}$ for all $x \in \mathbb{R}^S_{++}$.

(iii) $\nabla U^i(x) \in \mathbb{R}^S_{++}$ for all $x \in \mathbb{R}^S_{++}$.

(iv) For all $x \in \mathbb{R}^S_{++}$ $h \cdot D^2U^i(x)h < 0$ for all $h \in \mathbb{R}^S \backslash \{0\}$ such that $\nabla U^i(x)h = 0$.

Here $\nabla U^i(x)$ denotes the vector of partial derivates of U^i and $D^2U^i(x)$ is the Hessian matrix.

Let us first define the so-called 'Weak Axiom of Revealed Preferences' which is always fulfilled by individual asset demand functions, but which is not necessarily a property of the market demand function.

DEFINITION 3.1: $\theta : Q \to \mathbb{R}^J$ satisfies the *Weak Axiom of Revealed Preferences (WARP)* if $q \cdot \theta(\tilde{q}) \geq 0$ whenever $\tilde{q} \cdot \theta(q) \leq 0$.

Given Assumption U it is more convenient to consider the following version.

DEFINITION 3.2: $\theta : Q \to \mathbb{R}^J$ satisfies the *Differentiable Version of the Weak Axiom of Revealed Preferences* if whenever $\theta(q) \cdot v = 0$ for a vector $v \in \mathbb{R}^J$

$$v \cdot \partial_q \theta(q) v \leq 0.$$

Under the differentiability assumptions on the utility function Definition 3.1 and Definition 3.2 are essentially equivalent. This follows form a direct adaptation of the corresponding proof in the Arrow-Debreu model.

We have mentioned above that for a generic set of endowments the set of normalized equilibrium price systems is finite. The importance of WARP is that if it were to hold for market demand it would imply convexity of the equilibrium set (cf. Hildenbrand and Kirman (1988, Proposition 6.2.), which can be adapted to our setting). Hence, for a generic set of endowments equilibria are unique.

Regrettably, WARP is not an additive property, i.e. besides individual demand functions satisfy WARP, this is not necessarily true for the market demand. But a stronger property – restricted monotonicity – is additive if the restriction is with respect to the same normalizing vector across agents.

DEFINITION 3.3: The asset demand function $\theta : Q \to \mathbb{R}^J$ is *restricted monotonous with respect to* $n \in \mathbb{R}^J$ if for all $q, \tilde{q} \in Q$ with $(q - \tilde{q}) \cdot n = 0$ it holds that $(q - \tilde{q}) \cdot (\theta(q) - \theta(\tilde{q})) \leq 0$, where n is such that $q \cdot n > 0$ for all $q \in Q$.

Again, there is an analogous differentiable version.

DEFINITION 3.4: The asset demand function $\theta : Q \to \mathbb{R}^J$ is *restricted monotonous in the differentiable version with respect to* n if for all $v \in \mathbb{R}^J$ with $n \cdot v = 0$ for all $q \in Q$ it holds that $v \cdot \partial_q \theta(q) v \leq 0$, where n is such that $q \cdot n > 0$ for all $q \in Q$.

If the demand function θ is differentiable, then Definition 3.3 and Definition 3.4 are equivalent. Obviously, if individual demand functions are monotonous with respect to the same normalizing vector n, this is also true for the market demand. Besides monotonicity implies WARP.

These propositions can be shown by adaptation of the analogous proofs in the Arrow-Debreu model (cf. Mas-Colell (1985, Proposition 5.7.3. (i) and (iii))[3]). Therefore, restricted monotonicity of the market demand function for assets implies uniqueness of equilibria.

Another property which leads to uniqueness of the financial market equilibria is the so-called gross substitution property which requires that the demand for each asset is falling in its own price and increasing in the other asset prices.

DEFINITION 3.5: The asset demand function $\theta : Q \to \mathbb{R}^J$ satisfies the *gross substitution property* if the Jacobian matrix has negative diagonal and positive off-diagonal elements, i.e. for all $j \in J$ and all $q \in Q$ it holds that

$$\partial_{q_j} \theta_j(q) < 0$$

and

$$\partial_{q_k} \theta_j(q) > 0 \quad \text{for} \quad k \neq j.$$

[3]The proofs use slightly stronger forms of monotonicity and WARP, but a modification is simple.

Obviously, this property is additive, i.e. if the individual demand function exhibits the gross substitution property, this is also true for the market demand. Concerning uniqueness we obtain the following conclusion.

THEOREM 3.6: *If the market asset demand $\theta : Q \to \mathbb{R}^J$ satisfies the gross substitution property and if $Q \subseteq \mathbb{R}^J_{++}$, then financial market equilibria are unique.*

PROOF: The proof is similar to the proof in the Arrow-Debreu model (cf. Arrow and Hahn (1971, Chapter 9, Corollary 7, second proof, p. 223)), but it needs some further considerations about the underlying finite sequence of prices because one has to take care that it has to stay within the no-arbitrage price set Q. For a thorough treatment see Voß (1997, Theorem 3.2.10). *Q.E.D.*

The restriction in Theorem 3.6 to asset matrices for which all no-arbitrage price vectors are strictly positive is indeed a limiting assumption. Even a simple asset structure like

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

does not satisfy this property. What makes the restriction worse is the fact that there is even no asset structure with the same matrix span guaranteeing strictly positive no-arbitrage prices (cf. Theorem 2.3). It is not known whether Theorem 3.6 holds without the assumption of strictly positive asset prices.

