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Calculating the anti-causal part of the inverse of a causal, time-varying discrete time system in the framework of sensitivity minimization

ABSTRACT

An algorithm is presented to calculate a state space representation of the anti-causal part of the inverse of a causal, discrete, time-varying system. The state space representation given of the latter system has the same number of inputs and the same number of outputs. Furthermore, we assume the state dimension to be constant. The results of the algorithm may play an essential role in calculating the solutions of time-variant sensitivity minimization problems by reduction to a Nevanlinna-Pick interpolation problem, see [1] (cf. [3]). The algorithm is characterized by three recursive equations, one of which is a Riccati equation. In an example, we illustrate the calculation of the initial conditions of these recursive equations when the plant changes from one time-invariant condition to another one.

1. Introduction

Nevanlinna-Pick interpolation theory constitutes one of the approaches in solving H_∞ control problems for time-invariant systems, see e.g. [2]. One possible way to formulate a Nevanlinna-Pick interpolation problem for the prototype H_∞ control problem, namely the sensitivity minimization problem, is based on the decomposition of the inverse of the plant to be controlled into a stable and anti-stable part. For time-invariant systems this is generally a trivial matter.

It is known (see [2]) that also in the time-variant case, the sensitivity minimization problem may be reformulated as a Nevanlinna-Pick type interpolation problem. For this reformulation to be effective one needs to know explicitly the anticausal part of the inverse of the plant P. In [3] this part is assumed to be given. The present paper makes the next step and provides a computational procedure to determine the anticausal part of P^{-1} (if it exists) in terms of a time-variant realization of the given plant P. We restrict to the class of systems with constant state dimensions and m inputs and m outputs.

The algorithm to calculate the anticausal part of P^{-1} consists of three major steps:

i. Let $I_{\bullet m}$ denote the (block-)diagonal operator with the identity matrix I_m as it diagonal entries, then the first step of the algorithm is to embed the given system P into a J-unitary system Θ, with $J = \begin{bmatrix} I_{\bullet m} & \\ & -I_{\bullet m} \end{bmatrix}$, such that,

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & P \end{bmatrix}$$

and

$$\Theta J \Theta^* = J \qquad \Theta^* J \Theta = J$$

ii. In the second step we calculate the unitary system Σ that is related to Θ as follows,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Theta_{11} - \Theta_{12}P^{-1}\Theta_{21} & -\Theta_{12}P^{-1} \\ P^{-1}\Theta_{21} & P^{-1} \end{bmatrix}$$

The left hand side is the time-variant analogue of the Redheffer transform of Θ, defined in the invariant case in [2]. Figure 1 represents the input-output relationship between Σ and Θ.

iii. The final step is the calculation of the anti-causal part of Σ_{22}.

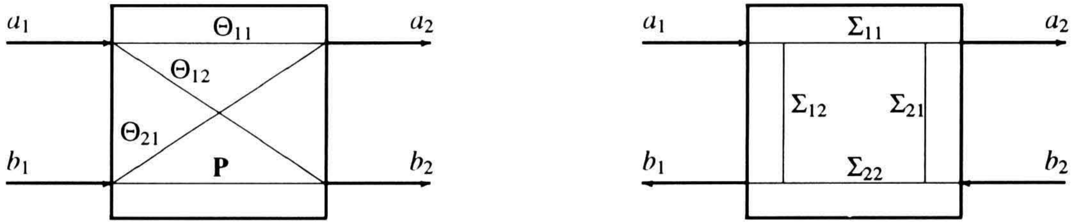

Figure 1: The J-unitary Θ section and corresponding unitary Σ section.

The key step of the algorithm is the first step and it is shown that a solution to this step exists when:

$$P^*P - I >> 0$$

that is there exists an $\varepsilon > 0$ such that for all (row) sequences u in ℓ_2:

$$||u(P^*P - I)u^*||_2 \leq \varepsilon||uu^*||_2$$

This condition can always be satisfied by scaling the plant P by some scalar parameter $\alpha > 1$ when $P^*P >> 0$. In that case, we obtain the anticausal part of P^{-1} by scaling the system obtained in the third step of the algorithm applied to αP by $\frac{1}{\alpha}$.

The outline of the paper is as follows. In section 2, we describe some basic concepts about the spaces and state space realizations that occur in this paper. The next two sections state the sensitivity minimization problem and outline how it can be transformed into a Nevanlinna-Pick interplation problem. These sections only serve as a motivation to the main contribution of this paper covered in section 5. In the latter section we present the algorithm to calculate the anticausal part of the inverse of the given plant. Section 6 illustrates the results of section 5 by means of a numerical simulation and finally section 7 concludes this paper with some remarks.

2. Some basic concepts about spaces and realizations

Spaces

The sequences analyzed in this paper are ℓ_2^p sequences. Here ℓ_2^p denotes the space of all row sequences $u = (u_j)_{j=-\infty}^{\infty}$ for which $\sum_{j=-\infty}^{\infty} ||u_j||^2 < \infty$. The zero-th element in the sequence must be distinguished, which we do by surrounding it with a square.

