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Recursive estimation in nonlinear dynamics: Application to the analysis of stage transition and chaos

Introduction

The recent spectacular progress in nonlinear dynamics has given rise to a paradigm shift in developmental psychology. Not only was some of the most important early work in nonlinear dynamics explicitly devoted to the analysis of developmental processes (e.g., Thom, 1975), but there also became available a new mathematical framework that enabled the unambiguous representation, and sometimes even the definitive solution, of old controversial issues in developmental theory (cf. Molenaar & Oppenheimer, 1985; Molenaar, 1986[a], 1986[b]). Perhaps the most important early contribution of nonlinear dynamics in this respect concerned the analysis of sudden qualitative transitions in the behavior of selforganizing systems (Thom, 1975; Nicolis & Prigogine, 1977; Haken, 1983). It was shown that such transitions (called bifurcations, catastrophes, or phase transitions) constitute the hallmark of selforganization in a way which closely resembles Piaget's epigenetic theory of stagewise cognitive development (Molenaar, 1986[b]). Thus, a principled mathematical theory of stage transitions became available that definitively settled a long controversy about the explanatory value of Piaget's theory (e.g., Brainerd, 1978). In retrospect this may not be such a surprizing coincidence, because Thom's (1975) catastrophe theory and Piaget's (1985) epigenetic theory were both inspired by Waddington's (1957) epigenetic evolutionary theory.

Our applied work within the new nonlinear dynamics paradigm of developmental psychology started with the elaboration of catastrophe models and methods for the analysis of stage transitions (van der Maas & Molenaar, 1992), including transitions occurring in stochastic developmental processes (Molenaar & Hartelman, 1996). Consecutively, this work was extended to the study of causal mechanisms underlying cognitive stage transitions, using artificial neural network models of information processing (e.g., Raijmakers, van der Maas, & Molenaar, 1996). Recently, another extension has been made in which specific dynamic models of phase transitions are directly fitted to observed time series (Molenaar & Raijmakers, 1997). The latter so-called recursive model fitting approach (to be defined below) provides new, powerful inductive tools that enable detailed micro-genetical analyses of nonlinear dynamical systems during transition.

The next step in this line of research concerning the analysis of stage transitions might involve an extension to chaotic processes. Chaos theory has already found interesting applications in various fields of psychology (e.g., Basar, 1990). Moreover, it turns out that the emergence of chaos in simple nonlinear models is associated with the occurrence of a cascade of bifurcations (Ott, 1993), while several distinct types of phase transitions are indicative of the emergence of specific chaotic regimes such as intermittency, crises, etc. (Beck & Schlögl, 1993; Ott, 1993). Given the importance of a small set of prototypical models in chaos theory (logistic map, Ikeda map, etc.), it would seem worthwhile to extend the recursive model fitting approach to the micro-genetical analysis of these chaotic maps during phase transitions. Before such an extension can be carried out, however, we will first have to deal with an entirely new phenomenon associated with chaotic systems, namely their pathological properties with respect to model fitting if the observed output is corrupted by measurement noise. This will be the topic of the remainder of this paper.

Statement of the problem

Suppose one wants to estimate the fractal dimension of the invariant set of a chaotic map on the basis of an observed time series which is corrupted by additive measurement noise. Of course, the fractal dimension in which one is interested pertains to the true (noise-free) trajectory of the chaotic map. From a traditional signal analytical point of view, the presence of additive measurement noise would not seem to present any special problems in the reconstruction of the true trajectory of the map underlying the observed time series. However, it was shown by Casdagli, Eubank, Farmer, & Gibson (1991) that the presence of even a negligible amount of measurement noise puts disproportionally severe limits on the fidelity with which the fractal dimension of the underlying strange attractor can be determined: 'It is now a well-known fact that chaos limits long-term predictability. We have shown that *when projected into lower dimensions, chaos may also impose limits to short term predicatbility*. For a dynamical system whose dimension and leading Lyapunov coefficient are sufficiently large, projection onto a low dimensional time series causes an explosion in the noise amplification. As a result, it is impossible to reconstruct localized states from measurements of any reasonable precision. The time series is unpredictable for times much less than the Lyapunov time and it becomes indistinguishable from one generated by a random process. This is true even when the dynamical system is known. Note that this is *not* true for nonchaotic systems ...' (Casdagli et al., 1991, p. 96. Italics in the original).

