MODELLING PRODUCER DECISIONS IN A SPATIAL CONTINUUM[1]

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The paper describes how stochastic optimization techniques can be used to model profit maximizing producer behaviour in a spatial continuum. The main methodological issues to be addressed are, first, that the representation of optimal allocations in a spatial continuum naturally leads to models that contain integrals over space, and secondly, that the resulting model tends to have a multi-level structure, i.e. requires solving nested optimization problems because it should combine the profit maximization by individual producers with market clearing at regional level. As an illustration, we specify four regional models. The first determines the optimal output level for factories that emit pollutants which reduce the crop output of neighbouring farmers. The main issue is to compute the associated level of compensation to be paid by the factories to the farmers. The second model deals with optimal zoning. It computes the optimal crop routing for farmers who can choose to sell their crop to factories situated at given locations. This is an optimization problem in functional space, which can be reformulated as a dual stochastic optimization problem. In the third model, the farmer has the possibility of routing his crop along different roads or distribution nodes to the various factories for processing. It can describe the optimal choice of distribution centres at given locations, around plants or cities, and produces optimal boundaries for the zones that supply to or buy from these centres. The fourth model deals with the problem optimal land consolidation, and distinguishes between consolidation processes with and without side-payments. To each of these four models we associate a stochastic quasi-gradient (SQG-)procedure for attaining the (global) optimum, which has a natural interpretation as a learning device for decentralized adaptive planning.

KEYWORDS: Stochastic quasi-gradient, spatial optimization, pollution control, land consolidation.

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1. INTRODUCTION

The growing concerns about the environmental and social sustainability of land-utilization patterns have led to a revival of interest in problems of optimal spatial allocation. With respect to the environmental aspects, questions include at the lower geographical scales, issues such as the optimal siting of intensive livestock industry in relation to the surrounding crop farms in the neighbourhood, and at a higher scale the impact of afforestation on temperature and rainfall. As regards social sustainability, one could, among many other, mention the questions how to organise squatter settlements around fast growing towns, and at village level, how

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to deal with the ongoing process of land fragmentation, whereby farmers sometimes have to cultivate isolated parcels of a few square meter in surface. Spatial aspects also play an essential role in the design of new transport infrastructure.

So far, a wealth of geographical data has been collected to deal with such problems, on climate, soils, settlement patterns and land cover. Part of these data originate from remote sensing, and have mainly been incorporated in Geographical Information Systems, which can produce highly detailed maps but contain only few and very simple decision support tools. As a rule, the geographically explicit tools either describe a separate optimal decision at every point on the map without accounting for spatial interdependencies, or limit themselves to optimization over a relatively small number of regional units, at the expense of geographical detail. The aim of this paper is to show that stochastic optimization techniques allow to represent spatial interdependencies while maintaining geographical detail.

The subject of optimization over a spatial continuum has been studied in location theory, in connection with the problem of optimal facility location. This problem usually amounts to finding the geographical location of an industrial facility that minimizes the cost of transporting goods to that location from a surrounding region or vice versa. The early location models are classical transportation models that only select the best out of a finite number of alternatives and treat the region as a finite number of fields identified by their barycentre (see e.g. Beck and Goodin (1982), for an application to dairy farms). Subsequent location/pricing models treat the site as a continuous choice variable and simultaneously calculate the consumer price at every point in the region on the basis of the distance from the facility (Hansen et al. (1987) and Drezner (1995) for a survey).

However, the question of optimal land use requires a broader treatment. It calls for an explicit spatial representation of land use itself, with surfaces being allocated to competing uses so as to maximize, say, the total revenue in the region, as opposed to the cost minimization of the location/pricing models. Consequently, from a modeling point of view new methodological challenges must be addressed. The first issue is that the representation of optimal allocations in a spatial continuum naturally leads to models that contain integrals over space, which in general cannot be eliminated analytically, in view of the variability in the underlying GIS-information. The second issue is that the resulting model, which combines optimal routing and profit maximizing decisions by individual producers, with market clearing at regional level, tends to have a multi-level structure, i.e. requires solving nested optimization problems.

The aim of this paper is to show that stochastic optimization techniques can be applied to address both issues. We specify four regional models in which profit maximizing producers are operating within a spatial continuum and compete on land and commodity markets. These producers are supposed to face given market prices when trading with the rest of the economy and the only endogenous prices to be considered are those prevailing in trade among producers, this in order to avoid the intricacies of price adjustment in a full general equilibrium setting. To every model we associate a decentralised, stochastic quasi-gradient (SQG-)procedure for attaining the (global) optimum (see Ermoliev (1988)). Since in this paper we are more interested in such algorithms as devices for decentralized adaptive learning than as efficient tools for computations, we only refer in passing to operational issues such as the computationally efficient choice of step-size.

The paper proceeds as follows. In Section 2 we describe by means of simple examples how stochastic optimization models can be used for planning problems in spatial continuum. Next, we deal with applications of increasing complexity. First, in Section 3, we study a model that determines the optimal output level for factories that emit pollutants which spread over the neighbouring environment, and reduce the crop output of local farmers. The main issue is to compute the associated level of compensation to be paid by the factories to the farmers. The technical difficulty is to deal with an integral in the objective of a convex program. Our second problem (Section 4) deals with optimal zoning. It computes the optimal crop routing for farmers who can choose to sell their crop to factories situated at given locations. We indicate that this zoning aspect generates an optimization problem in functional space, which can be reformulated as a dual stochastic optimization problem. The model of Section 5 extends this approach to include optimal routing: the crop can be sent to the various factories along different roads or distribution nodes. As this enables us to account for the cost of spreading over dispersed fields, we can use this model as a planning tool for optimal land consolidation and describe how the solution path generated by the SQG-algorithm can be interpreted as a sequence of land transactions with side-payments, that eventually converges to the optimum. However, in many practical situations it will be unrealistic to suppose that such side payments can be mobilized, because individual farmers are often reluctant to participate in land transactions if their farm becomes less profitable in the process. To deal with this case, we formulate an alternative model that maximizes the revenue without side payments of the least favoured. Section 6 concludes.

2. SOLVING SPATIAL PLANNING PROBLEMS BY STOCHASTIC OPTIMIZATION

To illustrate the difficulties that arise when modeling the decisions in a spatial continuum, let us start with two simple examples. The calculation of the barycentre of a given geographic region is a well known case. One might think of the possibly dispersed farmland whose "centre" has to be determined, say, in order to serve as the collection point for the harvest. Let X denote the region and $A \subseteq X$ the (not necessarily connected) land of a given farmer. The barycentre of A can now be calculated by minimizing

$$F(h) = \int_A \|x - h\|^2 dx, \tag{1}$$

where $\|\cdot\|$ is the Euclidean distance, which is equivalent to the stochastic optimization problem of minimizing the expectation function $F(h) = \int_A \|x-h\|^2 dG(x)$, where $G(x) = \frac{x}{\int_A dx}$ is a probability measure. This is an important point, as it connects the spatial planning problem to stochastic optimization.