An operator T acting from $\ell_2^p \to \ell_2^p$ can be represented by a block matrix (operator) $[T_{ij}]_{-\infty<i,j<\infty}$, where each block T_{ij} has size $p \times p$. The central $(0,0)$ block is surrounded by a square. The action of T on a sequence $t \in \ell_2^p$ is represented by the (vector-matrix) product tT. The space of bounded linear operators on ℓ_2^p is denoted by $\mathcal{X}^{p\times p}$ or simply $\mathcal{X}$ when no confusion exists on the dimension. The subclass of lower triangular operators and upper triangular operators is denoted respectively by $\mathcal{L}^{p\times p}$ and $\mathcal{U}^{p\times p}$. The class of block diagonla operators $\mathcal{D}^{p\times p}$ is defined as the intersection $\mathcal{D}^{p\times p} = \mathcal{L}^{p\times p} \cap \mathcal{U}^{p\times p}$.

A key element of $\mathcal{U}^{p\times p}$ is the bilateral (forward) shift operator Z. Its action on a sequence $x \in \ell_2^p$ is given as:

$$\left[\ \cdots, x_{-1}, \boxed{x_0}, x_1, \cdots\ \right] Z = \left[\ \cdots, x_{-2}, \boxed{x_{-1}}, x_0, \cdots\ \right]$$

We also need a diagonal shift operator: the k-th diagonal shift of $A \in \mathcal{X}$ is $A^{(k)} = Z^{*k}AZ^k$ and will shift the entries of A over k positions into the South-East direction: $(A^{(k)})_{i,j} = A_{i-k,j-k}$.

Realizations

The state space realizations analyzed in this paper are defined by means of diagonal operators [4]. Herefore, we assemble the matrices $\{A(k)\}$, $\{B(k)\}$ etc. as operators on spaces of sequences of appropriate dimensions, by defining $A = \operatorname{diag}(A(k))$, $B = \operatorname{diag}(B(k))$, $C = \operatorname{diag}(C(k))$ and $D = \operatorname{diag}(D(k))$. Let ℓ_2^m be the space of input sequences, ℓ_2^n the space of output sequences, and let ℓ_2^N be the space of the state sequences. If all operators $\{A(k)\}$, $\{B(k)\}$, etc. are uniformly bounded over k, then A, B, etc. may be viewed as bounded diagonal operators which together define a realization $\mathbf{T}$ of the map T between the input u and the output y,

$$\begin{aligned} xZ^{-1} &= xA + uB \\ y &= xC + uD \end{aligned}$$

For short we will use the notation

$$\mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

to represent the state space representation of a realization of T. Let the spectral radius of the operator AZ be strictly smaller than 1, then

$$T = D + BZ(I - AZ)^{-1}C$$

exist as a bounded upper operator. This class of systems will be members of the class of causal systems. Similarly, when the spectral radius of Z^*A is smaller than 1, then

$$T = D + B(I - Z^*A)^{-1}Z^*C$$

exists as a bounded lower operator. As such these systems are members of the class of anti-causal systems. Generally, if the spectral radius of both A_1Z and Z^*A_2 are strictly smaller then 1, then the class of mixed causal and anti-causal systems treated in this paper have a representation as,

$$T = D + B_1Z(I - A_1Z)^{-1}C_1 + B_2(I - Z^*A_2)^{-1}Z^*C_2$$

For a system $T \in \mathcal{U}$, with realization $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$, the controllability operator M_c satisfies:

$$M_c^{(-1)} = A^*M_cA + B^*B$$

and for this system the observability operator M_o satisfies

$$M_o = AM_o^{(-1)}A + CC^*$$

The system is defined to be minimal if both $M_c \gg 0$ and $M_o \gg 0$.

3. The sensitivity minimization problem

Assume the feedback configuration in figure 2. Here the plant P and the controller K are input-output maps in $\mathcal{X}^{m\times m}$. The plant P is assumed to be causal and given by the following finite dimensional linear, time-varying, state space representation:

$$\mathbf{P} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \qquad \begin{matrix} A \in \mathcal{D}^{N\times N}, C \in \mathcal{D}^{N\times m} \\ B \in \mathcal{D}^{m\times N}, D \in \mathcal{D}^{m\times m} \end{matrix}$$

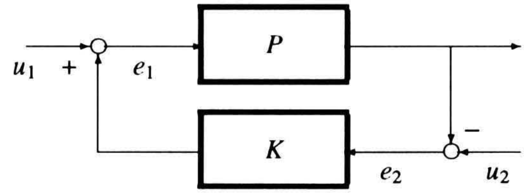

Figure 2:

and the task is to find a time-varying state space representation for the controller K.