Presently, we will consider whether this problem in reconstructing localized states from noisy series also generalizes to the estimation of parameters in an underlying chaotic map. To ease the comparison somewhat, the nonlinear Ikeda map which figures prominently in the examples given by Casdagli et al. (1991) will also be used in the simulation study presented below.

Recursive estimation

Parameter estimation in the nonlinear Ikeda map corrupted by additive measurement noise will be carried out by means of a recursive technique: the so-called extended Kalman filter (EKF). First we will give a concise introduction to the EKF in general terms, then its application to the Ikeda map will be detailed. For a more detailed description of the EKF, the reader is referred to Molenaar & Raijmakers (1997) and Molenaar (1994).

At a general level, recursive parameter estimation can be defined for a nonlinear state-space model represented by

$$\mathbf{x}(t+1) = \mathbf{f}_{t+1}[\mathbf{x}(t)] + \mathbf{w}(t)$$

$$\mathbf{y}(t) = \mathbf{h}_t[\mathbf{x}(t)] + \mathbf{v}(t)$$

where:

- $\mathbf{x}(t)$ is a latent q-variate state process
- $\mathbf{y}(t)$ is a p-variate manifest process
- $\mathbf{f}_{t+1}[.]$ and $\mathbf{h}_t[.]$ are nonlinear vector-valued time-varying functions that are sufficiently smooth (cf. Sage & Melsa, 1971, p. 93, footnote, for an exact specification of the smoothness conditions)
- $\mathbf{w}(t)$ is a q-variate white noise innovations process; $\mathrm{cov}[\mathbf{w}(t), \mathbf{w}(t+t')] = \delta(t')\mathbf{W}_t$, where $\delta(t') = 1$ if $t' = 0$ and $\delta(t') = 0$ otherwise
- $\mathbf{v}(t)$ is a p-variate white noise measurement error process; $\mathrm{cov}[\mathbf{v}(t), \mathbf{v}(t+t')] = \delta(t')\mathbf{V}_t$

It is noted that the set of unkown (possibly time-varying) parameters, denoted by the vector-valued process $\boldsymbol{\theta}(t)$, say, constitute part of the state $\mathbf{x}(t)$. Hence (recursive) estimates of $\mathbf{x}(t)$ include estimates of the parameters in $\boldsymbol{\theta}(t)$.

The recursive estimator of the state $\mathbf{x}(t)$, based upon observations $\mathbf{y}(t)$ up to time t, is denoted by $\mathbf{x}(t \mid t)$. The recursive nature of the estimator is apparent from the following schematic repreentation:

$$\mathbf{x}(t+1 \mid t+1) = \mathbf{f}_{t+1}[\mathbf{x}(t+1 \mid t)] + \mathbf{g}_{t+1}$$

where the q-variate vector $\mathbf{g}_{t+1}$ denotes the information gain due to the availability of $\mathbf{y}(t+1)$. Given suitable starting values $\mathbf{x}(0 \mid 0)$, this recursive scheme yields time-dependent estimates $\mathbf{x}(t \mid t)$, including $\boldsymbol{\theta}(t \mid t)$.

Recursive estimation in the Ikeda map

The Ikeda map, which is used extensively in the examples given by Casdagli et al. (1991), is a bivariate real-valued map defined as

$$x_1(t+1) = 1 + a[x_1(t)\cos(\phi_t) - x_2(t)\sin(\phi_t)]$$

$$x_2(t+1) = a[x_1(t)\sin(\phi_t) + x_2(t)\cos(\phi_t)]$$

where $\phi_t = b - 6/[1 + x_1(t)^2 + x_2(t)^2]$. Taking for instance a = .7 and b = .4, the trajectories generated by this map are chaotic.

The Ikeda map can be simply rewritten as a special instance of the nonlinear state-space model introduced in the previous section. Letting $x_3(t) = a$ and $x_4(t) = b$, we obtain:

$$x_1(t+1) = 1 + x_3(t)[x_1(t)\cos(\phi_t) - x_2(t)\sin(\phi_t)]$$

$$x_2(t+1) = x_3(t)[x_1(t)\sin(\phi_t) + x_2(t)\cos(\phi_t)]$$

$$x_3(t+1) = x_3(t)$$

$$x_4(t+1) = x_4(t)$$

where $\phi_t = x_4(t) - 6/[1 + x_1(t)^2 + x_2(t)^2]$. Notice that there is no innovations process $\mathbf{w}(t)$ influencing the latent state process $\mathbf{x}(t)$.