Now, we are in the position to use the presented properties to gain classes of economies for which financial market equilibria are unique. Since many of the proofs are quite long, but not very complicated, we do not give them in detail, but only present the ideas and give references for further information.

3.1. *Representative Agent Economies*

The assumption of a 'representative agent' is often made in financial market models, especially when seeking to determine simple formulae for asset prices. This amounts to assuming that the excess demand of the economy can be thought of as being derived from the optimization problem of only one agent. In technical terms, one assumes $I = 1$, hence neglecting the heterogeneity of agents. In effect, risk premiums of assets, for example, can be related to aggregate endowments, ω^I (cf. Magill and Quinzii (1996, Chapter 3) for an account of representative agent asset pricing). Indeed, under the assumption of a representative agent, risk premiums are well determined because the underlying competitive equilibrium is unique.

Since $I = 1$ market demand is generated by an individual demand function, hence it satisfies WARP. The existence of a representative agent is obvious for a one-agent economy. Moreover, as we will show later, even with several agents, for certain special classes of preferences the market demand can still be thought of as being derived from one hypothetical agent's decision problem, given the heterogeneity of agents is limited.

3.2. *Special Utility Functions*

3.2.1. *Quasi-linear Utility Functions*

Quasi-linear utility functions are characterized by indifference curves which are displacements of each other along one axis.

DEFINITION 3.7: A utility function $U^i : \mathbb{R} \times \mathbb{R}_+^{S-1}$ is *quasi-linear in the first good* if

$$U^i(x) = x_1 + V^i(x_{-1})$$

where $V^i : \mathbb{R}_+^{S-1} \to \mathbb{R}$ is continuous on $\mathbb{R}_+^{S-1}$, $\mathcal{C}^\infty$ on $\mathbb{R}_{++}^{S-1}$, strictly monotonic and strictly concave on $\mathbb{R}_{++}^{S-1}$. Besides for all $x_{-1} \in \mathbb{R}_{++}^{S-1} : \{\tilde{x}_{-1} \in \mathbb{R}_+^{S-1} \mid V^i(\tilde{x}_{-1}) \geq V^i(x_{-1})\} \subset \mathbb{R}_{++}^{S-1}$.

If we restrict attention to asset structures which incorporate first period consumption, from quasi-linearity of every agent's utility function in the first good, we obtain uniqueness of financial market equilibria. Since we have to use first order conditions, we have to require sufficiently large endowments today if we remain to restrict today's consumption to be non-negative.

THEOREM 3.8: *Let $[\mathbb{R}^S, (\mathbb{R}_+^S, U^i, \omega^i)_{i=1}^I, A]$ be a GEI-economy with first period consumption where consumers have quasi-linear utility functions. If ω_1^i is sufficiently large for all $i \in I$* [4]*, then financial market equilibria are unique.*

PROOF: Restricted monotonicity of the individual asset demand functions with respect to the first unit vector is obtained by applying the 'implicit function theorem' to the first order conditions (cf. Voß (1997, Theorem 4.1.7)). *Q.E.D.*

[4] An explicit condition is given in Voß (1997, p. 63), but this requires much work for little insight.

3.2.2. *Quasi-homothetic Utility Functions*

Another important class of utility functions is given by quasi-homothetic functions. These are those functions which exhibit indifference sets that are related by proportional expansions along rays through some point $\alpha \in \mathbb{R}^S$.

DEFINITION 3.9: A utility function $U^i : \mathbb{R}^S_+ - \{\alpha^i\} \to \mathbb{R}$ is *quasi - homothetic* if there is – given a constant vector $\alpha^i \in \mathbb{R}^S$ – a function $V^i : \mathbb{R}^S_+ \to \mathbb{R}$ such that for all $x \in \mathbb{R}^S_+ - \{\alpha^i\}$

$$U^i(x) = V^i(\alpha^i + x)$$

and V^i satisfies for all $y \in \mathbb{R}^S_+$

$$V^i(\lambda y) = \lambda^k V^i(y)$$

for all $\lambda \in \mathbb{R}_{++}$ and some $k \in \mathbb{R}$.

The subclass of additively separable quasi-homothetic utility functions is very close to the so-called HARA-class which are expected utility functions.[5]

Results concerning uniqueness of financial market equilibria are summarized in the following theorem.

THEOREM 3.10: *Let $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A]$ be a GEI-economy where the utility functions U^i are quasi-homothetic and $\omega^i + \alpha^i \gg 0$ for all $i \in I$. If one of the following conditions is satisfied, financial market equilibria are generically unique:*

(i) Preferences are identical homothetic, i.e. there is a function V such that $V^i = V$ for all $i \in I$, and besides either $\alpha^i + \omega^i \in \langle A \rangle$ for all $i \in I$ or the $\alpha^i + \omega^i$, $i = 1, \ldots, I$, are collinear.