The closed-loop system in figure 2 is described by the systems of equations:

$$\begin{cases} e_1 &= u_1 + e_2 K \\ e_2 &= u_2 - e_1 P \end{cases}$$

The overall closed-loop system is assumed to be *well-posed*, that is we can solve the above system of equations for the internal signals $[e_1\ ,\ e_2]$ in terms of the input respectively disturbance signals $[u_1\ ,\ u_2]$. This is indeed the case when the operator $(I+PK)$ has a bounded inverse and then the map H from $[u_1\ ,\ u_2]$ to $[e_1\ ,\ e_2]$ is in $\mathcal{X}^{m \times m}$ and given by:

$$H = \begin{bmatrix} (I+PK)^{-1} & -P(I+KP)^{-1} \\ K(I+PK)^{-1} & (I+KP)^{-1} \end{bmatrix}$$

An important notion of the closed-loop system is *internal stability*. This is defined next.

Definition 1: The system P in figure 2 is internally stabilized by the controller K if and only if H is causal, that is $H \in \mathcal{U}^{m \times m}$. □

The sensitivity map S is defined as the map from u_2 to e_2, that is is:

$$S = (I+KP)^{-1} \tag{1}$$

We are now in a position to state the sensitivity minimization problem: *For a given tolerance level $\Gamma \in \mathcal{D}$, with $\Gamma^*\Gamma > 0$ and an outer (i.e. stable and with stable inverse) weighting map $W \in \mathcal{U}$ find a compensator K (if any exist) such that:*

i. the closed-loop system H is internally stable,

ii. $$\|WS\Gamma^{-1}\| < 1 \tag{2}$$

Here $\|WS\Gamma^{-1}\|$ is the norm of $WS\Gamma^{-1}$ as a bounded operator on ℓ_2^m.

4. The sensitivity minimization problem as a Nevanlinna-Pick interpolation problem

In the time-varying context, the Youla parametrization of all internally stabilizing controllers remains valid. This is stated in the first theorem.

Theorem 1 *Let the plant $P \in \mathcal{U}$, then for any controller K internally stabilizing H there exists a $Q \in \mathcal{U}$ such that,*

$$K = (I - QP)^{-1}Q \tag{3}$$

Conversely, for any such Q the controller given by Eq. (3) internally stabilizes H.

Proof: The proof can be found in [1]. □

With the parametrization of K in (3), the sensitivity map S becomes:

$$S = I - QP \tag{4}$$

Suppose that in addition to the stability of P, its inverse $P^{-1} \in \mathcal{X}$ and decomposed as:

$$\begin{aligned} P^{-1} &= D_i + \underbrace{B_1 Z(I - A_1 Z)^{-1} C_1}_{\in \mathcal{U}} + \underbrace{B_2(I - Z^* A_2)^{-1} Z^* C_2}_{\in \mathcal{L}Z^{-1}} \\ &= D_i + P_{iu} + P_{i\ell} \end{aligned} \tag{5}$$

then we have from Eq. (4):

$$(I - S) = QP \quad \in \mathcal{U} \tag{6}$$

Using the expression for P^{-1} in Eq. (5), we can state that:

$$QPP^{-1} = QP(D_i + P_{iu} + P_{i\ell}) \quad \in \mathcal{U}$$

and hence,

$$QPP_{i\ell} \quad \in \mathcal{U}$$

or with Eq. (6),

$$(I - S)P_{i\ell} \quad \in \mathcal{U} \tag{7}$$

With the expression for $P_{i\ell}$ in Eq. (5) and the definition of $J \in \mathcal{X}$ as $\begin{pmatrix} I_{\bullet m} & 0 \\ 0 & -I_{\bullet m} \end{pmatrix}$, the Hermitian transpose of this last expression can be denoted as:

$$C_2^* Z(I - A_2^* Z)^{-1} \begin{bmatrix} B_2^* \Gamma^* & B_2^* \end{bmatrix} J \begin{bmatrix} \Gamma^{-*} S^* \\ I \end{bmatrix} \quad \in \mathcal{L} \tag{8}$$

Based on this outline, the task in the sensitivity minimization problem can be stated more precisely as: *Find the sensitivity map $(S\Gamma^{-1}) \in \mathcal{U}$ such that it satisfies the interpolation condition in (8) and the norm condition in (2) for $W = I$.*

A completely analogous interpolation problem results when using an inner-outer factorization of P. The solution to this interpolation problem for $W \neq I$ is treated in [1]. In the present paper we focus on a computational procedure to calculate the anti-causal part of P^{-1}.

A similar reduction of the time-variant sensitivity minimization problem to a Nevanlinna-Pick interpolation problem was given in [3]. The reduction in [3] does however not rely on the Youla parametrization and in addition requires the inverse of the plant P to be given in a special (diagonal) series representation. The sensitivity minimization problem in Eq. (8) only requires a state space realization of the anti-causal part of P^{-1}. In the next section we present an algorithm to calculate such a realization starting from a given state space realization of P. The calculations are completely performed in a state space framework.