The relationship between $\mathbf{x}(t)$ and the univariate manifest process y(t) is given by

$$y(t) = x_1(t) + v(t)$$

where v(t) is taken to be univariate Gaussian white measurement noise with variance V. Clearly, the 2-dimensional Ikeda map is projected onto a lower dimensional (univariate) time series, in accordance with the Casdagli et al. (1991) scenario. The recursive estimate $\mathbf{x}(t \mid t)$ includes the time-dependent estimates $x_3(t \mid t)$ and $x_4(t \mid t)$ of a and b, respectively.

Simulation experiment

Using the state-space representation of the Ikeda map as specified in the previous section, noise corrupted y(t) series, t=1,2,...,1000, were generated using the following parameter values: a = .7, b = .4, V = 0, .001, .01 or .1. In what follows, only the case in which the measurement noise variance V = .01 will be considered, because this yields a signal-to-noise ratio of the same order as for which Casdagli et al. (1991) found pathological reconstruction results. The signal-to-noise ratio is defined as SNR = $\mathrm{var}[x_1(t)]/\mathrm{var}[y(t)]$. For V = .01, SNR = .974.

Figure 1A shows the recursive estimate of a (the true value of which is a = .7). Figure 1B shows the recursive estimate of b (the true value of which is b = .4). It appears that the recursive estimates quickly stabilize within a 95% confidence interval about their true values. More specifically, the standard errors of a and b at t = 1000 are .00852 and .01235, respectively. The one-step-ahead prodeiction error variance is .03087.

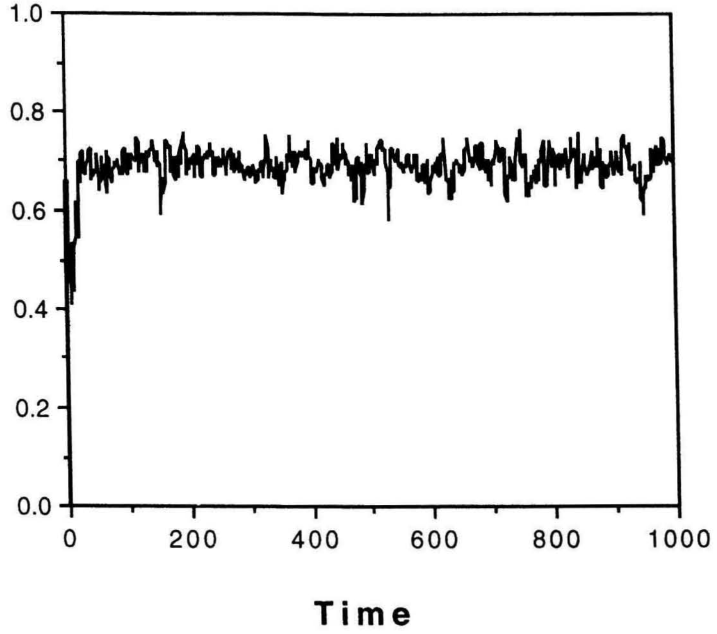

Fig. 1A. Recursive estimate of a in Ikeda map a-true = .7; V = .01; SNR = .974.

