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The Time-Varying Two-Sided Nudelman Interpolation Problem and Its Solution

ABSTRACT

This paper concerns the time-varying analogue of the two-sided Nudelman interpolation problem. Generalizations to the time-varying case of the results for rational matrix functions are presented in both the discrete and continuous time setting.

0. Introduction

For rational matrix functions the two-sided Nudelman interpolation problem can be stated as follows (cf., [6], Section 18.4). Let Δ denote the open right half plane Π^+ or the open unit disk $\mathbb{D}$, and consider the following interpolation data:

$$\omega = (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma). \tag{0.1}$$

Here A_π and A_ζ are square matrices of sizes $n_\pi \times n_\pi$ and $n_\zeta \times n_\zeta$, respectively, which have all their eigenvalues in Δ, the matrix B_+ has size $n_\zeta \times m$ and the pair (A_ζ, B_+) is a full range pair, which means that

$$\operatorname{Im}\ (B_+ \quad A_\zeta B_+ \quad \dots \quad A_\zeta^{n_\zeta - 1} B_+) = \mathbb{C}^{n_\zeta},$$

the matrix C_- has size $r \times n_\pi$ and the pair (C_-, A_π) is a null kernel pair, i.e.,

$$\bigcap_{j=1}^{n_\pi} \operatorname{Ker} C_- A_\pi^{j-1} = \{0\},$$

the items C_+ and B_- are matrices of sizes $m \times n_\pi$ and $n_\zeta \times r$, respectively, and Γ is an $n_\zeta \times n_\pi$ matrix. The rational matrix version of the *two-sided Nudelman interpolation problem* asks to find all rational $m \times r$ matrix functions F that do not have poles in Δ, that satisfy the interpolation conditions

$$\sum_{z_0\in\Delta} \mathrm{Res}_{z=z_0} F(z)C_-(zI - A_\pi)^{-1} = C_+,$$

(0.2) $$\sum_{z_0\in\Delta} \mathrm{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ F(z) = -B_-,$$

$$\sum_{z_0\in\Delta} \mathrm{Res}_{z=z_0} (zI - A_\zeta)^{-1} B_+ F(z) C_-(zI - A_\pi)^{-1} = \Gamma,$$

and that meet the following norm constraint:

(0.3) $$\sup_{z\in\Delta} \|F(z)\| < 1.$$

As usual, $\mathrm{Res}_{z=z_0}W(z)$ stands for the residue of the rational matrix function $W(z)$ at z_0. The norm in (0.2) is the usual operator norm for a matrix. If there is a solution of the problem (0.2)-(0.3), then necessarily Γ satisfies the Sylvester equation

(0.4) $$\Gamma A_\pi - A_\zeta \Gamma = B_-C_- + B_+C_+,$$

and therefore in what follows we assume that (0.4) holds.

To state the solution of the Nudelman problem (0.2)-(0.3), we need to introduce the *Pick matrix* $\Lambda(\omega)$ associated with the data ω in (0.1), namely

(0.5) $$\Lambda(\omega) = \begin{pmatrix} S_1 & \Gamma^* \\ \Gamma & S_2 \end{pmatrix},$$

where S_1 and S_2 are the unique (necessarily Hermitian) solutions of the Lyapunov equations

$$\begin{aligned} S_1A_\pi + A_\pi^* S_1 &= C_-^*C_- - C_+^*C_+ \\ S_2A_\zeta^* + A_\zeta S_2 &= B_+B_+^* - B_-B_-^* \end{aligned}$$

for the case $\Delta = \Pi^+$, or where S_1 and S_2 are the unique (necessarily Hermitian) solutions of the Stein equations

$$\begin{aligned} S_1 - A_\pi^* S_1 A_\pi &= C_-^*C_- - C_+^*C_+ \\ S_2 - A_\zeta S_2 A_\zeta^* &= B_+B_+^* - B_-B_-^* \end{aligned}$$

if $\Delta = \mathbb{D}$. Now assume that $\Lambda = \Lambda(\omega)$ is invertible and introduce an auxiliary matrix function $\Theta(z)$, by setting

$$\Theta(z) = I + \begin{pmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{pmatrix} \begin{pmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & (zI + A_\zeta^*)^{-1} \end{pmatrix} \Lambda(\omega)^{-1} \begin{pmatrix} -C_+^* & C_-^* \\ B_+ & B_- \end{pmatrix}$$

if $\Delta = \Pi^+$, or

$$\Theta(z) = D + \begin{pmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{pmatrix} \begin{pmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & (I - zA_\zeta^*)^{-1} \end{pmatrix} \Lambda(\omega)^{-1} \begin{pmatrix} A_\pi^{*-1}C_+^* & -A_\pi^{*-1}C_-^* \\ B_+ & B_- \end{pmatrix} D$$

where, fixing any $\alpha \in \mathbb{C}$ with $|\alpha| = 1$,

$$D = I - \begin{pmatrix} C_+ & B_+^*A_\zeta^{*-1} \\ C_- & -B_-^*A_\zeta^{*-1} \end{pmatrix} \Lambda(\omega)^{-1} \begin{pmatrix} (I - \alpha A_\pi)^{-1} & 0 \\ 0 & (\alpha I - A_\zeta)^{-1} \end{pmatrix} \begin{pmatrix} -C_+^* & C_-^* \\ B_+ & B_- \end{pmatrix}$$