5. Determination of the anti-causal part of P^{-1}

5.1. Embedding of a system P in a J-unitary system Θ

5.1.1. The embedding problem

Let the system $P \in \mathcal{U}^{m\times m}$ be represented by the rational transfer function operator $D + BZ(I - AZ)^{-1}C$. Then the embedding problem is to find the diagonal operators (of appropriate dimension) C_1, B_1, D_{11}, D_{12} and D_{21} in combination with the similarity transformation X, the state signature J_X and the signature operator J_2, such that:

$$
\begin{bmatrix} X^{-*(-1)} & & \\ & I & \\ & & I \end{bmatrix}
\begin{bmatrix} A^* & B_1^* & B^* \\ C_1^* & D_{11}^* & D_{21}^* \\ C^* & D_{12}^* & D^* \end{bmatrix}
\begin{bmatrix} X^* & & \\ & I & \\ & & I \end{bmatrix}
\begin{bmatrix} J_X & & \\ & I & \\ & & -I \end{bmatrix}
$$

$$
\begin{bmatrix} X & & \\ & I & \\ & & I \end{bmatrix}
\begin{bmatrix} A & C_1 & C \\ B_1 & D_{11} & D_{12} \\ B & D_{21} & D \end{bmatrix}
\begin{bmatrix} X^{-(-1)} & & \\ & I & \\ & & I \end{bmatrix}
=
\begin{bmatrix} J_X^{(-1)} & & \\ & J_2 & \\ & & -I \end{bmatrix}
\tag{9}
$$

Here $X^{-(-1)}$ denotes $(X^{-1})^{(-1)}$ and J_2 represents a signature operator, with signature to be determined in section 5.2.4, see Remark 1.

5.1.2. Relationships to be satisfied

From Eq. (9) we derive,

$$
\begin{bmatrix} A^*X^*J_XX & B_1^* & -B^* \\ C_1^*X^*J_XX & D_{11}^* & -D_{21}^* \\ C^*X^*J_XX & D_{12}^* & -D^* \end{bmatrix}
\begin{bmatrix} A & C_1 & C \\ B_1 & D_{11} & D_{12} \\ B & D_{21} & D \end{bmatrix}
=
\begin{bmatrix} X^{*(-1)}J_X^{(-1)}X^{(-1)} & & \\ & J_2 & \\ & & -I \end{bmatrix}
$$

Let $X^* J_X X$ be denoted by M, then we derive the following two sets of equations from the last expression:

$$A^*MA + B_1^*B_1 - B^*B = M^{(-1)} \tag{10}$$

$$A^*MC + B_1^*D_{12} - B^*D = 0 \tag{11}$$

$$C^*MC + D_{12}^*D_{12} - D^*D = -I \tag{12}$$

and,

$$\begin{bmatrix} A^*M & B_1^* & -B^* \\ C^*M & D_{12}^* & -D^* \end{bmatrix} \begin{bmatrix} C_1 \\ D_{11} \\ D_{21} \end{bmatrix} = 0 \tag{13}$$

$$C_1^*MC_1 + D_{11}^*D_{11} - D_{21}^*D_{21} = J_2 \tag{14}$$

Denote Eqs. (12) and (11) respectively as,

$$D_{12}^*D_{12} = D^*D - I - C^*MC \tag{15}$$

$$B_1^*D_{12} = B^*D - A^*MC \tag{16}$$

The question is when can this set of equations (10)-(15)- (16)-(13)-(14) be solved for the unknowns?

5.1.3. A sufficient condition

In this section we assert that when:

$$P^*P >> I$$

the operator M defined in the previous subsection can be computed as the solution of a Riccati difference equation. This assertion can be proved using the extension of the positive real lemma to the present time varying context, such as presented in [7]. We state this lemma next without proof. For a proof we refer to [7].

Lemma 2 *(The time-varying positive real lemma)*
Let $T \in \mathcal{U}$ be given as,

$$T = L + GZ(I - FZ)^{-1}K$$

with $\begin{bmatrix} F & K \\ G & L \end{bmatrix}$ *a minimal realization of T, and let $(T^* + T)^{\pm 1} \in \mathcal{X}$, then $T^* + T >> 0$ if and only if there exists diagonal operators R, Q, B with $R >> 0$ and $Q >> 0$ satisfying the following relationships:*

$$R^{(-1)} = F^*RF + B^*QB \tag{17}$$

$$B^*Q = G^* - F^*RK \tag{18}$$

$$Q = L + L^* - K^*RK \tag{19}$$

To apply this Lemma, we first derive a state space representation for $P^*P - I$ in terms of the minimal realization of P. This is done in the next Lemma.

Lemma 3 *Let $P \in \mathcal{U}$ be given as:*

$$P = D + BZ(I - AZ)^{-1}C$$

with $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ *a minimal realization of P, and let M_c be the controllability operator satisfying:*

$$M_c^{(-1)} = A^*M_cA + B^*B \tag{20}$$

then,

$$P^*P-I = D^*D-I+C^*M_cC+C^*(I-Z^*A^*)^{-1}Z^*(B^*D+A^*M_cC)+(D^*B+C^*M_cA)Z(I-AZ)^{-1}C \tag{21}$$

Proof: The proof is straightforward and directly follows from Lemma 1 of [7]. □

Although the given realization of P is minimal, this does not necessary hold for the realization $\begin{bmatrix} A & C \\ D^*B + C^*M_cA & D^*D - I + C^*M_cC \end{bmatrix}$. When the latter system indeed is not minimal, Lemma 1 cannot be applied. However, as shown in [7], Lemma 1 can be extended for non-uniformly controllable systems. In that case the solution R to Eqs. (17- 19) can only be guaranteed to be positive semidefinite.