(ii) The vectors $\alpha^i + \omega^i$, $i = 1, \ldots, I$, are collinear and elements of $\langle A \rangle$.

PROOF: The conditions (i) or (ii) on the vectors $\alpha^i + \omega^i$ allow a transformation of consumers' maximization problems such that together with the homotheticity property we get the existence of a representative consumer (under the additional assumptions of identical preferences resp. the collinearity of the vectors $\alpha^i + \omega^i$, $i = 1, \ldots, I$). So, the results of Section 3.1 can be applied (cf. Detemple and Gottardi (1998, Theorems 3.1 and 4.1) and Voß (1997, Theorem 4.1.5)). *Q.E.D.*

[5] For a discussion of expected utility functions and especially HARA-functions cf. Section 3.3.

3.2.3. *Quadratic Utility Functions*

Quadratic utility functions are also a special type of expected utility functions (cf. Section 3.3). In the finance literature they are applied especially in combination with the Capital Asset Pricing Model because they allow to compute the utility out of mean and variance according to the underlying probability measure.

DEFINITION 3.11: A utility function $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ is a *quadratic utility function* if

$$U^i(x) = \sum_{s=1}^{S} \rho^i_s (x_s - \frac{1}{2}\alpha^i x^2_s)$$

where $\rho^i \in \mathbb{R}^S_{++}$, $\sum_{s=1}^S \rho^i_s = 1$ and $\alpha^i \in \mathbb{R}_{++}$.

Quadratic utility functions are not monotonic everywhere, they have some satiation point. Therefore, we have to assure that this point is not attainable. This is realized by the assumption

$$1 - \alpha^i \omega_s > 0 \;\forall\, i \in I \;\wedge\; \forall\, s \in S$$

where $\omega := \sum_{i=1}^I \omega^i$.

Moreover, uniqueness is only obtained for interior equilibria since quadratic utility functions do not satisfy Assumption U.

THEOREM 3.12: *Let* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)^I_{i=1}, A]$ *be a GEI-economy with quadratic utility functions which satisfy the assumption of monotonicity. If there is a* $\rho \in \mathbb{R}^S_{++}$ *such that* $\rho^i = \rho$ *for all* $i \in I$, *i.e. if there is an objective probability measure, then there is at most one interior financial market equilibrium.*

PROOF: Using the market clearing property of equilibria it is possible to compute the unique equilibrium price vector out of the first order conditions to the maximization problems. *Q.E.D.*

3.3. *Expected Utility Hypothesis*

In financial markets models additive separability comes as a by-product of the well known expected utility hypothesis, which amounts to assuming that

$$U^i(x^i) = \sum_{s=1}^{S} \rho^i_s u^i(x^i_s), i = 1, \ldots, I.$$

According to this assumption agents' beliefs and their attitudes towards risk can be separately described by some probability measure ρ^i on the state space $\{1, \ldots, S\}$

and some von Neumann-Morgenstern utility function for expenditure $u^i : \mathbb{R}_+ \to \mathbb{R}$. With respect to uniqueness of financial market equilibria the expected utility hypothesis becomes very powerful when it is combined with further restrictions on the heterogeneity of beliefs, the functional forms of the risk aversion and possibly some restrictions on the distribution of agents' endowments.

3.3.1. *HARA-Functions*

Following Arrow (1970) and Pratt (1964), agents' absolute risk aversion is defined as a function of state income $x_s \in \mathbb{R}_+$:

$$ARA^i(x_s) = -\frac{u^{i''}(x_s)}{u^{i'}(x_s)}.$$

DEFINITION 3.13: Agents have *hyperbolic absolute risk aversion (HARA)*, if $(ARA^i(x_s))^{-1}$ is a linear function in state income x_s, i.e. $[ARA^i(x_s)]^{-1} = \alpha^i + \beta^i x_s$ where $\alpha^i \in \mathbb{R}_+$ and $\beta^i \in \mathbb{R}$.

Note that this defines a second order differential equation for the von Neumann-Morgenstern utility function u^i. It is well known that a solution is given by

$$u^i(x_s) = \begin{cases} \frac{(\beta^i)^2(\alpha^i+\beta^i x_s)^{\frac{\beta^i-1}{\beta^i}}}{\beta^i-1} & \text{for} \quad \beta^i \neq 0 \text{ and } \beta^i \neq 1, \\ -\alpha^i \exp\left(-\frac{x_s}{\alpha^i}\right) & \text{for} \quad \beta^i = 0, \\ \ln(\alpha^i + x_s) & \text{for} \quad \beta^i = 1. \end{cases}$$

For $\beta^i = -1$ the HARA-function is equal to the quadratic utility function. For $\beta^i \geq 0$ all HARA-functions are quasi-homothetic. Therefore, we can apply the results of Section 3.2.2. The functions V^i are identical for all consumers if $\beta^i = \beta$ for some β and the beliefs are homogenous, i.e. there is $\rho \in \mathbb{R}^S_{++}$ such that $\rho^i = \rho$ for all $i \in I$ (cf. Gorman (1953)). On the other hand, all additively separable quasi-homothetic utility functions are HARA-functions (cf. Pollak (1971) and Voß (1997)).