if $\Delta = \mathbf{D}$ and both A_π and A_ζ are nonsingular, or

$$\Theta(z) = I + (z - z_0) \begin{pmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{pmatrix} \begin{pmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & (I - zA_\zeta^*)^{-1} \end{pmatrix} \cdot$$

$$\cdot \Lambda(\omega)^{-1} \begin{pmatrix} (I - z_0 A_\pi^*)^{-1} C_+^* & -(I - z_0 A_\pi^*)^{-1} C_-^* \\ (A_\zeta - z_0 I)^{-1} B_+ & (A_\zeta - z_0 I)^{-1} B_- \end{pmatrix},$$

where z_0 is a chosen point on the unit circle, if $\Delta = \mathbf{D}$ and at least one of A_π and A_ζ is singular. We shall refer to Θ as the *matrix function determined from the data* ω.

We are now ready to state the main result (from [6]) on the two-sided Nudelman interpolation problem.

THEOREM 0.1. *The two-sided Nudelman interpolation problem* (0.2) – (0.3) *has a solution if and only if the Pick matrix* $\Lambda(\omega)$ *is positive definite. In this case, the set of all solutions* F *is given by*

$$F = (\Theta_{11} G + \Theta_{12})(\Theta_{21} G + \Theta_{22})^{-1},$$

where G *is an arbitrary* $m \times r$ *rational matrix function with no poles on* Δ *satisfying the norm constraint* $\sup_{z \in \Delta} \|G(z)\| < 1$ *and*

$$\Theta(z) := \begin{pmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{pmatrix}$$

is the matrix function associated with the interpolation data ω.

The main aim of this paper is to present the generalization of Theorem 0.1 to the time-varying case. In the discrete time setting ($\Delta = \mathbf{D}$) this means that the role of rational matrix functions is taken over by doubly infinite lower triangular matrices which appear as input-output operators of finite dimensional time-varying discrete time linear systems. In the continuous time case the rational matrix functions are replaced by integral operators of the second kind of Volterra type, which also appear as input-output operators, but now of time-varying continuous time systems. The method which we employ to solve the time-varying two sided Nudelman interpolation problem is based on ideas similar to those used in [6] for the time-invariant case.

The present paper has the character of a survey paper. It consists of two chapters. The first introduces the discrete time version of the time-varying two sided Nudelman interpolation problem and presents its solution. The second chapter gives the solution of the same problem in the continuous time setting. Both chapters start with a section of preliminary character in which the time-varying analogue of the residue calculus of complex function theory is developed further.

I. Discrete Time Nudelman Interpolation

I.1. Residue calculus and generalized point evaluation in discrete time

In this section we present some preliminaries on shift expansions and time-varying residue calculus. Throughout this paper m and r are given positive integers. We let $\mathcal{B}^{m\times r}$ denote the set of all doubly infinite block matrices $F = [F_{ij}]_{i,j=-\infty}^{\infty}$ with each block F_{ij} a matrix of size $m \times r$ over $\mathbb{C}$ such that matrix multiplication by F on the left defines a bounded linear operator from ℓ_2^r into ℓ_2^m. Here ℓ_2^k is the space of norm square summable sequences $(x_n)_{n=-\infty}^{\infty}$ with entries in $\mathbb{C}^k$ written as column vectors. We will have use of the following subspaces:

$$\begin{aligned}
\mathcal{L}^{m\times r} &= \text{lower triangular elements of } \mathcal{B}^{m\times r} = \{F = [F_{ij}] \in \mathcal{B}^{m\times r} \mid F_{ij} = 0 \text{ for } i < j\}; \\
\mathcal{U}^{m\times r} &= \text{upper triangular elements of } \mathcal{B}^{m\times r} = \{F = [F_{ij}] \in \mathcal{B}^{m\times r} \mid F_{ij} = 0 \text{ for } i > j\}; \\
\mathcal{D}^{m\times r} &= \text{diagonal elements of } \mathcal{B}^{m\times r} = \mathcal{L}^{m\times r} \cap \mathcal{U}^{m\times r}.
\end{aligned}$$

Our interest is to develop further the time-varying analogue of the function theory on the unit disc started in [1], [2]. The time-invariant case corresponds to the restriction to block Toeplitz matrices $F = [F_{i-j}]_{i,j=-\infty}^{\infty}$, which are associated to the Laurent series of functions $F(z) = \sum_{j=-\infty}^{\infty} z^j F_j$ on the unit circle. A key tool in the function theory analysis is the coefficient F_{-1} of z^{-1} in the Laurent series. Assuming that the function $F(z) = \sum_{j=-\infty}^{\infty} F_j z^j$ is meromorphic in the open unit disk $\mathbf{D}$ we have by elementary complex analysis that

$$F_{-1} = \frac{1}{2\pi i} \int_{\mathbf{T}} F(z)\, dz = \sum_{z_0 \in \mathbf{D}} \mathrm{Res}_{z=z_0} F(z)$$

where $\mathrm{Res}_{z=z_0} F(z)$ is the residue of $F(z)$ at the point z_0 in $\mathbf{D}$. For the time-varying case, where $F = [F_{ij}]$ is a general block matrix representing a bounded operator form ℓ_2^r into ℓ_2^m, there are two analogues of the Laurent series of interest, namely