Hence, based on the last two Lemma's we have the following theorem.

Theorem 4 *Let $P^*P - I >> 0$, let $P \in \mathcal{U}$ be given as:*

$$P = D + BZ(I - AZ)^{-1}C$$

with $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ *a minimal realization of P, let $(P^*P - I)^{\pm 1} \in \mathcal{X}$, then there exists $R \in \mathcal{D}$ and ≥ 0 satisfying:*

$$R^{(-1)} = A^*RA + (B^*D - A^*(R - M_c)C)(D^*D - I - C^*(R - M_c)C)^{-1}(D^*B - C^*(R - M_c)A) \tag{22}$$

with

$$D^*D - I - C^*(R - M_c)C >> 0 \tag{23}$$

and M_c satisfying Eq. (20).

Proof: From Lemma 2, we can take the state space representation $[F, G, K, L]$ in Lemma 1 as:

$$F = A \quad G = D^*B + C^*M_cA \quad K = C \quad L + L^* = D^*D - I + C^*M_cC$$

Hence by Lemma 1 there exists an R satisfying the conditions stipulated in the theorem. By the same Lemma the condition in Eq. (23) is also satisfied. □

Corollary 5 *Let the conditions of Theorem 2 be satisfied, and let the operator $M \in \mathcal{D}$ be defined as:*

$$M = R - M_c \tag{24}$$

then, M satisfies the Riccati equation:

$$A^*MA + (B^*D - A^*MC)(D^*D - I - C^*MC)^{-1}(D^*B - C^*MA) - B^*B = M^{(-1)} \tag{25}$$

as well as Eqs. (10-12).

Proof: With M defined as in Eq. (24), Eq. (23) and Eq. (15) show that:

$$D_{12}^*D_{12} >> 0$$

Hence $D_{12}^*D_{12} \in \mathcal{D}^{m \times m}$ is invertible.

To prove the second part, subtract Eq. (20) from Eq. (22) to see that M defined in Eq. (24) satisfies Eq. (25). That the latter Riccati equation results from Eq. (10-12) can easily be seen, using the fact that:

$$D_{12}(D_{12}^*D_{12})^{-1}D_{12} = I_{\bullet m}$$

in Eq. (10) as follows,

$$A^*MA + B_1^*D_{12}(D_{12}^*D_{12})^{-1}D_{12}B_1 - B^*B = M^{(-1)}$$

Hence with Eqs. (15-16), the latter equation precisely becomes the Riccati equation in Eq. (25). □

The solution M to Eq. (25) determines the similarity transformation X, the state signature J_X and the operators D_{12} and B_1 up to an orthogonal transformation on the left. In this paper, we will assume that the solution M derived in Corollary 1 is invertible. In that case, the similarity transformation X is invertible and we can define the quantities:

$$\begin{bmatrix} XAX^{-(-1)} & XC \\ B_1X^{-(-1)} & D_{12} \\ BX^{-(-1)} & D \end{bmatrix}$$

5.1.4. Completing the embedding problem

In section 5.3 we will explicitly show that the only quantities necessary in the calculation of the anti-causal part of P^{-1} are those obtained at the end of the previous section. Hence, completing the embedding problem is not necessary. Nevertheless, in this section we discuss the completion of the embedding problem. This discussion is held on a local time scale. That is we look for a series of matrices V_k (of appropriate dimensions) with independent columns such that:

$$\begin{bmatrix} A_k^* & B_{1,k}^* & B_k^* \\ C_k^* & D_{12,k}^* & D_k^* \end{bmatrix} \begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} V_k = 0 \tag{26}$$

and then set the remaining unknown matrix triplet $\begin{bmatrix} C_{1,k} \\ D_{11,k} \\ D_{21,k} \end{bmatrix}$ equal to $V_k U_k$, with the matrix U_k and $J_{2,k}$ determined from:

$$V_k^* \begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} V_k = U_k^{-*} J_{2,k} U_k^{-1} \tag{27}$$

We now study the existence of the matrices V_k, U_k and $J_{2,k}$. Since, M_k are invertible for $\forall k$, we can easily deduce from the relationship:

$$\begin{bmatrix} A_k^* & B_{1,k}^* & B_k^* \\ C_k^* & D_{12,k}^* & D_k^* \end{bmatrix} \begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} \begin{bmatrix} A_k & C_k \\ B_{1,k} & D_{12,k} \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} M_{k+1} & \\ & -I_m \end{bmatrix}$$

derived from Eq. (9) using the equality $M = X^* J_X X$, that the columns of the matrix $\begin{array}{c|c} & \begin{array}{cc} N & m \end{array} \\ \begin{array}{c} N \\ m \\ m \end{array} & \begin{bmatrix} A_k & C_k \\ B_{1,k} & D_{12,k} \\ B_k & D_k \end{bmatrix} \end{array}$

are independent. Therefore, since the matrix $\begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix}$ is a square invertible matrix, the same holds for the columns of the matrix,

$$\begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} \begin{bmatrix} A_k & C_k \\ B_{1,k} & D_{12,k} \\ B_k & D_k \end{bmatrix}$$

Hence we can select m column vectors that form the matrix V_k, such that Eq. (26) is satisfied.