The solution[2] $h^* = \frac{\int_A x dx}{\int_A dx}$ can be computed by discretization, numerical integration, either directly, or via sequential Monte Carlo simulation. Discretization is common in GIS-packages. It approximates (1) by a grid of N points x^j, and computes the barycentre as the average coordinate, $h = \frac{\sum_j x^j}{N}$. Sequential Monte Carlo simulation proceeds as follows. Sample at iteration $t = 1, 2, \dots$ a point $x(t)$ at random and independently, such that Prob.$[x(t) \in A] = \frac{\int_A x dx}{\int_A dx}$, and define the approximate solutions

$$h(t+1) = h(t) - \rho_t \left(h(t) - x(t)\right), \quad t = 1, 2, \dots \tag{2}$$

Notice that (2) is an SQG-algorithm, since $2\left(h(t) - x(t)\right)$ is the gradient of the random (sample) function under the sign of the integral in (1). If the step-size ρ_t satisfies:

$$\rho_t \geq 0, \ \sum_{t=1}^{\infty} \rho_t = \infty, \ \sum_{t=1}^{\infty} \rho_t^2 < \infty, \tag{3}$$

then the sequence $h(t)$ will converge to h^*, with probability 1. This requirement will be met for $\rho_t = \frac{1}{t}$, which corresponds to the sample mean calculation:

$$h(t) = \frac{1}{\sum_{i=1}^{t} x(i)}. \tag{4}$$

The advantage of (2) as opposed to (4) is that it does not require storing a large bundle of information, since in (2) the estimation of h^* proceeds sequentially, with a simple updating rule after each new observation. Procedure (2) is actually an SQG-algorithm for minimizing the integral (1).

Clearly, in an economic context the squared Euclidean distance will seldom be the relevant concept, as transport cost minimization would call for, say, minimization of average distance itself rather than average squared distance. Moreover, problem (2) allows for infinitely many routes from point x to the variable "home" h^*. It seems more realistic to treat this home or market outlet as fixed (let this be point b for base) and to let the farmer choose the shortest (or cheapest) route, by searching for collection points h^i, $i = 1, \dots, r$.

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[2]Calculation is coordinate-wise: $h_1 = \frac{\int_A (x_1, 0) dx}{\int_A (1,0) dx}$, $h_2 = \frac{\int_A (0, x_2) dx}{\int_A (0,1) dx}$.

The associated zoning pattern is defined by a function $i(x)$ which indicates that point i attracts x: $i(x) = \mathrm{argmin}_i \|x - h^i\| + \|h^i - b\|$. This function can be shown in a GIS zone-map. Other "statistics" could include, for example, the total cost

$$z_0 = \int_A \min_i \left[\|x - h^i\| + \|h^i - b\|\right] dG(x), \tag{5}$$

where $G(x)$ might reflect the demand for transport (i.e. the crop output at point x) and the traffic on the various roads:

$$z_i = \int_A \delta_i(x) dG(x), \quad i = 1, \ldots, r,$$

where $\delta_i(x) = 1$ if $i(x) = i$ and 0 otherwise. These values z_0, $z_1, \ldots, z_r$ can be estimated by discretization but also as minimizers of

$$F(z) = \left(z_0 - \int_A \min_i \left[\|x - h^i\| + \|h^i - b\|\right] dG(x)\right)^2 + \sum_i \left(z_i - \int_A \delta_i(x) dG(x)\right)^2. \tag{6}$$

The associated SQG-procedure, which is similar to (2), will sequentially process a large amount of information and with probability 1 yield a consistent estimate of z. The examples discussed can all be dealt with relatively easily by discretization methods, because their first-order optimality conditions yield an explicit solution for the unknown parameters q. We now turn to models that do not possess this special property.

3. OPTIMAL LAND USE IN THE PRESENCE OF POLLUTION

We consider industrial plants whose pollution negatively affects neighbouring land users. If the emissions consisted only of pollutants such as CO_2-gases that tend to dissipate quickly into the atmosphere, the analysis could focus on reduction of aggregate emissions, and there would be no need for a locational study. However, most emissions have a definite local component whose cumulative effects depend on location specific factors such as soil type, hill slope, crop coverage and climatological factors.

The spreading of emissions of a given pollutant is naturally represented via two-dimensional density functions. These are defined over the geographical region under consideration and measure the incidence (fraction) of pollutant emitted by a factory that is located at given site and generates depositions at every point in the region. The factory might be a chemical plant or an intensive livestock farm. The decision problem is to confront every factory with the land users in its neighbourhood (actually in its environment) who suffer from the emissions, i.e. to maximize total income or welfare of the region while internalizing the environmental effects, and this naturally leads to optimization over a spatial continuum. Polluters will pay compensation to the land users. This will affect their profitability, and size of operation at every site, as well as the revenues and cropping patterns of the land users around them.

3.1. *Model Formulation*

The model is specified as follows. We denote the given geographical space of region by the set $X \subset \mathbf{R}^2_+$ and consider S fixed factory sites indexed s, located at $b^s \in X$. A factory at site s makes use of an emission permit for a quantity q^s_k of pollutant k, for $k = 1, \ldots, K$. This use of permits is taken to increase its revenue $R^s(q^s)$ from the factory like an ordinary input; formally:

Assumption R (Revenue function of factory at fixed site s): For every site s, the revenue function $R^s\colon \mathbf{R}^K_+ \to \mathbf{R}$, $R^s(q^s)$ is increasing, continuously differentiable with respect to q^s, it has uniformly bounded derivative, and is strictly concave in the input q^s.

The emissions are dispersed around the factory by various physical processes such as winds and groundwater flows. Let the density function $\psi^s_k(x)$, defined over X describe the dispersal of emissions according to $c_k(x) = \sum_s q^s_k \psi^s_k(x)$, the incidence of pollutant k at location x, where x is a two- or three-dimensional vector. This reduces environmental quality at point x, leaving less natural inputs $h(x) = \omega(x) - c(x)$ for crops, where ω is the given resource availability, and affecting the revenue from crop farming at location x. The local revenue function $r(h, x)$ satisfies:

Assumption r (revenue function of farm at point x): the revenue function $r\colon \mathbf{R}^K \times X \to \mathbf{R}$, $r(h, x)$ is integrable in x and, almost everywhere in X, continuously differentiable w.r.t. the input h; it has uniformly bounded derivative, and is strictly concave in h.

