P. Bruinsma, A. Dijksma, H. de Snoo

Models for generalized Carathéodory and Nevanlinna functions

ABSTRACT

Operator valued Carathéodory functions and Nevanlinna functions with κ negative squares have operator representations involving unitary or selfadjoint relations in Pontryagin spaces. When $\kappa = 0$ they reduce to the classical integral representations of the Nevanlinna-Riesz-Herglotz type by means of the spectral representation of the unitary or selfadjoint relations involved. In this paper it is shown that these operator representations can be obtained systematically by means of the realization of Schur functions with κ negative squares.

1. Introduction

Let $\mathbf{L}(\mathbf{F})$ be the algebra of all bounded linear operators on a Hilbert space $\mathbf{F}$ with inner product $[.,.]$. An $\mathbf{L}(\mathbf{F})$ valued function $\mathsf{K}(z,w)$ defined for z and w in some set Ω is said to have κ negative squares, where $\kappa \in \mathbb{N}$, if $\mathsf{K}(z,w)^* = \mathsf{K}(w,z)$ for all $z, w \in \Omega$ and if, moreover, for every choice of $n \in \mathbb{N}$, of elements $f_1, \ldots, f_n \in \mathbf{F}$ and of points $z_1, \ldots, z_n \in \Omega$, the $n \times n$ heritian matrix with ij-th entry $[\mathsf{K}(z_i, z_j)f_i, f_j]$ has at most κ negative eigenvalues and exactly κ negative eigenvalues for some choice of $n, f_1, \ldots, f_n$ and $z_1, \ldots, z_n$.

A function $\Phi : \mathbb{D} \to \mathbf{L}(\mathbf{F})$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$, belongs to the class $\mathbf{C}_\kappa(\mathbf{F})$ of all (generalized) Carathéodory functions, if it is meromorphic on $\mathbb{D}$, holomorphic at 0 and if the kernel

$$\frac{\Phi(z) + \Phi(w)^*}{1 - z\bar{w}}, \quad z, w \in \mathcal{D}(\Phi),$$

has κ negative squares. Here and elsewhere in this paper $\mathcal{D}(\Phi)$ denotes the domain of holomorphy of the function Φ in $\mathbb{D}$, which contains 0. Functions of this class have operator representations involving unitary operators in Pontryagin spaces. When $\kappa = 0$ they reduce to the classical integral representations of the Nevanlinna-Riesz-Herglotz type by means of the spectral representation of the unitary operator involved. These operator representations were first proved by Kreĭn and Langer in [18], [19], [20]. They used a construction, to which we refer as the ε-method and which goes back to Kreĭn (see [17]), to obtain a Pontryagin space and a unitary operator to represent the function Φ.

In this paper we prove the operator representation of functions belonging to the class $\mathbf{C}_\kappa(\mathbf{F})$ in a different way, namely via the realization of (generalized) Schur functions. A function $\Theta : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ belongs to the class $\mathbf{S}_\kappa(\mathbf{F})$ of all (generalized) Schur functions if it is meromorphic on $\mathbb{D}$, holomorphic at 0 and if the kernel

$$\frac{I-\Theta(w)^*\Theta(z)}{1-z\bar{w}}, \quad z, w \in \mathcal{D}(\Theta),$$

has κ negative squares. Such functions turn out to be characteristic functions of unitary colligations in which the state space is a Pontryagin space of index κ. This has been proved by a number of authors in different ways, including the ε-method; see [2] for a list of references. In [3] it is shown that in particular the theory of reproducing kernels, which goes back to Aronszjan (see [4]), leads to the construction of unitary colligations with concrete state spaces, consisting of meromorphic functions.

We show that the class $\mathbf{C}_\kappa(\mathbf{F})$ of all Carathéodory functions corresponds to the subclass of all functions Θ in the Schur class $\mathbf{S}_\kappa(\mathbf{F})$ for which $I-\Theta(0)$ is invertible. The realization of Θ as a characteristic function then leads to the operator representation of Carathéodory functions announced above. In the scalar case Kreĭn and Langer have traversed the opposite route: using the operator representation of matrix valued Carathéodory functions they constructed a unitary colligation for matrix valued Schur functions; see [20] and [21].