3.3.2. *Small Relative Risk Aversion*

Allowing for any degree of heterogeneity of beliefs but still not sacrificing uniqueness of financial market equilibria is possible by assuming small values of agents' relative risk aversion,

$$RRA^i(x_s) := -\frac{x_s u^{i''}(x_s)}{u^{i'}(x_s)},$$

if this assumption is combined with some restrictions on the asset structure and on the distribution of endowments.

Mitjushin-Polterovich-Theorem

In the case of expected utility functions it is possible to gain results in the GEI-model which are analogous to the 'Mitjushin-Polterovich-Theorem' in the Arrow-Debreu model (cf. Mitjushin and Polterovich (1978)): under special assumptions the asset demand functions are restricted monotonous with respect to the endowment vectors. Regrettably, for restricted monotonicity of the market demand and therefore for uniqueness the strong assumption of collinear endowments is necessary.

The Assumption UU on $u^i : \mathbb{R}_+ \rightarrow \mathbb{R}$ which implies Assumption U for U^i is given in the following.

ASSUMPTION UU

(i) u^i *is continuous on* $\mathbb{R}_+$ *and* $\mathcal{C}^\infty$ *on* $\mathbb{R}_{++}$.

(ii) $\lim_{x \to 0} u^{i\prime}(x) = \infty$.

(iii) $u^{i\prime}(x) > 0$ *for all* $x > 0$.

(iv) $u^{i\prime\prime}(x) < 0$ *for all* $x > 0$.

THEOREM 3.14: *Let* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A]$ *be a regular GEI-economy with expected utility functions* U^i *such that*

- $RRA^i(x_s) < 4$ *for all* $x_s > 0$,
- *there exists a fundamental set* $R \subseteq S$, *i.e.* $R \subseteq S$ *with* $\mid R \mid = J$ *and* A_R *is invertible such that* $A_{S\setminus R}A_R^{-1} \geq 0$,
- $\omega^i, i = 1, \ldots, I$, *are collinear,*

then financial market equilibria are unique.

PROOF: The expected utility hypothesis together with the existence of fundamental sets allows to transform the consumers' maximization problems such that the well-known Mitjushin-Polterovich-Theorem can be applied and restricted monotonicity of the asset demand functions θ^i with respect to normalizing vectors $A_R^{-1}\omega_R^i$ is gained. Therefore, collinear endowments assure restricted monotonicity of the market demand with respect to some normalizing vector which – together with the finiteness of the equilibrium price set (regularity) – implies uniqueness (cf. Bettzüge (1998)). *Q.E.D.*

REMARK: For general utility functions satisfying Assumption U, a modified version of the 'Mitjushin-Polterovich-Theorem' can be applied if endowments are spanned (cf. Hens (1995, p. 168) and Voß (1997, Corollary 4.2.6)).

Gross Substitution

Willing to assume that agents' relative risk aversion is even not greater than one, we can dispense with the disturbing assumption of collinear endowments. Results along this line are based on a theorem by Varian (1985), which derives the gross substitution property of market demand under a similar condition within the Arrow-Debreu model. Transferring this theorem to incomplete financial markets was so far only possible under severe restrictions on the asset structure.

THEOREM 3.15: *Let* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A]$ *be a GEI-economy where* $U^i, i = 1, \ldots, I$, *satisfies the expected utility hypothesis and* $RRA^i(x_s) \leq 1$ *for all* $x_s > 0$. *If* A *is non-negative and weakly separating, i.e. for all* $s \in S$, $A^j_s > 0$ *for one* $j \in J$ *and* $A^k_s = 0$ *for all* $k \in J \setminus \{j\}$, *then financial market equilibria are unique.*

PROOF: Applying the 'implicit function theorem' to the first order conditions of consumers' maximization problems the special structure of A allows to gain the gross substitution property for the individual and therefore market asset demand functions which implies uniqueness (cf. Voß (1997, Corollary 4.3.4)). *Q.E.D.*

Two Assets

In the case of only two assets besides non-negativity there are no further restrictions on A necessary to get the gross substitution property. But uniqueness can be proven even without the non-negativity assumption.

THEOREM 3.16: *Let* $[\mathbb{R}^S, (\mathbb{R}^S_+, U^i, \omega^i)_{i=1}^I, A]$ *be a GEI-economy with two assets and with expected utility functions* U^i *such that* $RRA^i(x_s) \leq 1$ *for all* $x_s > 0$. *Then financial market equilibria are unique.*

PROOF: Applying the 'implicit function theorem' to the first order conditions of consumers' maximization problems gives monotonicity of the individual asset demand functions with respect to the same unit vector, and therefore of market demand which implies uniqueness (cf. Becker (1995, pp. 40–41) and Bettzüge

(1997, Proposition 2.2.)). *Q.E.D.*

The next section will demonstrate that restrictions on the relative risk aversion are not sufficient for uniqueness of equilibria: further requirements have to be put on the economy. This is a very important observation since as mentioned above - in the Arrow-Debreu model - no further restrictions are necessary.