(1.1)
$$F = \sum_{\nu=-\infty}^{\infty} S^{\nu} F_{[\nu]} = \sum_{\nu=-\infty}^{\infty} F_{\{\nu\}} S^{\nu}.$$

Here S stands for the block lower triangular forward shift on ℓ_2 of the appropriate dimension, and the coefficients $F_{[\nu]}$ and $F_{\{\nu\}}$ are block diagonal operators from $\mathcal{D}^{m\times r}$ of which the main diagonal entries are given by

(1.2)
$$(F_{[\nu]})_{jj} = F_{j+\nu,j}, \qquad (F_{\{\nu\}})_{jj} = F_{j,j-\nu}.$$

The convergence in (1.1) has to be understood entrywise. The coefficients $F_{[\nu]}$ and $F_{\{\nu\}}$ are uniquely determined by the entrywise convergence of the series and the fact that $F_{[\nu]}$ and $F_{\{\nu\}}$ are block diagonal operators. We call the series expansions in (1.1) the *left* and *right shift expansions* of F, respectively.

We define the *left* and *right nonstationary total* $\mathbf{D}$*-residue maps* $\mathcal{R}es_L$ and $\mathcal{R}es_R$ from $\mathcal{B}^{m\times r}$ into $\mathcal{D}^{m\times r}$, by

(1.3)
$$\mathcal{R}es_L(F) = F_{[-1]}, \qquad \mathcal{R}es_R(F) = F_{\{-1\}},$$

where $F_{[-1]}$ and $F_{\{-1\}}$ are the block diagonal operators in $\mathcal{D}^{m\times r}$ arising, respectively, from the left and right shift expansions of F as in (1.1). In general, note that

$$F = \sum_{\nu=-\infty}^{\infty} (S^{\nu} F_{[\nu]} S^{-\nu}) S^{\nu}$$

and $S^{\nu}F_{[\nu]}S^{-\nu}$ belongs to $\mathcal{D}^{m\times r}$. Thus, from the uniqueness mentioned above, it follows that $F_{\{\nu\}} = S^{\nu}F_{[\nu]}S^{-\nu}$ for each integer ν. In particular, we have the following simple relation between the two residue maps:

$$\mathcal{R}es_R(F) = S^{-1}\mathcal{R}es_L(F)S. \tag{1.4}$$

If $F(z) = \sum_{j=0}^{\infty} z^j F_j$ is a scalar function which is analytic on the closure of the open unit disc $\mathbf{D}$ and $z_0 \in \mathbf{D}$, then the value $F(z_0)$ at the point z_0 is also given by

$$F(z_0) = \frac{1}{2\pi i}\int_{\mathbf{T}} (z - z_0)^{-1} F(z)\, dz = \sum_{w\in \mathbf{D}} \mathrm{Res}_{z=w}(z - z_0)^{-1} F(z)$$

In terms of the Toeplitz matrix $F = [F_{i-j}]$ associated with the function $F(\cdot)$ we have therefore

$$F(z_0)I = \mathcal{R}es_L((S - z_0I)^{-1}F) = \mathcal{R}es_R(F(S - z_0I)^{-1}).$$

In this form the point evaluation carries over to the time-varying case.

Let $F = [F_{ij}]_{i,j=-\infty}^{\infty}$ be a general element of $\mathcal{L}^{m\times r}$, and let $A \in \mathcal{D}^{r\times r}$ be a diagonal matrix such that the spectral radius $\rho(S^{-1}A)$ of $S^{-1}A$ is strictly less than 1. We may then define $F^{\wedge R}(A)$ (the *right point evaluation* of F at A) by

$$F^{\wedge R}(A) = \mathcal{R}es_R(F(S - A)^{-1}). \tag{1.5}$$

Note that the condition $\rho(SA^{-1}) < 1$ guarantees that $S - A = S(I - S^{-1}A)$ is invertible as an operator on ℓ_2^r. Similarly, if $Z \in \mathcal{D}^{m\times m}$ is such that $\rho(ZS^{-1}) < 1$, then we may define the *left point evaluation* of F at Z by

$$F^{\wedge L}(Z) = \mathcal{R}es_L((S - Z)^{-1}F). \tag{1.6}$$

More explicitly,

$$\begin{aligned} F^{\wedge R}(A) &= \textstyle\sum_{j=0}^{\infty} F_{\{j\}}S^j(S^{-1}A)^j \quad \textit{if} \quad F = \sum_{j=0}^{\infty} F_{\{j\}}S^j, \\ F^{\wedge L}(Z) &= \textstyle\sum_{j=0}^{\infty}(ZS^{-1})^j S^j F_{[j]} \quad \textit{if} \quad F = \sum_{j=0}^{\infty} S^j F_{[j]}. \end{aligned}$$

The following proposition (cf., [2], Proposition 1.1) characterizes the right and left point evaluations.