The results for Carathéodory functions carry over immediately to Nevanlinna functions. A function $N : \mathbb{C}^+ \to \mathbf{L}(\mathbf{F})$, where $\mathbb{C}^+$ denotes the upper halfplane in $\mathbb{C}$, belongs to the class $\mathbf{N}_\kappa(\mathbf{F})$ of all (generalized) Nevanlinna functions if it is meromorphic in $\mathbb{C}^+$ and if the kernel

$$\frac{N(\ell)-N(\lambda)^*}{\ell-\bar{\lambda}}, \quad \ell, \lambda \in \mathcal{D}(N),$$

has κ negative squares. Now $\mathcal{D}(N)$ stands for the domain of holomorphy of the function N in $\mathbb{C}^+$. Such functions have operator representations involving selfadjoint relations in Pontryagin spaces. In the positive definite case this representation has been proved directly by Langer and Textorius in [22], using the ε-method. In the indefinite case a representation with a reproducing kernel Pontryagin space was obtained in [1] (see also [14]), by a reduction to the ε-method. An integral representation for matrix functions of the class $\mathbf{N}_\kappa(\mathbf{F})$ was obtained by Daho and Langer in [8].

In Section 2 we prove some simple facts about unitary colligations and their characteristic functions. The isometric part of the basic operator of the colligation in the state space and its characteristic function is studied in Section 3. The representation of Carathéodory functions is considered in Section 4 and that of Nevanlinna functions in Section 5.

We use the following notation. If $\mathbf{K}$ is a Banach space and $T \in \mathbf{L}(\mathbf{K})$, the set of bounded linear operators on $\mathbf{X}$, then $\rho(T)$ denotes the resolvent set of T and $\sigma(T)$ denotes the spectrum of T, whereas $\rho_i(T)$ stands for

$$\rho_i(T) = \{z \in \mathbf{D} \mid z = 0 \text{ or } 1/z \in \rho(T)\}.$$

If $\mathbf{K}$ is a Pontryagin space of index κ and T is a contraction in $\mathbf{L}(\mathbf{K})$, then $\rho_i(T)$ is equal to $\mathbf{D}$ with the exception of at most κ nonzero points, which are poles of the resolvent operator $z \to (I-zT)^{-1}$. For basic facts about Pontryagin spaces and operators in Pontryagin spaces we refer to [5], [6], [16].

2. Schur functions

Let $\mathbf{F}$ be a Hilbert space and let $\mathbf{K}$ be a Pontryagin space of index κ. Let

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathbf{K} \\ \mathbf{F} \end{pmatrix} \to \begin{pmatrix} \mathbf{K} \\ \mathbf{F} \end{pmatrix}$$

be a bounded operator, acting in the orthogonal direct sum $\mathbf{F} \oplus \mathbf{K}$. The collection $(U; \mathbf{K}, \mathbf{F}) = (T, F, G, H; \mathbf{K}, \mathbf{F})$ is called a *colligation* with state space $\mathbf{K}$ and inner space $\mathbf{F}$. We will assume that the colligation is *unitary*, i.e., we assume that the operator U is unitary. The characteristic function $\Theta = \Theta_U$ of the colligation is defined by

$$\Theta(z) = H + zG(I - zT)^{-1}F, \quad z \in \rho_i(T).$$

Clearly, Θ is a meromorphic function on $\mathbf{D}$ with values in $\mathbf{L}(\mathbf{F})$. In fact, it is holomorphic on $\rho_i(T)$, and in particular at 0. The following identities are straightforward to check:

$$(2.1) \quad \frac{I - \Theta(w)^*\Theta(z)}{1 - z\bar{w}} = F^*(I - wT)^{-*}(I - zT)^{-1}F, \quad z, w \in \rho_i(T),$$

$$(2.2) \quad \frac{I - \Theta(z)\Theta(w)^*}{1 - z\bar{w}} = G(I - zT)^{-1}(I - wT)^{-*}G^*, \quad z, w \in \rho_i(T).$$