4. COBB-DOUGLAS PREFERENCES

4.1. *The Merits of Cobb-Douglas Preferences*

In the context of financial markets agents are said to have Cobb-Douglas preferences if they satisfy the expected utility hypothesis and their von Neumann-Morgenstern utility functions are logarithmic, i.e. if

$$U^i(x^i) = \sum_{s=1}^{S} \rho_s^i \ln(x_s^i) \text{ for all } x^i \in \mathbb{R}_{++}^S.$$

Note that Cobb-Douglas preferences satisfy Assumption UU on $\mathbb{R}_{++}^S$. Since besides endowments are strictly positive, it does not matter that utility is not defined on the boundary of $\mathbb{R}_{+}^S$.

They are often used when markets are complete because of their following merits:

Complete Markets

Cobb-Douglas preferences satisfy the Assumption UU from the previous section on $\mathbb{R}_{++}^S$ which implies that the 'Linear Pricing Rule', $q^T = \nabla U^i(x^i)A$, is a necessary and sufficient condition for utility maximization. Traders' investment decisions based on Cobb-Douglas preferences reduce to fixing the shares which determine, how period one income $\pi \cdot \omega^i \in \mathbb{R}_{++}$ should be distributed, i.e. $x_s^i(\pi) = \rho_s^i \frac{\pi \cdot \omega^i}{\pi_s}$, for all $\pi \in \mathbb{R}_{++}^S, s = 1, \ldots, S$. From these demand functions it is easy to see that the gross substitution property is satisfied and hence that competitive equilibria are unique. Moreover, equilibrium prices are easily computed since they are the solution to the linear system of equations $\sum_{i=1}^I \rho_s^i(\pi \cdot \omega^i) = \sum_{i=1}^I \pi_s \omega_s^i, s = 1, \ldots, S$.

Incomplete Markets

When markets are incomplete, economies with Cobb-Douglas preferences lose all the nice properties mentioned above.

To see this we start with the following simple example taken from Voß (1997).

EXAMPLE 4.1: Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad U(x) = \sum_{s=1}^{4} \frac{1}{4} \ln(x_s).$$

Then the excess demand functions for goods are given by

$$z_1(\pi) = \frac{\pi_2 + \pi_3 + \pi_4}{4\pi_1} - \frac{3}{4},$$
$$z_2(\pi) = \frac{\pi_1 + \pi_3 + \pi_4}{4\pi_2} - \frac{3}{4},$$
$$z_3(\pi) = z_4(\pi) = -\frac{1}{2} + \frac{\pi_1 + \pi_2}{2\,(\pi_3 + \pi_4)},$$

and the Jacobian has the sign-pattern

$$\partial_\pi z(\pi) = \begin{pmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & - \\ + & + & - & - \end{pmatrix}.$$

The intuition underlying this example is that – when markets are incomplete – the asset payoff matrix may introduce some complementarities among consumption demand, x, hence destroying the gross substitution property. Indeed note that in this example the asset matrix is such that demand has to be the same in the last two states. This clearly shows that gross substitution cannot be expected for consumption demand. However, if gross substitution was to hold for asset demand then, arguing via the financial market equilibrium concept, this would be sufficient for the uniqueness of equilibria. There are examples where gross substitution does not hold for consumption demand while it still holds for asset demand (cf. Voß (1997)). But in Example 4.1 gross substitution fails as well for asset demand.

EXAMPLE 4.1 (CONTINUED): Asset demands are

$$\theta_1 = \frac{q_2 - 3q_1}{4q_1}, \quad \theta_2 = \frac{3q_2 - q_1 - 4q_3}{4(q_3 - q_2)}, \quad \theta_3 = \frac{3q_2q_3 - 2q_2^2 + 3q_1q_3 - 2q_1q_2}{4q_3(q_3 - q_2)}.$$

Hence for $4q_3 = 3q_2$

$$\partial_{q_1}\theta_3 = -\frac{1}{3q_2} < 0.$$

REMARK: Note that the gross substitution property is not invariant with respect to different asset structures representing the same asset span. Indeed, in Example 4.1 instead of A we could also consider the asset matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\langle B \rangle = \langle A \rangle$. Then, by the proof of Theorem 3.15 excess demand satisfies gross substitution.

4.2. *Multiplicity of Competitive Equilibria with Cobb-Douglas Preferences*

Recall those uniqueness results mentioned in Section 3 which apply to the case of Cobb-Douglas preferences.

First note that for Cobb-Douglas preferences the relative risk aversion is equal to one. Moreover, Cobb-Douglas preferences belong to the HARA-class. Hence, with homogenous beliefs equilibria are unique. Moreover, with heterogeneous beliefs uniqueness is obtained if one of the following conditions is satisfied:

- There are only two assets.
- The asset payoff matrix is weakly separating and non-negative.
- Endowments are collinear and there exists a fundamental set.
- Endowments are collinear and elements of the asset span.

As the following example demonstrates, in general, however when markets are incomplete there can be multiple competitive equilibria even if all agents have Cobb-Douglas preferences.