PROPOSITION 1.1. *Let $F \in \mathcal{L}^{m\times r}$, $A \in \mathcal{D}^{r\times r}$ and $Z \in \mathcal{D}^{m\times m}$, and assume that the spectral radii of SA^{-1} and ZS^{-1} are strictly less than 1. Then $F^{\wedge R}(A)$ and $F^{\wedge L}(Z)$ are the unique elements of $\mathcal{D}^{m\times r}$ such that*

$$(F - F^{\wedge R}(A))(S - A)^{-1} \in \mathcal{L}^{m\times r}, \quad (S - Z)^{-1}(F - F^{\wedge L}(Z)) \in \mathcal{L}^{m\times r}.$$

I.2. The time-varying two-sided Nudelman interpolation problem in discrete time

For the discrete time time-varying two-sided Nudelman interpolation problem the given interpolation data form a set

$$\omega = (C_+, C_-, A; Z, B_+, B_-; \Gamma) \tag{2.1}$$

consisting of diagonal block matrices with the following properties. The diagonal block matrices A and Z belong to $\mathcal{D}^{n\times n}$ and $\mathcal{D}^{k\times k}$, respectively, and the spectral radii of $S^{-1}A$ and ZS^{-1} are strictly less than one. Here, as before, S is the block forward shift on ℓ_2 of the appropriate dimension. The diagonal block matrix C_- belongs to $\mathcal{D}^{r\times n}$, and the pair (C_-, A) is required to be *uniformly observable* which means that the diagonal block matrix

$$\sum_{j=0}^{\infty}(A^*S)^j C_-^* C_-(S^{-1}A)^j \tag{2.2}$$

acts as a positive definite operator on ℓ_2^n. The diagonal block matrix B_+ belongs to $\mathcal{D}^{k\times n}$, and the pair (Z, B_+) is required to be *uniformly controllable*, i.e.,

$$\sum_{j=0}^{\infty}(ZS^{-1})^j B_+ B_+^*(SZ^*)^j \tag{2.3}$$

is a positive definite operator on ℓ_2^k. Finally, C_+, B_- and Γ are assumed to be in $\mathcal{D}^{m\times n}$, $\mathcal{D}^{k\times r}$ and $\mathcal{D}^{k\times n}$, respectively.

The time-varying two-sided Nudelman interpolation problem asks to find all $F \in \mathcal{L}^{m\times r}$ such that

$$\mathcal{R}\mathrm{es}_R FC_-(S-A)^{-1} = C_+, \tag{2.4}$$

$$\mathcal{R}\mathrm{es}_L (S-Z)^{-1}B_+F = -B_-, \tag{2.5}$$

$$\mathcal{R}\mathrm{es}_L[(S-Z)^{-1}B_+FC_-(S-A)^{-1}] = \Gamma, \tag{2.6}$$

and F meets the following norm constraint:

$$\|F\| < 1. \tag{2.7}$$

Since F is required to be lower triangular, FC_- and B_+F must also be lower triangular, and hence the left hand side of (2.4) is equal to $(FC_-)^{\wedge R}(A)$ and that of (2.5) is equal to $(B_+F_-)^{\wedge L}(Z)$. But then it follows that from Proposition 1.1 that (2.5) and (2.6) may be reformulated as

$$(FC_- - C_+)(S-A)^{-1} \in \mathcal{L}^{m\times n}, \tag{2.5'}$$

$$(S-Z)^{-1}(B_+F + B_-) \in \mathcal{L}^{k\times r}. \tag{2.6'}$$

One can show that if the interpolation problem (2.4)-(2.6) has a solution $F \in \mathcal{L}^{m\times r}$, then necessarily Γ must satisfy the equation

$$\Gamma A - ZS^{-1}\Gamma S = B_+C_+ + B_-C_-. \tag{2.8}$$

In more detail, if $\Gamma = \text{diag}\,(\Gamma_\nu)$, $A = \text{diag}\,(A_\nu)$, $Z = \text{diag}\,(Z_\nu)$, $B_\pm = \text{diag}\,(B_{\pm,\nu})$ and $C_\pm = \text{diag}\,(C_{\pm,\nu})$, then (2.8) means that the sequence (Γ_ν) satisfies the time-varying Sylvester equation

$$\Gamma_\nu A_\nu - Z_\nu \Gamma_{\nu+1} = B_{+,\nu}C_{+,\nu} + B_{-,\nu}C_{-,\nu}, \qquad \nu \in \mathbb{Z}.$$

We are now ready to state our main theorem for the discrete case.