In Assumptions R and r, the requirement of continuous differentiability can be relaxed, but it will be seen to offer the advantage of ensuring uniqueness of the market clearing prices. Concavity implies that the damage becomes more serious at higher pollution intensities. The converse assumption of increasing returns in pollution would open up the possibility of polluting without further damage an already fully spoiled area, but this would introduce non-convexity. Yet the formulation allows to promote the factories that are surrounded by land (or water) whose revenue is relatively insensitive to pollution.

Concavity is taken to be strict to keep the optimal q^* unique. Finally, in Assumption r, integrability in x is obviously necessary for integral calculations, while uniform boundedness of the derivative to q is a requirement of the SQG-procedure itself, which could be relaxed but is maintained for simplicity. Below we will comment on the qualification "almost everywhere."

The problem is now to define the optimal production levels, while maximizing the revenue in the region and to determine the compensatory payments of the factories to the other users of environmental resources. The revenue maximizing regional model is:

$$\max \sum_s R^s(q^s) + \int r(h(x), x)dG(x)$$

$$q^s \geq 0, \text{ all } s, \ h(x), \text{ all } x, \quad \text{subject to} \tag{7}$$

$$h_k(x) + \sum_s q_k^s \psi_k^s(x) \leq \omega_k(x), \quad \text{for } k = 1, \ldots, K, \text{ all } x,$$

where $G(x)$ is a distribution on the domain X, and the symbols under the maximization denote choice variables. This is a convex program, since the objective is concave and the constraint set linear. It has an infinite number of constraints. Let us briefly characterize the solution of this problem. We assume that $h(q, x)$ is everywhere on X positive in the optimum. Since $R^s(q^s)$ is increasing in q^s and the density is nonnegative, the constraint holds with an equality and we can substitute $h_k(q, x) = \omega_k(x) - \sum_s q_k^s \psi_k^s(x)$, which leads to the unconstrained problem:

$$\max_{q^s \geq 0, \forall s} \left\{ \sum_s R^s(q^s) + \int r(h(q, x), x)dG(x) \right\} \tag{8}$$

and yields the same optimal solution as the decomposed problems:

$$\max_{q^s} \left\{ R^s(q^s) - p^s q^s \right\}, \text{ for every } s \tag{9}$$

and

$$\max_h \left\{ r(h, x) - p(x)h \right\}, \text{ for all } x, \tag{10}$$

where $p(x)$ clears the balance

$$h_k(x) + \sum_s q_k^s \psi_k^s(x) = \omega_k(x), \tag{11}$$

and clearing prices satisfy

$$p_k^s = \int p_k(x)\psi_k^s(x)dG(x), \tag{12}$$

and

$$p_k(x) = \frac{\partial r(h,x)}{\partial h_k}. \tag{13}$$

Hence, the compensation payments satisfy:

PROPOSITION 3.1 (COMPENSATION PAYMENTS BY POLLUTERS): *Under Assumptions R and r, the regional revenue maximization (7) implies that:*

(1) *every factory s pays a total transfer $\sum_k p_k^s q_k^s$ to farmers;*
(2) *every spot x receives $\sum_s \sum_k p_k(x)\psi_k^s(x)q_k^s$.*

PROOF: Follows directly from (12, 13) and the first-order optimality conditions:

$$\frac{\partial R^s(q^s)}{\partial q_k^s} q_k^s = \int \frac{\partial r(h(q,x),x)}{\partial h_k}\psi_k^s(x)dG(x)\; q_k^s.$$

Q.E.D.

This proposition implies that the receipts of a crop farmer from factory s will increase with (i) the emission q_k^s, (ii) the dispersion density $\psi_k^s(x)$ to location x, and (iii) the marginal damage $\frac{\partial r(h,x)}{\partial h_k}$ at location x. To put it differently, farmer x sells his resource endowment $\omega_k(x)$ to himself as $h_k(q,x)$, and to the factory as $\psi_k^s(x)q_k^s$, at the same price $p_k(x)$. Conversely, factory s buys endowments at prices p_k^s and will have to restrict its emissions (and possibly even close down) if its pollution dissipates to locations x where the damage is important, either because the location itself is vulnerable or because other factories can pollute it less harmfully (i.e. can obtain more revenue from a marginal unit of pollution q_k^s). Therefore, this can be interpreted as a location model, even though location b^s is fixed.

3.2. *Solution Procedure*

Problem (7) can be solved by an SQG-process. Define the function

$$f(q,x) = \sum_s R^s(q^s) + r(h(q,x),x), \tag{14}$$

where $h_k(q,x) = \omega_k(x) - \sum_s q_k^s\psi_k^s(x)$, with gradient

$$\frac{\partial f(q,x)}{\partial q_k^s} = \frac{\partial R^s(q^s)}{\partial q_k^s} - \frac{\partial r(h,x)}{\partial h_k}\psi_k^s(x). \tag{15}$$

The spatial optimization problem (7) is now to maximize

$$F(q) = \int f(q,x)dG(x), \tag{16}$$

on the compact convex set Q, which is taken to be specified as $Q = \{q \mid 0 \leq q \leq \overline{q}\}$. The SQG-algorithm considers a sequence of random drawings $x(t)$ from the set X and starting from a given $q(1) = q^1 \in Q$, adjusts $q(t)$ according to:

$$q(t+1) = \Pi_Q \left(q(t) + \rho_t \frac{\partial f(q(t), x(t))}{\partial q} \right), \quad t = 1, 2, \ldots, \tag{17}$$

where Π_Q is the projection operator on Q (i.e. the point in Q nearest to the projected point), and the scalar step-size ρ_t converges to zero according to some appropriate step-size rule. From the general results on the convergence of SQG-methods it follows that if $f(q,x)$ is differentiable and concave in q, almost everywhere on G, then process (17) converges, with probability 1, to the optimal solution of (16), for step-size ρ_t chosen as in (3). The representation in a continuum has special attraction for spatial problems because, once an optimal value q^* has been estimated, GIS-tools allow to produce, say, an "altitude" map of the farm revenues $r(h(q^*,x),x)$. Furthermore, once problem (16) has been solved, the results can be compared in a spatially explicit manner with those that do not allow for compensation where the permits have zero price and the factories maximize $R^s(q^s)$, while the land-users are confronted with given pollution levels.