EXAMPLE 4.2: There are two agents, five states and three assets. The agents' utility functions are

$$\begin{aligned} U^1(x^1) &= 0.1 \ln x_1^1 + 0.2 \ln x_2^1 + 0.3 \ln x_3^1 + 0.1 \ln x_4^1 + 0.3 \ln x_5^1, \\ U^2(x^2) &= 0.3 \ln x_1^2 + 0.2 \ln x_2^2 + 0.1 \ln x_3^2 + 0.3 \ln x_4^2 + 0.1 \ln x_5^2. \end{aligned}$$

They hold the initial endowments

$$\omega^1 = \begin{pmatrix} 1.5942379526 \cdot 10^{-2} \\ 2.9374491276 \cdot 10^{-3} \\ 6.1116956313 \cdot 10^{-2} \\ 2.6125379370 \cdot 10^{-1} \\ 1.3112543254 \cdot 10^{-1} \end{pmatrix}, \omega^2 = \begin{pmatrix} 6.5484744962 \cdot 10^{-3} \\ 6.1159308802 \cdot 10^{-2} \\ 4.6047779430 \cdot 10^{-3} \\ 7.1874620630 \cdot 10^{-1} \\ 7.6887456746 \cdot 10^{-1} \end{pmatrix}.$$

The asset matrix is given by[6]

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5.7551080261 & -11.211070031 \\ -49.735960347 & -14.564562225 & 1.4276396811 \end{pmatrix}.$$

This economy has at least the following three equilibria:

$$(\bar{x}^1, \bar{x}^2, \bar{\theta}^1, \bar{\theta}^2, \bar{q}) = \left(\begin{pmatrix} 3.1569068369 \cdot 10^{-3} \\ 2.0847535004 \cdot 10^{-2} \\ 5.6794692704 \cdot 10^{-2} \\ 4.0 \cdot 10^{-1} \\ 5.0 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.9333947185 \cdot 10^{-2} \\ 4.3249222926 \cdot 10^{-2} \\ 8.9270415519 \cdot 10^{-3} \\ 5.8 \cdot 10^{-1} \\ 4.0 \cdot 10^{-1} \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1.2785472689 \cdot 10^{-2} \\ 1.7910085876 \cdot 10^{-2} \\ -4.3222636089 \cdot 10^{-3} \end{pmatrix}, \begin{pmatrix} 1.2785472689 \cdot 10^{-2} \\ -1.7910085876 \cdot 10^{-2} \\ 4.3222636089 \cdot 10^{-3} \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 \\ 1.1 \\ 1.6 \end{pmatrix} \right),$$

$$(\hat{x}^1, \hat{x}^2, \hat{\theta}^1, \hat{\theta}^2, \hat{q}) = \left(\begin{pmatrix} 3.2964558373 \cdot 10^{-3} \\ 1.6548013477 \cdot 10^{-2} \\ 5.2816066068 \cdot 10^{-2} \\ 4.2 \cdot 10^{-1} \\ 5.5 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.9194398185 \cdot 10^{-2} \\ 4.7548744453 \cdot 10^{-2} \\ 1.2905668188 \cdot 10^{-3} \\ 5.6 \cdot 10^{-1} \\ 3.5 \cdot 10^{-1} \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1.2645923689 \cdot 10^{-2} \\ 1.3610564349 \cdot 10^{-2} \\ -8.3008902451 \cdot 10^{-3} \end{pmatrix}, \begin{pmatrix} +1.2645923689 \cdot 10^{-2} \\ -1.3610564349 \cdot 10^{-2} \\ +8.3008902451 \cdot 10^{-3} \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 \\ 1.6 \\ 1.1 \end{pmatrix} \right),$$

$$(\tilde{x}^1, \tilde{x}^2, \tilde{\theta}^1, \tilde{\theta}^2, \tilde{q}) = \left(\begin{pmatrix} 5.53593779 \cdot 10^{-3} \\ 2.98288576 \cdot 10^{-3} \\ 4.72273200 \cdot 10^{-2} \\ 4.06826530 \cdot 10^{-1} \\ 6.28208646 \cdot 10^{-1} \end{pmatrix}, \begin{pmatrix} 1.69549162 \cdot 10^{-2} \\ 6.11138722 \cdot 10^{-2} \\ 1.84944143 \cdot 10^{-2} \\ 5.73173470 \cdot 10^{-1} \\ 2.71791354 \cdot 10^{-1} \end{pmatrix}, \right.$$

[6] Note that this asset matrix does not satisfy the assumption of desirable portfolios, i.e. Assumption A. As we demonstrate in the next section this 'defect' is innocuous since all agents will still satisfy their budget restrictions with equality if at least one of the assets has a positive price.

$$\begin{pmatrix} -1.04064417 \cdot 10^{-2} \\ 4.54366309 \cdot 10^{-5} \\ -1.38896363 \cdot 10^{-2} \end{pmatrix}, \begin{pmatrix} 1.04064417 \cdot 10^{-2} \\ -4.54366309 \cdot 10^{-5} \\ 1.38896363 \cdot 10^{-2} \end{pmatrix},$$
$$\left.\begin{pmatrix} -1 \\ 11.3031102 \\ 0.786198913 \end{pmatrix}\right).$$

In the next section we explain how Example 4.2 has been found.