THEOREM 2.1. *Assume we have given interpolation date ω as in (2.1) such that (2.8) is satisfied. Let $H_R \in \mathcal{D}^{n\times n}$ and $H_L \in \mathcal{D}^{k\times k}$ be the unique solutions of the respective time-varying Stein equations*

$$S^{-1}H_RS - A^*H_RA = C_-^*C_- - C_+^*C_+, \tag{2.9}$$

$$H_L - Z(S^{-1}H_LS)Z^* = B_+B_+^* - B_-B_-^*, \tag{2.10}$$

and consider the 2×2 operator matrix

$$\Lambda = \begin{pmatrix} H_R & \Gamma^* \\ \Gamma & H_L \end{pmatrix} : \ell_2^n \oplus \ell_2^k \to \ell_2^n \oplus \ell_2^k. \tag{2.11}$$

Then there exists a solution $F \in \mathcal{L}^{m\times r}$ of the interpolation problem (2.4) – (2.6) satisfying the norm constraint (2.7) if and only if Λ is a positive definite operator on ℓ_2^{n+k}, and in this case all $F \in \mathcal{L}^{m\times r}$ satisfying (2.4) – (2.7) are given by

$$F = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}, \tag{2.12}$$

where G is any element in $\mathcal{L}^{m\times r}$ such that $\|G\| < 1$ and the operator matrix

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} : \ell_2^m \oplus \ell_2^r \to \ell_2^m \oplus \ell_2^r$$

is given by

$$\Theta = \delta + \gamma(S - A)^{-1}\beta^{(1)} + J\beta^*(I - SZ)^{-1}S\beta^{(2)}.$$

Here

$$\gamma = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}, \qquad \beta = (\,B_+ \quad B_-\,), \qquad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : \ell_2^m \oplus \ell_2^r \to \ell_2^m \oplus \ell_2^r,$$

and $\delta \in \mathcal{D}^{(m+r)\times(m+r)}$, $\beta^{(1)} \in \mathcal{D}^{n\times(m+r)}$ and $\beta^{(2)} \in \mathcal{D}^{k\times(m+r)}$ are arbitrary solutions of

$$\begin{pmatrix} \Gamma_{\nu+1}^* & -A_\nu^*H_{R,\nu} & \gamma_n^*J \\ Z_\nu H_{L,\nu+1} & -\Gamma_\nu & \beta_n \end{pmatrix} \begin{pmatrix} \beta_\nu^{(2)} \\ \beta_\nu^{(1)} \\ \delta_\nu \end{pmatrix} = 0$$

and

$$\left(\beta_\nu^{(2)*} \quad \beta_\nu^{(1)*} \quad \delta_\nu^* \right) \begin{pmatrix} H_{l,\nu+1} & 0 & 0 \\ 0 & H_{R,\nu} & 0 \\ 0 & 0 & J \end{pmatrix} \begin{pmatrix} \beta_\nu^{(2)} \\ \beta_\nu^{(1)} \\ \delta_\nu \end{pmatrix} = J,$$

where ν *runs over* $\mathbf{Z}$.

If A and Z are boundedly invertible, then the formula for Θ can be given more explicitly as

$$\Theta = \tilde{\delta} + (\gamma \quad \beta^* Z^{-*}) \begin{pmatrix} (S-A)^{-1} & 0 \\ 0 & (S-Z^{-*})^{-1} \end{pmatrix} \Lambda^{-1} \begin{pmatrix} A^{-*}\gamma^* J \\ \beta \end{pmatrix} \tilde{\delta}$$

where $\tilde{\delta} \in \mathcal{D}^{(m+r)\times(m+r)}$ is any solution of the congruence relation

$$\tilde{\delta}^{-*} J \tilde{\delta} = J - (J\gamma A^{-1} \quad Z^*) \Lambda^{-1} \begin{pmatrix} A^{-*}\gamma^* J \\ \beta \end{pmatrix}.$$

Instead of the suboptimal norm constraint (2.7) we may also look for solutions that are just contractions, i.e., solutions that satisfy

$$\|F\| \le 1. \tag{2.13}$$

One can prove that the interpolation problem (2.4)-(2.6) has a solution satisfying (2.13) if and only if Λ in (2.11) is positive semi-definite. Furthermore, if Λ is assumed to be positive definite, then all contractive solutions of (2.4)-(2.6) are obtained by formula (2.11) where the free parameter now runs over all $G \in \mathcal{L}^{m\times r}$ that are contractive.

If in (2.1) the block diagonal matrices C_+, C_-, A and Γ are vacuous, then the two-sided Nudelman problem reduces to the one-sided problem (2.5), (2.7), and in this case Theorem 2.1 follows from the results in Section 4 of [2]. (see also [10], where the doubly infinite block matrices $F = [F_{ij}]_{i,j=-\infty}^{\infty}$ are allowed to have operator entries). A full proof of Theorem 2.1 will be given in [4].

II. Continuous Time Nudelman Interpolation

II.1. Generalized point evaluation in continuous time

In the continuous time setting (generalized) points are square matrix functions with entries from $L_\infty(\mathbf{R})$, the space of essentially bounded measurable functions on $\mathbf{R}$, and the role of the shift operator is taken over by the operator of differentiation $\frac{d}{dt}$, which will be denoted by Δ. The domain $\mathcal{D}(\Delta)$ of $\Delta = \frac{d}{dt}$ is the set of all functions $f \in L_2^p(\mathbf{R})$ (where p may be any positive integer) such that f is absolutely continuous on finite intervals and $\Delta f = f' \in L_2^p(\mathbf{R})$. Here $L_2^p(\mathbf{R})$ stands for the space of all square (Lebesgue) integrable vector functions with values in $\mathbf{C}^p$.