With V_k determined, it is always possible to determine form Eq. (27) a non-singular square matrix U_k and a signature matrix $J_{2,k}$ with possibly +1's, −1's and zeros on the diagonal. In the next lemma it is shown that no zeros can be present in the signature $J_{2,k}$.

Lemma 6 *Let $V_k \in \mathbf{R}^{N_k+2m \times N_k-N_{k+1}+m}$ with independent columns, satisfy the following relationship:*

$$\begin{bmatrix} A_k^* & B_{1,k}^* & B_k^* \\ & V_k^* & \\ C_k^* & D_{12,k}^* & D_k^* \end{bmatrix} \begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} \begin{bmatrix} A_k & & C_k \\ B_{1,k} & V_k & D_{12,k} \\ B_k & & D_k \end{bmatrix} = \begin{bmatrix} M_{k+1} & & \\ & J_{2,k} & \\ & & -I_m \end{bmatrix} \tag{28}$$

and let the matrices M_k be invertible for $\forall k$, then the matrix $\begin{bmatrix} A_k & & C_k \\ B_{1,k} & V_k & D_{12,k} \\ B_k & & D_k \end{bmatrix}$ *is square and invertible.*

Proof: Suppose invertibility does not hold, hence:

$$\exists \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq 0 : \begin{bmatrix} A_k & & C_k \\ B_{1,k} & V_k & D_{12,k} \\ B_k & & D_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{29}$$

However, from Eq. (28) we derive that:

$$\begin{bmatrix} A_k^* & B_{1,k}^* & B_k^* \end{bmatrix} \begin{bmatrix} M_k & & \\ & I_m & \\ & & -I_m \end{bmatrix} \begin{bmatrix} A_k & & C_k \\ B_{1,k} & V_k & D_{12,k} \\ B_k & & D_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = M_{k+1}x_1$$

Since, M_{k+1} is non-singular $x_1 = 0$. In the same way, we can prove that $x_3 = 0$. Hence (29) holds if,

$$V_k x_2 = 0$$

However, since the columns of V_k are independent $x_2 = 0$ and the proof of the lemma is completed. □

The result of the above lemma show that the signature matrix $J_{2,k}$ cannot have zeros on the diagonal.

Remark 7 *Let $\#_+(M_k)$ denote the number of $+1$'s in the inertia of M_k and similarly $\#_-(M_k)$ denoting the number of -1's, then we derive fromEq. (28) and the result of Lemma 3 that,*

$$\begin{aligned} \#_+(M_k) + m &= \#_+(M_{k+1}) + \#_+(J_{2,k}) \\ \#_-(M_k) + m &= \#_-(M_{k+1}) + \#_-(J_{2,k}) + m \end{aligned} \tag{30}$$

In [8] it was shown that when the state dimension is constant for $\forall k$ and when the transition matrix of P^{-1} is non-singular, $\#_-(M_k)$ remains constant. Therefore, the last relationship shows that,

$$\#_-(J_{2,k}) = 0 \tag{31}$$

and hence $\#_+(J_{2,k}) = m$.

5.2. The calculation of the corresponding unitary Σ system

With the help of figure 2, the state representation of the J-unitary system Θ determined in the previous subsection can be denoted as:

$$\begin{bmatrix} x & a_1 & b_1 \end{bmatrix} \begin{bmatrix} XAX^{-(-1)} & XC_1 & XC \\ B_1X^{-(-1)} & D_{11} & D_{12} \\ BX^{-(-1)} & D_{21} & D \end{bmatrix} = \begin{bmatrix} xZ^{-1} & a_2 & b_2 \end{bmatrix} \tag{32}$$

Let us consider, similar to the exposure given in [4], a more general form of this state representation. Namely, the same set of equations hold when the input, output and state quantities belong to $\mathcal{X}_2^{1\times m}$, $\mathcal{X}_2^{1\times m}$ and $\mathcal{X}_2^{1\times N}$ respectively.