Hence each of the kernels on the lefthand side has at most κ negative squares. If $\kappa = 0$, then T is a Hilbert space contraction, Θ is holomorphic on $\mathbf{D}$ and for each $z \in \mathbf{D}$ the operator $\Theta(z) \in \mathbf{L}(\mathbf{F})$ is a contraction. The unitary colligation U is called *closely connected* if

$$\mathbf{K} = \text{c.l.s.}\{(I - zT)^{-1}Ff + (I - wT^*)^{-1}G^*g \mid f, g \in \mathbf{F}, \ z \in \rho_i(T), \ w \in \rho_i(T^*)\}.$$

Here c.l.s. stands for closed linear span. Hence, if the unitary colligation U is closely connected, then Θ belongs to the class $\mathbf{S}_\kappa(\mathbf{F})$. Let $\mathbf{K}'$ be a Pontryagin space and let

$$U' = \begin{pmatrix} T' & F' \\ G' & H' \end{pmatrix} : \begin{pmatrix} \mathbf{K}' \\ \mathbf{F} \end{pmatrix} \to \begin{pmatrix} \mathbf{K}' \\ \mathbf{F} \end{pmatrix}$$

be a unitary operator. Then the unitary colligation $(U'; \mathbf{K}', \mathbf{F})$ is called *isomorphic* to the unitary colligation $(U; \mathbf{K}, \mathbf{F})$ if there exists a unitary operator Z from $\mathbf{K}'$ to $\mathbf{K}$ such that

$$\begin{pmatrix} Z^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} T' & F' \\ G' & H' \end{pmatrix}.$$

It then follows that $Z^*TZ = T'$, $Z^*F = F'$, $GZ = G'$ and $H = H'$. Hence the characteristic function of $(U'; \mathbf{K}', \mathbf{F})$ coincides with the characteristic function Θ_U of $(U; \mathbf{K}, \mathbf{F})$. We recall the following well-known theorem, see [3], [12].

THEOREM 2.1. *Let Θ be a Schur function of class $\mathbf{S}_\kappa(\mathbf{F})$. Then there exist a Pontryagin space $\mathbf{K}$ of index κ and a unitary colligation $(U; \mathbf{K}, \mathbf{F}) = (T, F, G, H; \mathbf{K}, \mathbf{F})$, such that Θ is the characteristic function of $(U; \mathbf{K}, \mathbf{F})$ and $\mathcal{D}(\Theta) = \rho_i(T)$. The colligation can be chosen closely connected, in which case it is uniquely determined, up to isomorphisms.*

In particular, if $\Theta : \mathbf{D} \to \mathbf{F}$ is holomorpic on $\mathbf{D}$ and $\Theta(z)$ is a contraction for each $z \in \mathbf{D}$, then Θ belongs to $\mathbf{S}_0(\mathbf{F})$ and is the characteristic function of a unitary colligation in which the state space is a Hilbert space; see [7, Theorems 3.2, 3.3], [23].

If Θ is the characteristic function of the unitary colligation $(U; \mathbf{K}, \mathbf{F})$, then it is straightforward to show that

$$(2.3) \quad P_{\mathbf{F}}(I - zU)^{-1}|_{\mathbf{F}} = (I - z\Theta(z))^{-1}, \quad z \in \rho_i(U) \cap \rho_i(T).$$

Here $P_{\mathbf{F}}$ stands for the orthogonal projection onto $\mathbf{F}$. The lefthand side is commonly called the *compressed resolvent* of U to $\mathbf{F}$.