As an alternative to the SQG-method, one might consider using deterministic techniques after having approximated the function $F(q)$ by its sample mean:

$$F^N(q) = \frac{1}{N} \sum_{i=1}^{N} f(q, x(i)), \tag{18}$$

where the values $x(i)$ are independent samples from $G(x)$. For most of the models discussed in this paper, however, the value N would have to be extremely large, and this would require a large number of terms $f(q, x(i))$, which might, moreover, be highly non-smooth and not available explicitly. For the SQG-procedure (17), the convergence principle can be understood as follows. Since the direction $\frac{\partial f(q(t),x(t))}{\partial q}$ coincides "on average" with the gradient $\frac{F(q)}{\partial q}$, i.e.

$$\int \frac{\partial f(q(t), x(t))}{\partial q} dx = \frac{\partial \int [f(q(t), x) dx]}{\partial q}, \tag{19}$$

on average the value $F(q)$ increases from one iteration to the next, and for a step-size moving to zero at appropriate speed, the sequence $\{q(t)\}$ generated by adjustment rule (15) will converge to the optimal solution. In fact this rule can be viewed as a stochastic decentralisation procedure. It has an "evolutionary" interpretation as a decentralised learning process. Once a point $x(t)$ has been selected by at random within the region (the mutation), the vector $q(t) \in Q$ gradually changes its composition (the selection) so as to improve performance (survival). Whereas a deterministic gradient method would use the gradient $\int \frac{\partial f(q(t),x)}{\partial q} dG(x)$ and thus assume the distribution $G(x)$ to be known to the planner, this SQG-adjustment only improves fitness from the perspective of the randomly chosen point $x(t)$, without use of any information on the distribution or on the value F at any other point x. Thus, the process is perfectly decentralised and derives its coordination (optimality of allocations) from the infinite repetition of events and a relatively "naive" adjustment to current pressure as expressed through the quasi-gradient. And, while the single point has a negligible influence on the final outcome, the optimal solution is eventually determined by the shape of the distribution $G(x)$, jointly with the function $f(q,x)$ and the set Q.

Let us briefly return to the qualification, "almost everywhere" in conjunction with the "integrability in x" of Assumption r. This is a significant relaxation of the usual demands that differentiability and concavity properties should apply everywhere on the domain. It is important in a spatial context, as it allows for the representation of natural discontinuities due to, say, rivers or roads. Technically, the role of this qualification is expressed in the following simple lemma:

LEMMA 3.2 (MEAN VALUE): *Consider the distribution $G(x)$ and the function $f: Q \times X \to \mathbf{R}$, $f(q,x)$, which is integrable w.r.t. x. If, almost everywhere on X (the points where this does not hold have measure zero), $f(q,x)$ is concave w.r.t. q, then $F(q) = \int f(q,x)dG(x)$ is concave on Q.*

PROOF: Choose two arbitrary points q_1 and q_2 from Q and consider an arbitrary convex combination $q_3 = \lambda q_1 + (1-\lambda)q_2$, for any $\lambda \in [0,1]$. Then, by concavity w.r.t. q, $\lambda f(q_1,x) + (1-\lambda)f(q_2,x) \leq f(q_3,x)$ almost everywhere on $G(x)$. Therefore, $\lambda \int f(q_1,x)dG(x) + (1-\lambda)\int f(q_2,x)dG(x) \leq \int f(q_3,x)dG(x)$ and $F(q)$ is concave on Q. *Q.E.D.*

4. ZONING PROBLEM

Assigning land to factories located at given sites is known as zoning. Instead of describing a physical flow of pollutants from a point to a surface, zoning assigns surfaces to given locations. Assume that farm output is carried from the fields to the collection point, which might be a factory, but also a city and even the farmer's own homestead. Every field is assumed to grow the most profitable crop (combination) at every spot and the harvest is carried to a central site to be chosen so as to yield the highest revenue after accounting for transportation costs. Thus, fields are optimally associated to sites.

4.1. *Model Formulation*

Let the variable q denote, as before, the input into the factory. We also define a physical flow $y_k^s(x)$ of harvested crop k from point x to site s, located, say, at point b^s. The model will, as in (7) have an integral in its objective but, in addition, the objective will contain the spot-specific endogenous variable $y_k^s(x)$. This introduces the difficulty that the exogenously given density function should be replaced by an endogenous routing decision.

Transportation costs are denoted by $w_k^s(x)$, a function that should be integrable on $G(x)$. They might be a function of the Euclidean distance between x and b^s. The formulation could easily be generalized to account for possible asymmetry in transportation costs (up-hill/down-hill), but here we disregard this aspect for convenience. Let q^s now denote the aggregate flow of crop output to point b^s, to be used as input by the factory, and let the revenue function $R^s(q^s)$ satisfy Assumption R.

Without loss of generality, we now write the revenue function at spot x in a more explicit form as $r(h, \delta, x)$, where δ is the residual production factor which, in view of the strict concavity in Assumption r, can be taken to be essential for production (i.e. $r(h, 0, x)$ is equal to zero) and guarantees homogeneity of degree one in (h, δ). This factor serves to ensure full specialization of all crops from one spot to a single destination. We rephrase Assumption r accordingly:

Assumption r1 (revenue function of farm at point x): the revenue function $r: \mathbb{R}^K \times [0,1] \times X \to \mathbb{R}$, $r(h, \delta, x)$ is integrable in x and, almost everywhere on X, continuously differentiable w.r.t. the inputs (h, δ); it has uniformly bounded derivative, is concave, homogeneous of degree one in (h, δ), strictly concave in h, increasing in δ, and such that $r(h, 0, x) = 0$.

Finally, let p_k^s denote the purchasing price that clears the commodity market k at site s, whose balance reads:

$$q_k^s - \int y_k^s(x) dG(x) \leq 0, \text{ all } k \text{ and } s. \tag{20}$$

To represent this model formally, we proceed in the way that is usual in location analysis and replace the unknown supply function $y_k^s(x)$ by a well specified function of prices, as follows. For prices p^s define the profit functions:

$$\Pi^s(p^s) = \max_{q^s \geq 0} \left(R^s(q^s) - \sum_k p_k^s q_k^s \right), \quad \text{all } s, \tag{21}$$

which is convex non-increasing in input price p^s (see Varian, 1992). For spot x, the profit from routing the crop to site s, is:

$$\pi^s(p^s - w^s(x), x) = \max_{y \geq 0} \sum_k (p_k^s - w_k^s(x)) y_k + r(-y, \delta, x), \tag{22}$$

for $\delta = 1$, and is convex nondecreasing in output price $(p^s - w^s(x))$. Competitive market prices p^s solve the master problem:[3]

$$\min_{p^s \geq 0,\ \forall s} \sum_s \Pi^s(p^s) + \int \max_s \pi^s(p^s - w^s(x), x)dG(x). \tag{23}$$

This program finds prices p^s that minimize the aggregate producer surplus, which is the dual to the (primal) maximization of aggregate producer surplus in terms of commodity allocations. We use the dual problem because it is a standard convex program, whereas the primal is a problem in functional space. We now check that the objective of dual problem (23) has aggregate excess supply as its derivative and, hence, reaches its minimum at prices that clear the markets.