Let $A \in L_\infty^{p\times p}(\mathbf{R})$. We shall use the symbol A also for the operator of multiplication on $L_2^p(\mathbf{R})$ by A. Thus $Af = A(\cdot)f$. It follows that $\Delta - A$ is a well-defined unbounded operator on $L_2^p(\mathbf{R})$ with domain equal to the domain of Δ. By $\tau_A(t,s)$ we denote the transition matrix associated with the differential equation

$$x'(t) = A(t)x(t), \qquad -\infty < t < \infty. \tag{1.1}$$

We call A an *anti-stable time-varying* point if there exists constants $M > 0$ and $0 < a < 1$ such that

$$(1.2) \qquad \|\tau_A(t, s)\| \le M a^{s-t}, \quad s \ge t.$$

In this case the differential operator $\Delta - A$ is boundedly invertible, and for $\varphi \in L_2^p(\mathbf{R})$ we have

$$(1.3) \qquad ((\Delta - A)^{-1}\varphi)(t) = -\int_t^\infty \tau_A(t, s)\varphi(s)\, ds, \quad t \in \mathbf{R}.$$

Generalized point evaluations will be defined for operators $F : L_2^m(\mathbf{R}) \to L_2^r(\mathbf{R})$ that belong to the *nonstationary* $m \times r$ *Wiener algebra* $\mathcal{W}^{m\times r}$, that is, operators F that admit a representation of the form

$$(1.4) \qquad (F\varphi)(t) = D(t)\varphi(t) + \int_{-\infty}^\infty f(t, s)\varphi(s)\, ds, \quad t \in \mathbf{R},$$

where D is an $m \times r$ matrix function whose entries are in $L_\infty(\mathbf{R})$ and the kernel function f is an $m \times r$ matrix function whose entries are measurable functions on $\mathbf{R} \times \mathbf{R}$ such that

$$\sup_{t-s=\alpha} \|f(t, s)\| \le \ell_f(\alpha),$$

for some $\ell_f \in L_1(\mathbf{R})$. An operator $F \in \mathcal{W}^{m\times r}$ is called *lower triangular* (notation: $F \in \mathcal{LW}^{m\times r}$) if the kernel function f in (1.4) is zero a.e. on $s \ge t$. In this case

$$(1.5) \qquad (F\varphi)(t) = D(t)\varphi(t) + \int_{-\infty}^t f(t, s)\varphi(s)\, ds, \quad t \in \mathbf{R}.$$

By $\mathcal{UW}^{m\times r}$ we denote the *upper triangular* F's in $\mathcal{W}^{m\times r}$, i.e., those $F \in \mathcal{W}^{m\times r}$ for which the kernel function f is zero a.e. on $s \le t$. Note that $(\Delta - A)^{-1}$ in (1.3) belongs to $\mathcal{UW}^{p\times p}$.

Now, let $A \in L_\infty^{r\times r}(\mathbf{R})$ be an anti-stable time varying point with associated transition matrix $\tau_A(t, s)$, and let $F \in \mathcal{LW}^{m\times r}$ be given by (1.5). We define the *right point evaluation* of F at A to be the function

$$F^{\wedge R}(A)(t) = D(t) + \int_{-\infty}^0 f(t, t+\alpha)\tau_A(t+\alpha, t)\, d\alpha, \quad t \in \mathbf{R}.$$

Similarly, for an anti-stable time varying point $Z \in L_\infty^{m\times m}(\mathbf{R})$ with associated transition matrix $\tau_Z(t, s)$ we define the *left point evaluation* of F at Z to be the function

$$F^{\wedge L}(Z)(t) := D(t) + \int_0^\infty \tau_Z(t, t+\alpha) f(t+\alpha, t)\, d\alpha, \quad t \in \mathbf{R}.$$

It turns out that both $F^{\wedge R}(A)$ and $F^{\wedge L}(Z)$ are $m\times r$ matrix functions with entries from $L_\infty(\mathbf{R})$. The following proposition (cf., [3], Proposition 1.4) characterizes the left and right point evaluations.

PROPOSITION 2.1. *Let $F \in \mathcal{LW}^{m\times r}$, and let $A \in \mathcal{L}_\infty^{r\times r}(\mathbf{R})$ and $Z \in \mathcal{L}_\infty^{m\times m}(\mathbf{R})$ be anti-stable time varying points. Then $F^{\wedge R}(A)$ and $F^{\wedge L}(Z)$ are the unique elements in $L_\infty^{m\times r}(\mathbf{R})$ such that*

$$(F - F^{\wedge R}(A))(\Delta - A)^{-1} \in \mathcal{LW}^{m\times r}, \quad (\Delta - Z)^{-1}(F - F^{\wedge L}(Z)) \in \mathcal{LW}^{m\times r}.$$

In analogy with the time invariant case, if K is an integral operator in $\mathcal{W}^{m\times r}$ with kernel function $k(t,s)$, it is natural to define the *nonstationary total right half plane residue* of K to be the function

$$\mathcal{R}es^{+}(K)(t) := \lim_{h\downarrow 0} -\frac{1}{h}\int_{-h}^{0} k(t+\alpha, t)\, d\alpha.$$