According to the signature operator J_X, the state space sequence ℓ_2^N decomposes into two complementary space sequences $\ell_2^N = \ell_2^{N_+} \oplus \ell_2^{N_-}$. Let any state sequence $x \in \mathcal{X}_2^{1\times N}$ be partitioned accordingly into $x = \begin{bmatrix} x_+ & x_- \end{bmatrix}$, with $x_+ \in \mathcal{X}_2^{1\times N_+}$ and $x_- \in \mathcal{X}_2^{1\times N_-}$, then the state space representation of Θ can be written as:

$$\begin{bmatrix} x_+ & x_- & a_1 & b_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{21} & \gamma_{22} \\ \beta_{11} & \beta_{12} & \delta_{11} & \delta_{12} \\ \beta_{21} & \beta_{22} & \delta_{21} & \delta_{22} \end{bmatrix} = \begin{bmatrix} x_+Z^{-1} & x_-Z^{-1} & a_2 & b_2 \end{bmatrix}$$

Using the J-unitary property of this state space realization, it is shown in [4] that the corresponding unitary system Σ always exists and that the following block-diagonal operators are well-defined:

$$\begin{bmatrix} F_{11} & H_{11} \\ G_{11} & K_{11} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{11} & \delta_{11} \end{bmatrix} - \begin{bmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{bmatrix} \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{bmatrix}$$

$$\begin{bmatrix} F_{12} & H_{12} \\ G_{12} & K_{12} \end{bmatrix} = -\begin{bmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{bmatrix} \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} F_{21} & H_{21} \\ G_{21} & K_{21} \end{bmatrix} = \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{bmatrix}$$

$$\begin{bmatrix} F_{22} & H_{22} \\ G_{22} & K_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1}$$

The latter operators define the state representation of the corresponding unitary Σ section as:

$$\begin{bmatrix} x_+ & x_-Z^{-1} & a_1 & b_2 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & H_{11} & H_{12} \\ F_{21} & F_{22} & H_{21} & H_{22} \\ G_{11} & G_{12} & K_{11} & K_{12} \\ G_{21} & G_{22} & K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} x_+Z^{-1} & x_- & a_2 & b_1 \end{bmatrix} \tag{33}$$

5.3. Calculating a realization of the anti-causal part of P^{-1}

The final step in the calculation of the anti-causal part of P^{-1} is the calculation of the anti-causal part of Σ_{22}. For this we borrow a result from [4].

Let the input quantity $b_2 \in \mathcal{X}_2$ be decomposed as $b_2 = b_{2p} + b_{2f}$, with $b_{2p} = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(b_2) \in \mathcal{L}_2 Z^{-1}$ the 'past' part of the signal (with reference to its 0-th diagonal) and $b_{2f} = \mathbf{P}(b_2) \in \mathcal{U}_2$ the 'future' part, and similarly let the other input and output quantities of Σ be decomposed in a 'past' and 'future', then the anti-causal part is the transfer from b_{2f} to b_{1p}. A realization of this transfer has been determined in [4]. This result is summarized in the following proposition. Here we make use of the additional notation that $x_{[k]}$ denotes the k-th diagonal above the 0-th diagonal of the operator $x \in \mathcal{X}_2$.

Proposition 8 *Let the operators S and R define the following mapping:*

$$x_{-[0]}S = x_{+[0]} \quad \text{and} \quad x_{-[1]}S = x_{+[1]} \quad \text{for} \quad a_{1p} = b_{2p} = 0 \text{ and } a_{1[0]} = b_{2[0]} = 0$$

$$x_{+[0]}R = x_{-[0]} \quad \text{for} \quad a_{1f} = b_{2f} = 0$$

then both S and R are contractive and determined by the following recursions:

$$\begin{aligned} S^{(-1)} &= F_{21} + F_{22}(I - SF_{12})^{-1}SF_{11} \\ R &= F_{12} + F_{11}(I - R^{(-1)}F_{21})^{-1}R^{(-1)}F_{22} \end{aligned}$$

A state space realization for the strictly lower (anti-causal) part of Σ_{22} is given as:

$$\begin{aligned} x_{-[0]} &= x_{-[1]}^{(-1)}A_2 + b_{2[0]}B_2 \\ b_{1[0]} &= x_{-[1]}^{(-1)}C_2 \end{aligned}$$

with the diagonal operators (A_2, B_2, C_2) given as:

$$\begin{aligned} A_2 &= F_{22}(I - SF_{12})^{-1} \\ C_2 &= H_{22} + F_{22}(I - SF_{12})^{-1}SH_{12} \\ B_2 &= (G_{22} + G_{21}R^{(-1)}(I - F_{21}R^{(-1)})^{-1}F_{22})(I - SR)^{-1} \end{aligned}$$

6. An Example

In the present example, the following Single-Input Single-Output (SISO) system P is considered:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}_{k+1} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}_k \begin{bmatrix} -a_{1,k} & 1 \\ -a_{2,k} & 0 \end{bmatrix} + \kappa u_k \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{34}$$

$$y_k = \begin{bmatrix} x_1 & x_2 \end{bmatrix}_k \begin{bmatrix} -(b_{1,k} + b_{2,k}) \\ b_{1,k}b_{2,k} \end{bmatrix} + \kappa u_k \tag{35}$$