Continuous differentiability in Assumption r1 implies strict convexity in p of this objective (see Ginsburgh and Keyzer 1997, ch. 2), while strict concavity in Assumption R ensures continuous differentiability of $\Pi^s(p^s)$. Hence, by Hotelling's lemma, optimal inputs q^s satisfy

$$q^s = -\frac{\partial \Pi^s(p^s)}{\partial p^s} \tag{24}$$

while, almost everywhere on X, $\pi^s(p^s - w^s(x), x)$ is continuously differentiable, and outputs y^s are given by:

$$y^s(x) = \delta^s(x)\frac{\partial \pi^s(p^s - w^s(x), x)}{\partial p^s}, \tag{25}$$

where $\delta^s(x) = 1$ if routing to s yields maximal profit, and 0 otherwise. Notice that by construction, the routing will be the same for all commodities produced at point x. Finally, the first-order optimality conditions of (23) ensure satisfaction of the commodity balances:

$$\int y^s(x)dG(x) = q^s. \tag{26}$$

We also mention that by the conjugate function theorem, the problem in quantity terms dual to (23) can be written:

$$\max \sum_s R^s(q^s) + \int \left(\sum_s r(-y^s(x), \delta^s(x), x) - \sum_s \sum_k w_k^s(x) y_k^s(x) \right) dG(x)$$

[3]Although model (23) only deals with farm output, it is relatively straightforward to account for inputs, by treating outputs and inputs separately, in (21, 22), with their own prices. The profit functions will then have two arguments, the first for output p_o^s and the second for input prices p_i^s at market s, and read $\Pi^s(p_o^s, p_i^s)$ and $\pi^s(p_o^s - w^s(x), p_i^s + w^s(x), x)$, where the transport costs $w^s(x)$ are deducted from the output price and added to the market price, to arrive at farm level prices. These profit functions should, almost everywhere on X, be jointly convex in the first two arguments, increasing in output prices and decreasing in input prices, and continuously differentiable in x.

$$q^s \geq 0,\ y_k^s(x) \geq 0,\ \delta^s(x) \geq 0,\ \text{all } k,\ s,\ \text{all } x, \quad \text{subject to} \tag{27}$$

$$q_k^s - \int y_k^s(x)dG(x) \leq 0,\ \text{all } k \text{ and } s,\quad \sum_s \delta^s(x) = 1,$$

which is a problem in functional space.

4.2. *Solution Procedure*

Let us now turn to the solution procedure for model (23). The problem is to minimize

$$V(p) = \int v(p,x)dG(x) \tag{28}$$

on a compact convex price set $\mathcal{P} = \{0 < \underline{p} \leq p \leq \overline{p}\}$, for $v(p,x) = \sum_s \Pi^s(p^s) + \max_s \pi^s(p^s - w^s(x), x)$. Due to the $\max_s$-operator, the function $v(p,x)$ is nonsmooth but under Assumption r1 it is convex in p^s and any stationary point is a global optimum.

The solution procedure can be cast into an SQG-form for (decentralized) price adjustments. At step $t = 1, 2, \ldots$, the change of prices from $p(t)$ to $p(t+1)$ is initiated by the random choice of a location $x(t)$ from the distribution $G(x)$. This change decreases the aggregate profit function $v(p,x) = \sum_s \Pi^s(p^s) + \max_s \pi^s(p^s - w^s(x), x)$ at $x(t)$:

$$p(t+1) = \Pi_{\mathcal{P}}\left(p(t) - \rho_t \frac{\partial v(p(t), x(t))}{\partial p}\right),\quad t = 1, 2, \ldots, \tag{29}$$

where $\Pi_{\mathcal{P}}$ is the projection operator on $\mathcal{P}$. In fact, procedure (29) defines a stochastic Walrasian tatonnement of the type described in Ermoliev et al. (1997). Recall from (24) and (25) that since net supply is the derivative of the profit function, the value $-\frac{\partial v(p(t),x(t))}{\partial p}$ is simply the net demand of point $x(t)$, where the net demand from factories s is taken to be spread uniformly over the surface. As in the deterministic Walrasian tatonnement, the procedure changes prices in the direction of net demand until aggregate (i.e. expected) net demand becomes zero eventually and markets are cleared.

4.3. *Zoning*

The solution of model (23) will yield market clearing prices p^s, from which input demands q^s by factories and farm supplies $y_k^s(x)$ can be recovered via (24) and (25), respectively. This raises the question as to the properties of the optimal values $y_k^s(x)$. Indeed, under relatively mild assumptions about transportation costs

$w_k^s(x)$, and about the differentiability of revenue function $r(h, \delta, x)$, it is possible to show that, almost everywhere on X, $y_k^s(x)$ will be nonzero for at most one destination s and that this destination will be the same for all outputs k. This amounts to asserting that the model generates a specialized zoning, i.e. an integer valued mapping $s(x)$ that is single valued, almost everywhere on x, thus creating a unique association between spots and factories.

Assumption W (unit cost of transportation from x site s): The cost of transporting one unit of commodity k to site s, $w_k^s(x)$ is, almost everywhere on X, (i) nonnegative and continuously differentiable in x. Moreover, (ii) almost everywhere on X and for every k and pair $[s, s']$, $s \neq s'$, the difference $c_k(x) = w_k^s(x) - w_k^{s'}(x)$ has non-zero gradient whenever $c(x) = 0$.

Assumption r2 (differentiability): the revenue function $r(h, \delta, x)$ is almost everywhere on X continuously differentiable in x.

Various specific formulations of the cost functions, such as the Euclidean distance, guarantee satisfaction of Assumption W(ii). Geometrically, this is because two circles with different centres can only share points but no arcs. We can now state and prove:

PROPOSITION 4.1 (SPECIALIZATION OF SPOTS): *Let Assumptions R, r1, r2 and W hold. Then, (a) program (23) partitions the region X into zones with a common routing such that (a) no zone will produce for more than one factory, and (b) the number of zones is finite.*

PROOF: Continuous differentiability of the revenue functions ensures that program (23) has a unique solution p^*. Now for an arbitrary pair of sites $[s, s']$, $s \neq s'$, define the function $z(x) = \pi^s(p^{s*} - w^s(x), x) - \pi^{s'}(p^{s'*} - w^{s'}(x), x)$. By Assumptions W(i) and r2, the function $z(x)$ is continuously differentiable almost everywhere on X. Also partition x into (x_1, x_2), where x_1 refers to the first coordinate of x (a scalar) and x_2 the other coordinate (we assume for convenience that the space is two-dimensional). The proof proceeds in two stages.