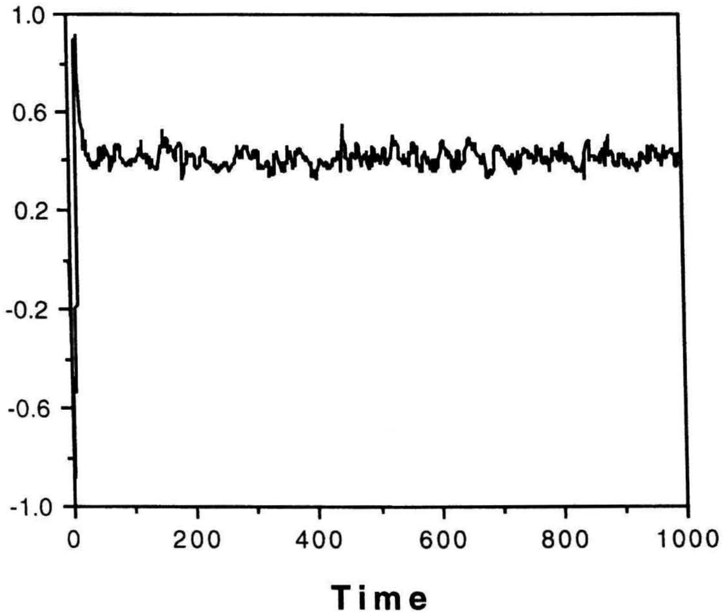

Fig. 1B. Recursive estimate of b in Ikeda map b-true = .4; V = .01; SNR = .974.

In closing this section, we present a result of applying the recursive estimator (EKF) to an instance of the Ikeda map with time-varying coefficient a(t):

$$a(t) = a(t-1) + .7/1000; \; t=1,\ldots,1000; \; a(0)=0$$

As before, b is constant (b=.4), and y(t), t=1,...,1000, is in all other respects generated like before. In particular, V=.01.

Figure 2 presents the recursive estimate of $x_3(t) = a(t)$. It is seen that the EKF nicely recovers the time-varying trajectory of a(t), which starts in a region where the Ikeda map is not chaotic and after some time enters a chaotic region (including the terminal value a(1000) = .7).

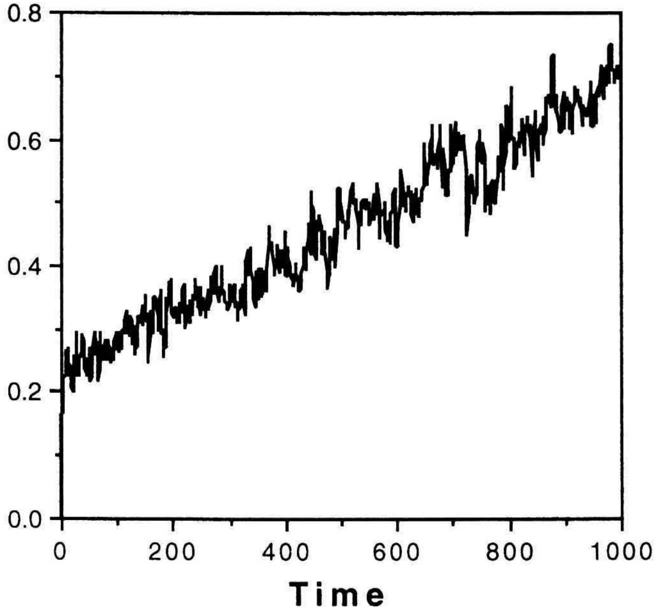

Fig. 2. Recursive estimate of a(t) in Ikeda map a(t) = a(t-1) + .7/1000; a(0) = 0 V = .01; SNR = .996.

Conclusion

Perhaps the main conclusion is that the Casdagli et al. (1991) results concerning the impossibility of reconstructing localized states in the presence of additive measurement noise do not seem to carry over to parameter estimation in the Ikeda map under the same noise conditions as considered by Casdagli et al. That is, the obtained recursive EKF parameter estimates do not appear to behave pathological and their approximate 95% confidence intervals contain the true values during almost all times. Yet this result should be considered as preliminary, requiring much further elaboration in almost all respects. To arrive at more definitive conclusions, large-scale Monte Carlo studies are required using alternative, high-dimensional state evolutions. Also our recursive estimation technique should be embedded in an expectation-maximization (EM) algorithm which will increase the fidelity of the parameter estimates thus obtained.

To further underpin our cautionary remarks concerning the strictly preliminary status of the results reported in this paper, we note some phenomena that would indeed seem to qualify our main conclusion in some important respects. To begin with, according to standard signal analytical principles one would expect the recursive estimate of a constant parameter to become constant itself as time increases. More specifically, the recursive estimates of the constant parameters a and b in the Ikeda map should converge to stable values as time increases. In fact, this convergence to stable estimated values should be quite fast, given the presence of only a small amount of measurement noise. Yet Figures 1A and 1B appear to tell a different story: the variation of the recursive estimates of a and b does not seem to wane as time proceeds. Clearly, this apparent lack of consistency requires further scrutiny before more definite conclusions can be drawn.