5. A NEW METHOD TO COMPUTE MULTIPLE COMPETITIVE EQUILIBRIA

Up to now the computation of competitive equilibria has been phrased as the following problem:

(P1) Given some characteristics of an economy compute some equilibrium for this economy.

The computation of competitive equilibria amounts to solving a nonlinear system of interdependent equations. If demand functions are given, or can at least be explicity derived from maximization of the utility functions subject to the budget constraints, then one could try to solve the system of excess demand equations $\theta(q) = 0$. Explicit knowledge of the demand functions can however seldomly be expected. To circumvent this problem we work on the so-called extended system which is the system of equations given by the first order conditions for utility maximization and the market clearing conditions. For instance with Cobb-Douglas preferences, when markets are incomplete, in general, demand functions need to be obtained as a solution to a polynomial equation of degree equal to the number of states S.

The most successful algorithms to compute competitive equilibria are 'homotopy algorithms' (see Garcia and Zangwill (1981) for a survey). The idea of a homotopy algorithm is to solve (P1) along a continuous path in the characteristics' space, the so-called homotopy path, which starts with a simple economy for which a solution is obvious and which then gradually updates the characteristics of the simple initial economy towards the economy for which the competitive equilibrium is actually wanted. A standard approach is to start with a single agent economy and then to gradually introduce the other agents out of which the economy in question consists. For the single agent economy the unique competitive equilibrium is obviously the no-trade allocation of initial resources, which is then gradually updated along the homotopy path towards the competitive equilibrium of the heterogeneous agent economy in question.

The advantage of homotopy algorithms is that they are almost always able to compute some competitive equilibrium. Their disadvantage with respect to finding multiple equilibria is however that for any given set of characteristics they are bound to find at most one competitive equilibrium. And this equilibrium has a positive index since along a homotopy the indices of the equilibria do not change (cf. Mas-Colell (1985)) and the initial equilibrium in the economy obviously has a positive index. Likewise the global Newton method (cf. Smale (1976)) is bound to compute equilibria with positive index only.

There is one important exception to this remark, namely the all-solutions algorithm of Drexler (1978) and Garcia and Zangwill (1979, 1981) which is only known to work for sure in the case of polynomials. See Li (1997) for a recent survey.

Finally, note that a first 'quick and dirty' idea to use homotopy algorithms to compute multiple equilibria for a given economy, say with two agents, is to run the homotopy algorithm twice; once starting with the first agent being the single agent in the initial economy and once starting with the second agent. If we are lucky, doing this we can find two different homotopies leading to two different equilibria.

The disadvantage of all the existing algorithms is their point of view to compute solutions for any given set of the economy's characteristics. To find multiple equilibria within some class of economies like those with Cobb-Douglas preferences, it is necessary to do an exhaustive search in the space of the economy's characteristics and for any such data run the algorithm which itself might rely on an exhaustive search of starting points.

Instead we propose to proceed in the following way. Note that we are not merely interested in computing equilibria of some particular given economy. We are interested in whether within some class of economies there are some particular economies having multiple equilibria. Therefore, we turn the question (P1) upside down:

(P2) Given two prospective equilibria, find some economy (within some class of economies) such that these equilibria are competitive equilibria for the economy found.

We solve (P2) by a trial and error procedure, which mimics a homotopy algorithm:

1. For some given economy solve (P1):

   Given some initial parameters of the utility functions, $\rho(0)$, the endowments, $\omega(0)$, and the asset structure, $A(0)$, compute an equilibrium asset price system, $q(0)$, and an equilibrium allocation of assets, $\theta(0)$, and of consumption bundles, $x(0)$.

2. Take the solution of Step 1 as the starting point for the procedure of computing the economy's characteristics rationalizing two asset market equilibria:

   Double the system of equations describing a solution to Step 1, i.e. $(\rho(0), \omega(0), A(0))$ is the initial solution to the "reverse" procedure of computing characteristics rationalizing the two initial equilibria $(q_1(0), \theta_1(0), x_1(0)) = (q_2(0), \theta_2(0), x_2(0))$.

3. Consider a homotopy in which the data of the second equilibrium in Step 2 are changed slightly away from the first equilibrium:

   Fix some new candidate equilibrium $(\hat{q}, \hat{\theta}, \hat{x})$ and define the homotopy

   $$((q_2(\lambda), \theta_2(\lambda), x_2(\lambda)) = \lambda(q_2(0), \theta_2(0), x_2(0)) + (1-\lambda)(\hat{q}, \hat{\theta}, \hat{x}),$$

   leaving the first equilibrium unchanged.

4. Update the characteristics of the economy along the homotopy considered in Step 3:

   Increase λ in small steps form 0 to 1 solving the system of equations in terms of the economy's characteristics (ρ, ω, A).

Following the Steps 1–4 we were eventually able to find the first two equilibria in Example 4.2 given in the previous section.