First, we keep x_2 fixed at an arbitrary value x_2^0 in X and characterize the fixed points for x_1, as defined by $z(x_1, x_2^0) = 0$, as follows. One possibility is that there are no solutions. This means that either $z(x_1, x_2^0) > 0$ (meaning that s dominates s') or $z(x_1, x_2^0) < 0$ (s' dominates s) for all points on the line $x_2 = x_2^0$. If solutions exist, these can be of two kinds: isolated or on closed segments along the line $x_2 = x_2^0$. Since X is compact, the number of closed segments must be finite, and the fixed point counting theorem in Ortega and Rheinboldt (1970, p. 150) says that, because $z(x)$ is continuously differentiable and X is compact, the number of isolated solutions is finite as well. We can therefore construct around the line $x_2 = x_2^0$, a band consisting of neighbourhood sets $N_j(x_2^0)$, indexed $j = 1, \ldots, J$ that fully cover this line and do not overlap.

Secondly, consider the perturbed fixed points close to x_2^0, say, at $x_2^0 + \epsilon$ within the band. If there were no fixed points at x_2^0, this may still be the case at $x_2^0 + \epsilon$ but this only means that the domination persists. Otherwise $z(x_1, x_2^0 + \epsilon) = 0$ might have some fixed points. Consider in the neighbourhood $N_j(x_2^0)$ of the j-th solution at x_2^0, the set of perturbed solutions $Z_j(x_2^0) = \{x_1 \mid z(x_1, x_2^0 + y) = 0,\ 0 \leq y \leq \epsilon,\ (x_1, x_2) \in N_j(x_2^0)\}$. If this segment persisted, the set $Z_j(x_2^0)$ would have positive surface but would contradict Assumption W(ii). If it does not persist, it can become an intersection or disappear altogether. In both cases $Z_j(x_2^0)$ has zero surface on X (property (a)) and partitions $N_j(x_2^0)$ in a finite number of zones (property (b)). Alternatively, assume that $N_j(x_2^0)$ is the neighbourhood of an isolated root. In this case, by the implicit function theorem, the perturbation defines, for positive ϵ chosen sufficiently small, a continuous function on $N_j(x_2^0)$ for which $Z_j(x_2^0)$ also has zero surface on X. Therefore, for all j, the set $Z_j(x_2^0)$ has zero surface partitions $N_j(x_2^0)$ in a finite number of zones. This holds in a band around every x and since $\epsilon > 0$, it follows that the unions of all sets $Z_j(x_2^0)$ has zero surface, and since it holds for an arbitrary pair $[s, s']$, the property holds for all pairs, proving (a) and (b). *Q.E.D.*

The result has an easy geometric interpretation. For given prices p, the function $z(x_1, x_2)$ can be thought of as a mountain. Consider the intersection of this mountain with a horizontal plane at given altitude (sea level). On this plane, mark the locations where the land that lies below sea level in blue, and those above sea level in black. Because the function is continuously differentiable, the resulting map will now delineate islands and lakes. The distinction will only be ambiguous if the mountain is exactly tangent to sea level. But this tangency will disappear after an arbitrarily small rise in sea level (property (a)). Assumption W(ii) implies that there are no such planes. Property (b) establishes that the number of islands and lakes will be finite.

4.4. *Errors in Evaluation of Gradient: Interpretation as a Learning Process*

So far, we have assumed that the stochastic gradients (net supplies) could be evaluated without errors. SQG-methods have the special attraction that they can accept errors in calculation. Let us denote an estimate of this gradient by $\zeta(t)$. We assume that both $\zeta(t)$ and the step-size ρ_t are defined on a probability space $(\Omega, \mathcal{F}, P)$ and that they are $\mathcal{F}_t$-measurable, where $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is a sequence of σ-fields (see e.g. Billingsley, 1986). This amounts to requiring that $\zeta(t)$ and ρ_t are defined on the basis of information from $\mathcal{F}_t$ available at time t, for example the history of prices $p(1), \ldots, p(t)$ and locations $x(1), \ldots, x(t)$.

We will suppose that the process learns from earlier mistakes and has the ability of gradually eliminating any bias eventually. For example, assuming an additive error $\epsilon(t)$ that is independent of the location $x(t)$, we could replace process (29) by:

$$p(t+1) = \Pi_{\mathcal{P}}\left(p(t) - \rho_t \zeta(t)\right), \quad t = 1, 2, \ldots \tag{30}$$

for

$$\zeta(t) = \frac{\partial v(p(t), x(t))}{\partial p} + \epsilon(t), \tag{31}$$

where $\epsilon(t)$ is an $\mathcal{F}_t$-measurable (random) vector of errors with conditional expectation:

$$\mathrm{E}[\epsilon(t) \mid \mathcal{F}_t] = b(t), \tag{32}$$

such that the following assumption holds (Ermoliev, 1988):

Assumption E (Error specification):
(i) bias elimination: $\|b(t)\| \leq \frac{\alpha_1}{t}$, for fixed $\alpha_1 > 0$,
(ii) bounded variance: $\mathrm{E}[\|\epsilon(t) - b(t)\|^2 \mid \mathcal{F}_t] \leq \alpha_2$, for fixed $\alpha_2 > 0$.
(iii) non-vanishing step-size: $\rho_t \geq 0$, $\sum_{t=1}^{\infty} \rho_t = \infty$, with probability 1,
(iv) step-size convergence: $\mathrm{E}\left[\sum_{t=1}^{\infty} \left(\rho_t \|b(t)\| + \rho_t^2 \mathrm{E}[\|\zeta(t)\|^2 \mid \mathcal{F}_t]\right)\right] < \infty$.

Requirement (i) ensures that the bias $\|b(t)\|$ in the conditional expectation of $\epsilon(t)$ is to vanish in the course of iterations. It means that gradually everyone learns to make unbiased estimates of desired net trades. The requirement (ii) of bounded variance means that errors should not be infinite. It usually is innocuous as the error can usually be assumed to satisfy $\|\epsilon(t)\| \leq C$ for some fixed $C > 0$. Requirements (iii) and (iv) are a generalization of condition (3). It is possible to develop this error specification in various directions, for example by allowing for location specific errors, whose bias only needs to vanish in the aggregate. Finally, notice that, since every point x has zero surface, the individual location has no incentive to behave strategically by communicating false signals. Yet it has no incentive to communicate correct signals either.

5. LAND CONSOLIDATION

The zoning approach can also be used to study land consolidation, with a similar learning interpretation for the associated SQG-process. When applied at village level, the zoning problem (23) allows to reassign land among farmers, and the potential revenue from this consolidation process can be evaluated. However, model (23) has the limitation that it does not account for aspects of parcel configuration: fragmentation, unbalanced shape, and lack of contiguity of parcels do not affect the cost of production. In reality fragmentation can be costly, because the farmer may have to spend much of his time commuting and transporting farm implements from one plot to the other. Here we extend the zoning model to account for the cost of fragmentation. The first model in this section determines the optimal parcel structure, while the second imposes an additional restriction on equitable distribution of gains from consolidation.