In this terminology we have

$$F^{\wedge R}(A) = \mathcal{R}es^{+}(F(\Delta - A)^{-1}), \quad F^{\wedge L}(Z) = \mathcal{R}es^{+}((\Delta - Z)^{-1}F).$$

II.2. Rational input-output operators

In this section we give the definition of a rational input-output operator from $L_2^r(\mathbf{R})$ into $L_2^m(\mathbf{R})$. Consider a time-varying system of the form

$$\Sigma \begin{cases} x'(t) & = A(t)x(t) + B(t)u(t), \quad t \in \mathbf{R} \\ y(t) & = C(t)x(t) + D(t)u(t), \end{cases}$$

where $A \in \mathcal{L}_\infty^{n\times n}(\mathbf{R})$, $B \in \mathcal{L}_\infty^{n\times r}(\mathbf{R})$, $C \in \mathcal{L}_\infty^{m\times n}(\mathbf{R})$ and $D \in \mathcal{L}_\infty^{m\times r}(\mathbf{R})$. We also assume that the differential equation

(2.1) $$x'(t) = A(t)x(t), \quad t \in \mathbf{R},$$

has a dichotomy. The latter means (see [9], [11], [8]) that there exists a projection P of the state space $\mathbb{C}^n$ and constants $M > 0$, $0 < a < 1$ such that

(2.2a) $$\|\tau_A(t,0)P\tau_A(0,s)\| \le Ma^{t-s}, \quad t \ge s,$$

(2.2b) $$\|\tau_A(t,0)(I-P)\tau_A(0,s)\| \le Ma^{s-t}, \quad s \ge t.$$

Here $\tau_A(t,s)$ is the transition matrix function associated with (2.1). In [7] it is proved that (2.1) admits a *dichotomy* if and only if $(\Delta - A)^{-1}$ exists as a bounded operator on $L_2^n(\mathbf{R})$, and in this case

$$((\Delta - M_A)^{-1}\varphi)(t) = \int_{-\infty}^{\infty} \gamma_A(t,s)\varphi(s)\, ds, \quad t \in \mathbf{R}.$$

where

(2.3) $$\gamma_A(t,s) = \begin{cases} \tau_A(t,0)P\tau_A(0,s), & t > s, \\ -\tau_A(t,0)(I-P)\tau_A(0,s), & t < s. \end{cases}$$

Note that A is an anti-stable time varying point if and only if (2.1) has dichotomy $P = 0$.

In the system equations Σ we impose the condition that $x \in L_2^n(\mathbf{R})$. Then for each $u \in L_2^r(\mathbf{R})$ we can solve uniquely for $x(t)$, namely, $x = (\Delta - A)^{-1}Bu$, and hence the output y is uniquely determined from the input u by the formula $y = T_\Sigma u$, where $T_\Sigma : L_2^r(\mathbf{R}) \to L_2^m(\mathbf{R})$ is given by

(2.4) $$T_\Sigma = D + C(\Delta - A)^{-1}B.$$

In (2.4) we see A, B, C and D as operators of multiplication defined by the corresponding functions. In other words,

$$(T_\Sigma\varphi)(t) = D(t)\varphi(t) + \int_{-\infty}^{\infty} C(t)\gamma_A(t,s)B(s)\varphi(s)\,ds, \qquad t \in \mathbf{R}.$$

From the dichotomy inequalities (2.2a,b) and formula (2.3) one sees that $T_\Sigma \in \mathcal{W}^{m\times r}$. Formula (2.4) is the analogue of the realization formula for a rational matrix function with no poles on the extended imaginary axis in the time-invariant case.

Any operator from $L_2^r(\mathbf{R})$ into $L_2^m(\mathbf{R})$ of the form (2.4) will be called a *rational input-output operator.* The class consisting of these operators, which was introduced in [3], will be denoted by $\mathcal{R}^{m\times r}$. We write $T \in \mathcal{LR}^{m\times r}$ if $T \in \mathcal{R}^{m\times r}$ and T is lower triangular, i.e., the kernel function of T is zero a.e. on $s \geq t$, and hence, $\mathcal{LR}^{m\times r} \subset \mathcal{LW}^{m\times r}$.

II.3. The time-varying two-sided Nudelman interpolation problem in continuous time

For the continuous time time-varying two-sided Nudelman interpolation problem the given interpolation data form a set

$$\omega = (C_+, C_-, A; Z, B_+, B_-; \Gamma). \tag{3.1}$$

consisting of matrix functions with entries from $L_\infty(\mathbf{R})$ that have the following properties. The matrix functions A and Z are anti-stable time-varying points of sizes $M\times M$ and $N\times N$, respectively. The matrix functions C_+ and B_+ have sizes $r\times M$ and $N\times m$, respectively, and the pairs (C_-, A) and (Z, B_+) satisfy the following uniform controllability/observability requirements:

$$\int_{-\infty}^{t} \tau_A(\alpha, t)^* C_-(\alpha)^* C_-(\alpha)\tau_A(\alpha, t)\,d\alpha \geq \delta I_M, \tag{3.2}$$

$$\int_{t}^{\infty} \tau_Z(t, \alpha)B_+(\alpha)B_+(\alpha)^* \tau_Z(t, \alpha)^*\,d\alpha \geq \delta I_N, \tag{3.3}$$

with $\delta > 0$ a positive number independent of t. Here τ_A and τ_Z are the transition matrix functions associated with the anti-stable time varying points A and Z, respectively, and I_p stands for the $p\times p$ identity matrix. The matrix functions C_+, B_- and Γ in (3.1) are assumed to have sizes $m\times M$, $N\times r$, and $N\times M$, respectively.