From the numerical values displayed in Example 4.2 one can infer which data we have taken as given and which have been computed by our algorithm. We took as given the agents' utility functions, the first ten payoffs of the asset matrix, the equilibrium allocations in the last two states as well as the asset prices.

Since for Cobb-Douglas preferences excess demand cannot explicitly be computed when markets are incomplete we worked on the extended system, which given the notation introduced in Section 2 reads as follows.

For all $i = 1, \dots, I$ it is

$$\sum_{s=1}^{S} \frac{\rho_s^i A_s^j}{x_s^i} = \lambda^i q^j, \qquad (j = 1, \dots, J)$$

$$x_s^i = \omega_s^i + A_s \theta^i, \quad (s = 1, \dots, S)$$

$$q \cdot \theta^i = 0.$$

Besides markets are cleared, i.e.

$$\sum_{i=1}^{I} \theta^i = 0.$$

Here commodity market clearing has been omitted since it follows from asset market clearing. The budget identity $q \cdot \theta^i = 0$ is assumed since we normalized one asset price to be positive. To be sure, we prove the following lemma.

LEMMA 5.1: *Let U satisfy the Assumption U, then*

$$\overset{\star}{\theta} \in \arg \max_{\substack{x \in \mathbb{R}^S_+ \\ \theta \in \mathbb{R}^J}} U(x) \quad s.t. \quad q \cdot \theta \leq 0 \quad \wedge \quad x \leq \omega^i + A\theta$$

satisfies $q \cdot \overset{\star}{\theta} = 0$, *if* $q_j > 0$ *for some* $j \in 1, \ldots, J$.

PROOF: Suppose not, i.e. $q \cdot \overset{\star}{\theta} < 0$. Consider $\widehat{\theta} = \overset{\star}{\theta} + \varepsilon e_j$, where e_j is the j-th unit vector in $\mathbb{R}^J$ and $\varepsilon \in \mathbb{R}$. Then for some small but positive ε, $\widehat{\theta}$ satisfies $q \cdot \widehat{\theta} \leq 0$ and $\omega^i + A\widehat{\theta} \geq 0$. From the first order condition we know

$$\nabla U(\omega^i + A\overset{\star}{\theta})A^j = q^j > 0.$$

Hence

$$U(\omega^i + A\widehat{\theta}) > U(\omega^i + A\overset{\star}{\theta})$$

for some ε small enough which is a contradiction to the maximizing property of $\overset{\star}{\theta}$ (cf. Magill and Quinzii (1996, Proposition 31.2)). *Q.E.D.*

In solving the extended system we solve the nonlinear system of independent equations, given by the first order conditions, in those variables which are not restricted in sign, i.e. in the asset payoffs in the last two states and the portfolio holding of agent 1 (those of agent 2 being then determined from the asset market clearing). We choose initial endowments and consumption in the first three states such that the budget restrictions and the non-negativity constraints on consumption are satisfied.

For the details of our computation, a source code of our programm can be obtained on request. It can however not be expected to find an exact solution to the extended system. Computers work with some precision but not perfectly. The numerical values of our example are exact up to the following maximal errors. To check the precision use the following values for the marginal utility of income λ^i, $i = 1, 2$, in the three equilibria,

$$\bar{\lambda}^1 = 3.6, \qquad \bar{\lambda}^2 = 2.085,$$
$$\hat{\lambda}^1 = 3.445, \qquad \hat{\lambda}^2 = 1.955,$$
$$\tilde{\lambda}^1 = 5.4417381, \quad \tilde{\lambda}^2 = 8.19317872 \cdot 10^{-2}.$$

Plugging in these λ-values together with the other equilibrium values it is obtained that the maximum errors, F, in the extended system are

$$\bar{F} = 4.44 \cdot 10^{-15},$$
$$\hat{F} = 2.84 \cdot 10^{-14},$$
$$\tilde{F} = 1.4 \cdot 10^{-14}.$$

Moreover the Jacobian at the three equilibria have the following determinants, D,

$$\bar{D} = -1.62 \cdot 10^8,$$
$$\hat{D} = +1.0637 \cdot 10^8,$$
$$\tilde{D} = -6.55 \cdot 10^9.$$

Hence the second equilibrium has a negative index!

Finally, we point out that the third equilibrium has been found by solving the standard computational problem (P1) with the characteristics of the economy found by (P2) taken as the input data. Then, on starting with agent 1 being the representative agent in the initial economy from which the homotopy algorithm starts, the first equilibrium is obtained and starting with the second agent the third equilibrium is obtained! Hence even this 'quick and dirty' way of computing multiple equilibria does work in our example.

6. SUMMARY

The uniqueness of competitive equilibria is very important for a sound foundation of financial market models. We have outlined that uniqueness can be achieved by appropriate restrictions on the agents beliefs combined with assumptions on their risk aversions as well as on the market structure. The general rule is that incompleteness of markets requires even more severe restrictions than complete markets. In particular it was shown that the assumption of Cobb-Douglas preferences is no longer sufficient to obtain the gross substitution property and that under this assumption multiple equilibria are possible when markets are incomplete. The latter was obtained by a novel approach to compute multiple competitive equilibria.

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