Another noteworthy phenomenon is that the so-called extended Kalman smoother does not seem to work for parameter estimation in the Ikeda map. The extended Kalman smoother can be interpreted as the application of the EKF forwards in time (as reported in this paper), followed by an another application backwards in time. It can be proved that this will yield optimal recursive estimates. Yet, for reasons which are presently unknown, only the extended Kalman filter (EKF) works. This is all the more surprizing because the Ikeda map has an explicit inverse. These and similar phenomena observed in the present simulation study indicate that the obtained results should be interpreted with great caution.

In closing, we would like to reiterate the importance of applications of recursive filtering techniques in the micro-genetical analysis of chaotic maps during phase transitions. Such applications can provide detailed empirical evidence of the existence of new varieties of selforganization in developmental processes. As alluded to erlier, this constitutes the next logical step in our ongoing work on statistical modeling of stage transitions in epigenetical processes. The approach presented in this paper addresses an important preliminary issue in this endeavour, that also in itself would seem to have profound and interesting implications for applied nonlinear dynamics.

References

Basar, E. (Ed.) (1990). *Chaos in brain function.* Berlin: Springer-Verlag.

Beck, C. and F. Schlogl (1993). *Thermodynamics of chaotic systems.* Cambridge: Cambridge University Press.

Brainerd, C.J. (1978). The stage question in cognitive-developmental theory. *Behavioral and Brain Sciences*, **2**, 173-213.

Casdagli, M., S. Eubank, J.D. Farmer and J. Gibson (1991). State space reconstruction in the presence of noise. In: D.K. Campbell, R.E. Ecke and J.M. Hyman (Eds.), *Nonlinear science: The next decade*, (pp. 52-98). Amsterdam: Elsevier.

Haken, H. (1983). *Advanced synergetics.* Berlin: Springer-Verlag.

Molenaar, P.C.M. (1986[a]). Issues with a rule-sampling theory of conservation learning from a structuralist point of view. *Human Development*, **29**, 137-144.
Molenaar, P.C.M. (1986[b]). On the impossibility of acquiring more powerful structures: A neglected alternative. *Human Development*, **29**, 245-251.
Molenaar, P.C.M. (1994). Dynamic latent variable models in developmental psychology. In A. von Eye and C.C. Clogg (Eds.), *Latent variables analysis: Applications for developmental research*, (pp. 155-180). Thousand Oaks: Sage.
Molenaar, P.C.M. and L. Oppenheimer (1985). Dynamic models of development and the mechanistic-organismic controversy. *New Ideas in Psychology*, **3**, 233-242.
Molenaar, P.C.M. and P. Hartelman (1996). Catastrophe theory of stage transitions in metrical and discrete stochastic systems. In A. von Eye and C.C. Clogg (Eds.), *Categorical variables in developmental research: Methods of analysis*, (pp. 107-130). San Diego: Academic Press.
Molenaar, P.C.M. and M.E.J. Raijmakers (1997). Fitting nonlinear dynamical models directly to observed time series. In K.M. Newell and P.C.M. Molenaar (Eds.), *Applications of nonlinear dynamics to developmental process modeling*. Hillsdale, NJ: Lawrence Erlbaum (to appear).
Nicolis, G. and I. Prigogine (1977). *Self-organization in nonequilibrium systems: From dissipative structures to order through fluctuations*. New York: Wiley.
Ott, E. (1993). *Chaos in dynamical systems*. Cambridge: Cambridge University Press.
Piaget, J. (1985). *The equilibration of cognitive structures*. Chicago: The University of Chicago Press.
Raijmakers, M.E.J., Van Der Maas, H.L.J. and Molenaar, P.C.M. (1996). Numerical bifurcation analysis of distance-dependent on-center off-surround shunting neural networks. *Biological Cybernetics*, **75**, 495-507.
Sage, A.P. and J.L. Melsa (1971). *Estimation theory with applications to communications and control*. New York: Academic Press.
Thom, R. (1975). *Structural stability and morphogenesis*. Reading: Benjamin.
Van der Maas, H.L.J. and P.C.M. Molenaar (1992). Stagewise cognitive development: An application of catastrophe theory. *Psychological Review*, **99**, 395-417.
Waddington, C.H. (1957). *The strategy of the genes: A discussion of aspects of theoretical biology*. London: Allen and Unwin.

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