Here κ is a scalar, constant scaling parameter, used to guarantee that the condition $P^*P >> I$ holds. This was the case when κ was chosen equal to 5. The other parameter values are fixed in the following way:

$$
a_{1,k} = \begin{cases} -1.5 & k < 0 \\ 0.0005k^2 + 0.022k - 1.5 & 0 \le k \le 10 \\ -1.2 & k > 10 \end{cases}
$$

$$
a_{2,k} = \begin{cases} 0.7 & k < 0 \\ 0.02k + 0.7 & 0 \le k \le 10 \\ 0.9 & k > 10 \end{cases}
$$

$$
b_{1,k} = \begin{cases} 1.5 & k < 0 \\ 0.2k + 1.5 & 0 \le k \le 10 \\ 1.7 & k > 10 \end{cases}
$$

$$
b_{2,k} = 0.7 \qquad -\infty < k < \infty
$$

With this model, the solution to Eq. (20) and Eq. (22) are computed. Since these are both recursive equations we pay some attention to the calculation of the initial conditions. The system is assumed to be constant for $k < 0$, hence, the matrix $M_{c,0}$ satisfies the (classical) Lyapunov equation:

$$
M_{c,0} = A_0^* M_{c,0} A_0 + B_0^* B_0
$$

Similarly the matrix R_0 satisfies the algebraic Riccati equation:

$$
R_0 = A_0^* R_0 A_0 + (B_0^* D_0 - A_0^*(R_0 - M_{c,0})C_0)(D_0^* D_0 - I - C_0^*(R_0 - M_{c,0})C_0)^{-1}(D_0^* B_0 - C_0^*(R_0 - M_{c,0})A_0)
$$

The solution to this algebraic Riccati equation can be obtained via the solution of a generalized eigenvalue problem, see [10].

With the solutions to Eq. (20) and Eq. (22), the solution to Eq. (25) is given as in Corollary 1. The signature matrix J_X in this case was equal to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the calculated state space representation of the Θ sections was $\begin{bmatrix} I_{\bullet 1} & & & \\ & -I_{\bullet 1} & & \\ & & I_{\bullet 1} & \\ & & & -I_{\bullet 1} \end{bmatrix}$ $(= J')$- unitary up to machine precision.

Using the obtained state space realization of the J'-unitary Θ-sections, the second major step was the calculation of a state space realization of the corresponding unitary Σ-section. Again these realizations were unitary up to machine precision. The calculations are performed for k running from 1 until some upper bound k_0 when the calculations have converged. In the present example k_0 was taken equal to 100.

Finally, the strictly anti-causal part of P^{-1} is calculated as described in Proposition 1. Again we pay attention to the calculation of the initial conditions to the recursions for the S and R operators.

The matrix S_k defined in Proposition 1 satisfies the following the following recursion:

$$S_{k+1} = F_{21,k} + F_{22,k}(I - S_k F_{12,k})^{-1} S_k F_{11,k}$$

Since for $k \le 0$, the matrices $F_{11,k}$, $F_{12,k}$, $F_{21,k}$ and $F_{22,k}$ are constant, S remains constant for $k \le 0$. Denote this quantity by S_0. Then S_0 satisfies the algebraic equation:

$$S_0 = F_{21,0} + F_{22,0}(I - S_0 F_{12,0})^{-1} S_0 F_{11,0}$$

Using the contractivity property of S_k, the solution to this equation is obtained by calculating S_k' according to the following recursive equation:

$$S_{k+1}' = F_{21,0} + F_{22,0}(I - S_k' F_{12,0})^{-1} S_k' F_{11,0}$$

using the initial conditions $S_0' = 0$. The recursions with the above equation are stopped when for some tolerance level ε, $\|S_{k+1}' - S_k'\|_2 \le \varepsilon$.

In the same way the recursion that determines R_k is initiated now starting from the other end $k = k_0$. The constancy of the matrices $F_{11,k}$, $F_{12,k}$, $F_{21,k}$ and $F_{22,k}$ for $k \ge k_0$ is thereby crucial.

Using the results of the previous two steps the operators A_2, C_2 and B_2 were calculated again as outlined in Proposition 1. To check the obtained results, we calculated the impulse response at time instant 0 of the system,

$$(P^{-1} - P_{i\ell}) \tag{36}$$

using the causal (non-stable) state space realization of both operators. Though the system matrices are bounded in both realizations, this need not to be the case in general.

According to the decomposition of P^{-1} as given in Eq. (5), this is equal to the impulse response of $D_i + P_s$, with $D_i \in \mathcal{D}$ and $P_s \in \mathcal{U}$, and hence has to be a stable causal impulse response. Let the impulse response of P^{-1} be denoted by y and that of $P_{i\ell}$ as w, then indeed the quantity $\frac{|y-w|}{|y|}$, not shown here for the sake of brevity, corresponds to the impulse response of a causal (stable) system.

7. Concluding remarks

In this paper an algorithm is derived to calculate a state space representation for the strictly anti-causal part of the inverse of a causal system P. A state space realization of the latter system is assumed to be known. It is assumed that the plant P is square and that the state dimension of the given state representation of P is constant.

The use of the algorithm is shown in formulating the prototype problem of robust control, namely sensitivity minimization, as a Nevanlinna-Pick interpolation problem.

Another decomposition of the system that leads to a similar interpolation problem is the outer-inner factorization. This is discussed in [1]. When using the recent strategy in [9] to do outer-inner factorization the key equation to be solved is a Lyapunov type of equation in stead of a Riccati equation, as is the case in the present paper.

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