In model (23), the cost of production only depends on the distance between the point x in the field and the homestead located at b^s. This might be unrealistic, as the farmer can visit his plots more easily if they are located close together than if they are dispersed over a wide area. Hence we must also account for the distance between plots. To obtain such a measure of spread, we allow every farm s to be the user of not more than, say, J distribution centres or roads indexed sj, as in Section 2. Farmer s could be thought of as taking his bullock cart with seeds or manure to some roadside location near his plots and distributing these inputs over his fields with manual labour, or conversely, collecting his crops with manual labour before loading them on the cart. We continue referring to b^s as the homestead but it might also be the location of the marketing post of farmer s, who could choose to live somewhere along this road.

The main assumption is that there is a given, finite number of alternative routes given to every farmer (which might be the same for different farmers), and that farmers have the right of passage through neighbouring fields, to ensure that lack of contiguity of parcels does not pose any problem. This amounts to assuming that land fragmentation and parcel shape only matter to the extent that they cause parcels to be spread more widely, but the number of parcels itself does not affect cost.

5.1. *A Model of Constrained Land Consolidation*

Formally, we assign at most J distribution centres or roads to every farm s. We can, for example, decompose the transportation cost from the field x to the homestead (or market) b^s into two parts: from x to h^{sj} and from h^{sj} to b^s. Thus, total transport costs per unit is a function $w_k^{sj}(x)$, say, as in Section 2:

$$w_k^{sj}(x) = a_k^j \|x - h^{sj}\| + b_k^s \|h^{sj} - b^s\|, \tag{33}$$

where a_k^j is a unit cost for transport, say, by foot, b_k^s a unit cost for transport by cart, and $\|\cdot\|$ denotes a vector-norm and measures the distance between x and b^s, and a distance between centre of operations and the homestead (for a road j that is inaccessible to farmer s, transportation costs will be given a prohibitively high value). This yields as an immediate extension of model (23)

$$\min_{p^s \geq 0,\ \forall s} \left[\sum_s \Pi^s(p^s) + \int \max_{sj} \pi^s(p^s - w^{sj}(x), x) dG(x) \right]. \tag{34}$$

At every point x, the farmers make a separate decision which distribution centre to use. It would be interesting to allow for endogenous determination of a single location h^s for every farm but the resulting profit maximization problem would not be concave with respect to this location.

5.2. *Regional Model with Fixed Land Base*

Program (34) determines an optimal schedule for land consolidation. Alternatively, to represent farm operations for a given pattern of land fragmentation, one may define the distribution $G^s(x)$ which specifies the given association between point x and farmer s:

$$\min_{p^s \geq 0,\ \forall s} \left[\sum_s \Pi^s(p^s) + \sum_s \int \max_{j} \pi^s(p^s - w^{sj}(x), x) dG^s(x) \right]. \tag{35}$$

This is actually a regional model with separate farms and a fixed land base per farm. Its solution will, for every farm, describe the optimal land allocation to various crops and the zones indexed j will become the fields of the farm. Moreover, every regional model will have a differentiable supply response function $q^s(p^s)$.

5.3. *Restrictions on Land Consolidation*

Land consolidation transactions obviously generate a change in the land distribution. A distinction can be made between a centralized and a decentralized process of transactions. In a centralized land consolidation process all participants start by pooling their land resources together. Next, one computes the optimum of (34), jointly with the cost for every site (household). At the end of the process, a compensation is paid to the losers, which is financed either from taxes on those who gain, or from external funds (village development), according to some agreed rule for sharing the surplus from cooperation. In many situations, the external funds to compensate losers directly or through development of infrastructure will be lacking. In addition, participants are often reluctant to accept compensation payments, because the consolidation process has lasting consequences and calls for recurrent payments which, in view of weak enforcement mechanisms, might not be forthcoming once the consolidation process has been completed. Moreover, farmers will rarely be able to provide side payments in cash.

Under such conditions, participants will tend to prefer a system that guarantees some fairly distributed gain in the direct revenue from farming, and maximizes the relative gain of the least-favoured. It is relatively straightforward to modify (34) accordingly. Let V_0^s denote the original income of farmer s in (34), which is assumed to be positive. We will solve:

$$\max \min_s \tfrac{1}{V_0^s}\left[R^s(q^s) + \int \left(\textstyle\sum_j r(-y^{sj}(x), \delta^{sj}(x), x) - \sum_j \sum_k w^{sj}(x) y_k^{sj}(x)\right) dG(x)\right]$$

$$q_k^s \geq 0,\ y_k^{sj}(x) \geq 0,\ \delta^{sj}(x) \geq 0 \text{ all } j,\ k,\ s, \text{ all } x, \quad \text{subject to} \tag{36}$$

$$q_k^s - \sum_j \int y_k^{sj}(x) dG(x) \leq 0, \text{ all } k \text{ and } s, \quad \sum_{sj} \delta^{sj}(x) = 1.$$

The structure of this problem differs from that of the earlier models because of the $\min_s$ operator. This requires some reformulation.

5.4. *Solution Procedure*

Let us define the S-dimensional simplex $\Lambda = \{(\lambda^1, \ldots, \lambda^S) \geq 0 \mid \sum_s \lambda^s = 1\}$ and represent the $\min_s \int f^s(\cdot) dG(x)$ part as $\min_{\lambda \in \Lambda} \int \sum_s \lambda^s f^s(\cdot) dG(x)$, because this enables us to compute the gradients directly. The variables λ^s can be interpreted as welfare weights (or as value-added tax rates incremented by unity). We also define scaled prices $\widetilde{p}^s = \lambda^s p^s$, and scaled profit functions terms

$$\widetilde{\Pi}^s(\widetilde{p}^s, \lambda^s) = \max_{q^s \geq 0} \frac{1}{V_0^s}\left[\lambda^s R^s(q^s) - \sum_k \widetilde{p}_k^s q_k^s\right], \quad \text{all } s, \tag{37}$$

which is convex non-increasing in scaled input price $\widetilde{p}^s$ and convex non-decreasing in welfare weight λ^s. For spot x, the scaled profit function is:

$$\widetilde{\pi}^{sj}(\widetilde{p}^s - \lambda^s w^s(x), \lambda^s, x) = \max_{y \geq 0} \frac{1}{V_0^s}\left[\sum_k \left(\widetilde{p}_k^s - \lambda^s w_k^{sj}(x)\right) y_k + \lambda^s r(-y, 1, x)\right]. \tag{38}$$