The time-varying continuous time two-sided Nudelman interpolation problem asks to find all $F \in \mathcal{LW}^{m\times r}$ such that

$$\mathcal{R}\mathrm{es}^+ FC_-(\Delta - A)^{-1} = C_+, \tag{3.4}$$

$$\mathcal{R}\mathrm{es}^+(\Delta - Z)^{-1}B_+F = -B_-, \tag{3.5}$$

$$\mathcal{R}\mathrm{es}^+[(\Delta - Z)^{-1}B_+FC_-(\Delta - A)^{-1}] = \Gamma, \tag{3.6}$$

and

$$(3.7)\qquad \|F\| < 1.$$

In (3.4)-(3.6) we view the data from (3.1) as multiplication operators defined by the corresponding functions.

One can show that in order for the problem (3.4)-(3.6) to have a solution $F \in \mathcal{LW}^{m\times r}$ it is necessary that the matrix function Γ is absolutely continuous on compact intervals and satisfies the following time-varying Sylvester equation:

$$(3.8)\qquad \frac{d}{dt}\Gamma(t) + \Gamma(t)Z(t) + A(t)\Gamma(t) = B_+(t)C_+(t) + B_-(t)C_-(t), \quad t \in \mathbf{R}, \text{ a.e..}$$

We are now ready to state our main theorem for the continuous time case.

THEOREM 2.1. *Assume we have given interpolation data ω as in (3.1) such that (3.8) is satisfied. Let $\Lambda_1(t)$ and $\Lambda_2(t)$ be the unique solutions of the time-varying Sylvester equations*

$$(3.9)\qquad \frac{d}{dt}\Lambda_1(t) + \Lambda_1(t)A(t) + A(t)^*\Lambda_1(t) = C_-(t)^*C_-(t) - C_+(t)^*C_+(t), \quad t \in \mathbf{R},$$

$$(3.10)\qquad \frac{d}{dt}\Lambda_2(t) + \Lambda_2(t)Z(t)^* + Z(t)\Lambda_2(t) = B_+(t)B_+(t)^* - B_-(t)B_-(t)^*, \quad t \in \mathbf{R},$$

and consider the 2×2 block matrix function

$$(3.11)\qquad \Lambda(t) := \begin{pmatrix} \Lambda_1(t) & \Gamma(t)^* \\ \Gamma(t) & \Lambda_2(t) \end{pmatrix}, \quad t \in \mathbf{R}.$$

Then there exists a solution $F \in \mathcal{LW}^{m\times r}$ of the interpolation problem (3.4)–(3.6) satisfying the norm constraint (3.7) if and only if there is a positive number $\delta > 0$ (independent of t) such that

$$(3.12)\qquad \Lambda(t) \geq \delta I_{M+N}, \quad t \in \mathbf{R}.$$

In this case all lower triangular rational input-output operators $F \in \mathcal{LR}^{m\times r}$ satisfying (3.4)–(3.7) are given by

$$(3.13)\qquad F = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}$$

where G is an arbitrary element of $\mathcal{LR}^{m\times r}$ with $\|G\| < 1$ and the operator matrix

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} : L_2^m(\mathbf{R}) \oplus L_2^r(\mathbf{R}) \to L_2^m(\mathbf{R}) \oplus L_2^r(\mathbf{R})$$

is the rational input-output operator Θ given by

$$\Theta = \begin{pmatrix} I_m & 0 \\ 0 & I_r \end{pmatrix} + \begin{pmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{pmatrix} \begin{pmatrix} (\Delta - Z)^{-1} & 0 \\ 0 & (\Delta + A^*)^{-1} \end{pmatrix} \Lambda^{-1} \begin{pmatrix} -C_+^* & C_-^* \\ B_+^* & B_-^* \end{pmatrix},$$

where Λ^{-1} stands for the operator of multiplication by $\Lambda(\cdot)^{-1}$.

One can show that the interpolation problem (3.4)-(3.6) has a contractive solution $F \in \mathcal{LW}^{m\times r}$ if and only if the matrix $\Lambda(t)$ defined by (3.11) is positive semi-definite for each $t \in \mathbf{R}$. Furthermore, if (3.12) holds, then all contractive solutions $F \in \mathcal{LR}^{m\times r}$ of (3.4)-(3.6) are obtained by formula (3.13) provided the free parameter runs over all contractions $G \in \mathcal{LR}^{m\times r}$.

If the matrix functions C_+, C_-, A and Γ in (3.1) are vacuous, then the two-sided Nudelman problem reduces to the one-side problem (3.5) and (3.7), and in this case Theorem 3.1 follows from the results in Section 5.1 of [3]. The full proof of Theorem 3.1 will appear in [5].

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