Noticing that $\widetilde{p} \in \mathcal{P}$ since $0 \leq \widetilde{p} \leq p$, the resulting nested formulation is:

$$\min_{\widetilde{p} \in \mathcal{P}, \lambda \in \Lambda} \int v(\widetilde{p}, \lambda, x) dG(x), \tag{39}$$

for

$$v(\widetilde{p}, \lambda, x) = \sum_s \left[\widetilde{\Pi}^s(\widetilde{p}^s, \lambda^s) + \max_{sj} \widetilde{\pi}^s(\widetilde{p}^s - \lambda^s w^{sj}(x), \lambda^s, x)\right], \tag{40}$$

which is jointly convex in $(\widetilde{p}, \lambda)$. We can solve this problem by an SQG-procedure similar to (29): at each step $t = 1, 2, \ldots$ the price vector $\widetilde{p}(t)$ is adjusted in the direction $\frac{\partial v(\tilde{p}(t),\lambda(t),x(t))}{\partial \tilde{p}}$ and $\lambda(t)$ in the direction $\frac{\partial v(\tilde{p}(t),\lambda(t),x(t))}{\partial \lambda}$. We propose the following decentralized stochastic adjustment procedure:

$$\tilde{p}(t+1) = \Pi_{\mathcal{P}}\left(\tilde{p}(t) - \rho_t \frac{\partial v(\tilde{p}(t), \lambda(t), x(t))}{\partial \tilde{p}}\right), \tag{41}$$

$$\lambda(t+1) = \Pi_{\Lambda}\left(\lambda(t) - \rho_t \frac{\partial v(\tilde{p}(t), \lambda(t), x(t))}{\partial \tilde{p}}\right), \quad t = 1, 2, \ldots, \tag{42}$$

where ρ_t is the step-size that is supposed to satisfy (3). Note that, as in (29), the price adjustment is a stochastic Walrasian tatonnement, but in addition, the weights are adjusted so as to give higher weight to those who stand to gain less, relative to the original situation V_0^s. We summarize this as a proposition.

PROPOSITION 5.1 (DECENTRALISED LAND CONSOLIDATION WITHOUT LOSERS): *Let Assumptions R, r1, r2 and W hold. Then, with probability 1, procedure (41, 42) with step-size ρ_t satisfying (3) converges to the (global) optimum of (39, 40) which also solves (36).*

PROOF: Since $v(\tilde{p}, \lambda, x)$ is jointly convex in $(\tilde{p}, \lambda)$ on the compact, convex domain $\mathcal{P} \times \Lambda$, convergence follows as in Section 2. *Q.E.D.*

Procedure (41, 42) is remarkably simple and transparant, despite the apparent complexity of problem (36). We start from given $\tilde{p}(t)$ and $\lambda(t)$ and select a point $x(t)$ at random. Next, we choose the most profitable destination sj, for all commodities produced at x to be shipped to. Finally, we compute the SQG as derivative w.r.t. $\tilde{p}^s$ and λ^s of the associated "total profit" function $[\tilde{\Pi}^s(\tilde{p}^s, \lambda^s) + \tilde{\pi}^s(\tilde{p}^s - \lambda^s w^{sj}(x), \lambda^s, x)]$ (if there are several, choose, say, the one with lowest index value).

6. CONCLUSION

The approach described in this paper is currently applied in two projects. The first implements the model of Section 3 with pollution to a region in Poland where the ammonia emissions from the intensive livestock industry currently reduce the crop yields of surrounding farms. The second application studies, on the basis of detailed household surveys, the causes of land fragmentation and the scope for land consolidation in two villages in India.

With respect to the latter application, we have so far treated land as a perfectly divisible commodity and this enabled us to reach relatively strong conclusions about the convergence of the processes of land allocation and consolidation, even when restrictions are being imposed on transactions. However, in reality land transactions are difficult and the question to be faced is, therefore, whether the models and adjustment processes presented here bypass important elements.

First, in reality land is always indivisible to some extent. Farmers own plots of different size and quality and transactions relate to plot surfaces, not to points, precisely because agents cannot keep on transacting forever. Now, if we drop the divisibility assumption and partition the surface X into N fixed parcels, the optimal land consolidation model (39, 40) becomes

$$\min_{\tilde{p}\in\mathcal{P},\lambda\in\Lambda} V^N(\tilde{p},\lambda) \tag{43}$$

for

$$V^N(\tilde{p},\lambda) = \frac{1}{N}\sum_{i=1}^{N}\sum_s \tilde{\Pi}^s(\tilde{p}^s,\lambda^s) + \max_{sj}\tilde{\pi}^{sj}(\tilde{p}^s - \lambda^s w^{sj}(x^i),\lambda^s,x^i). \tag{44}$$

To solve this problem, i.e. to clear the land market, the decentralized SQG-procedure (29) can be used as before, now with $x(t)$ sampled from the discrete distribution $\{x^1,\ldots,x^N\}$ instead of $G(x)$. The difference between the calculated optimal $V^N(\tilde{p},\lambda)$ and $V(\tilde{p}',\lambda)$ will estimate the welfare loss from maintaining a given parcel size as well as the associated price distortion $(\tilde{p}-\tilde{p}')$. A similar discretization approach can be used for the other models in this paper.

Secondly, if parcels are large, the resulting optimum would often be significantly lower than the unconstrained one, even for divisible parcels. This might frustrate the progress of transactions, because some potential gainers might prefer the status quo to a time consuming negotiation process that only brings marginal improvement, and that might even end before a Pareto-efficient solution is reached, i.e. as soon as the least-favoured cannot improve their position.

Thirdly, by treating the locations h^{sj} of the distribution centres as given for every farm, we avoid the nonconvexity that emerges as soon as this variable is made endogenous. Though convergence of the SQG-procedure is preserved when prices and locations are adjusted simultaneously, it will be to a stationary value that is not necessarily the global optimum. This could by itself already explain the inability of a decentralized transaction process to converge to a Pareto-efficient solution.

Finally, even though under process (41, 42) no one will be worse off eventually, in the course of the adjustment process some participants might lose while others gain. Hence some participants might want to step out: the losers because they fear further losses and the winners because they want to consolidate their position. Thus, the process of transactions might end prematurely for lack of participants. It is possible to adjust the SQG-procedure accordingly, by restricting the random selection of points to an appropriately defined part of the distribution. One might also look for "matching" pairs of transactions. However, this raises complex issues of strategic behaviour and mechanism design and is a topic for further investigation.

Another challenge for further research is to characterize the type of non-convexities and discontinuities that can be "healed" by the integral operation. Lemma 3.2 has illustrated the advantage of modelling in a spatial continuum, as opposed to a discretized space, since this allows to neglect specific discontinuities and non-convexities, as characterized by the "almost everywhere"-qualifier. However, we have only considered problems for which this qualification is given as part of the problem specification. It would seem useful to formulate conditions on the model itself which allow to neglect non-convexities and discontinuities that appear in it, because of the healing effect of integration. In the model of pollution control this might allow to study increasing returns in pollution, and in the land consolidation model to determine endogenously the location of distribution centres.

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