In the rest of this section we assume that $\mathbf{K}$ is a Pontryagin space, $\mathbf{F}$ is a Hilbert space and $(U; \mathbf{K}, \mathbf{F})$ is a unitary colligation with matrix representation $U = \begin{pmatrix} T & F \\ G & H \end{pmatrix}$. With $(U; \mathbf{K}, \mathbf{F})$ we associate the linear relation V in $\mathbf{K}$ defined by

$$V = \{\{P_{\mathbf{K}}h, P_{\mathbf{K}}Uh\} \in \mathbf{K} \oplus \mathbf{K} \mid h \in \mathbf{K} \oplus \mathbf{F}, (I-U)h \in \mathbf{K}\}.$$

For $h \in \mathbf{K} \oplus \mathbf{F}$ with $(I-U)h \in \mathbf{K}$ we have $P_{\mathbf{F}}h = P_{\mathbf{F}}Uh$, and hence for $\{P_{\mathbf{K}}h, P_{\mathbf{K}}Uh\}, \{P_{\mathbf{K}}k, P_{\mathbf{K}}Uk\} \in V$,

$$\begin{aligned}
&[P_{\mathbf{K}}h, P_{\mathbf{K}}k] - [P_{\mathbf{K}}Uh, P_{\mathbf{K}}Uk] = \\
&\quad = [h, k] - [P_{\mathbf{F}}h, P_{\mathbf{F}}k] - [P_{\mathbf{K}}Uh, P_{\mathbf{K}}Uk] \\
&\quad = [h, k] - [P_{\mathbf{F}}Uh, P_{\mathbf{F}}Uk] - [P_{\mathbf{K}}Uh, P_{\mathbf{K}}Uk] \\
&\quad = [h, k] - [Uh, Uk] = 0.
\end{aligned}$$

It follows that V is an isometric relation.

LEMMA 2.2. *If $I - H$ is invertible, then*

(i) $V = T + F(I-H)^{-1}G$ *and is unitary,*

(ii) $VG^*(I-H)^{-*} = F(I-H)^{-1}$, $V^*F(I-H)^{-1} = G^*(I-H)^{-*}$,

(iii) $(I - zV)^{-1}F(I-H)^{-1}(I - \Theta(z)) = (I - zT)^{-1}F, \quad z \in \rho_i(V) \cap \rho_i(T)$,

(iv) $(I - \Theta(z))(I-H)^{-1}G(I - zV)^{-1} = G(I - zT)^{-1}, \quad z \in \rho_i(V) \cap \rho_i(T)$.

Proof. Straightforward calculations show (i) and (ii). Now we prove (iii):

$$\begin{aligned}
&F(I-H)^{-1}(I-\Theta(z)) = \\
&\quad = F(I-H)^{-1}(I - H - zG(I-zT)^{-1}F) \\
&\quad = (I - zF(I-H)^{-1}G(I-zT)^{-1})F \\
&\quad = (I - zT - zF(I-H)^{-1}G)(I-zT)^{-1}F \\
&\quad = (I - zV)(I-zT)^{-1}F.
\end{aligned}$$

A similar calculation yields (iv):

$$\begin{aligned}
&(I-\Theta(z))(I-H)^{-1}G = \\
&\quad = (I - H - zG(I-zT)^{-1}F)(I-H)^{-1}G \\
&\quad = G(I - z(I-zT)^{-1}F(I-H)^{-1}G) \\
&\quad = G(I-zT)^{-1}(I - zT - zF(I-H)^{-1}G) \\
&\quad = G(I-zT)^{-1}(I-zV).
\end{aligned}$$

This completes the proof of the lemma.

A contraction $K \in \mathbf{L}(\mathbf{K})$ is called *completely nonunitary* if the only subspace $\mathbf{K}_0$ of $\mathbf{K}$ such that $K\mathbf{K}_0 = \mathbf{K}_0$ and the restriction $K|_{\mathbf{K}_0}$ of K to $\mathbf{K}_0$ is isometric, is the trivial subspace. Since the colligation $(U; \mathbf{K}, \mathbf{F})$ is unitary and $\mathbf{F}$ is a Hilbert space, the operators T and T^* in $\mathbf{L}(\mathbf{K})$ are contractions. It is known that the unitary colligation $(U; \mathbf{K}, \mathbf{F})$ is closely connected if and only if T is completely nonunitary, and if and only if T^* is completely nonunitary; see [12]. Let $\mathbf{R}$ be a closed linear subspace of $\mathbf{K}$. The unitary operator V is said to be $\mathbf{R}$-*minimal*, if

$$\mathbf{K} = \text{c.l.s.}\{(I - zV)^{-1}h \mid h \in \mathbf{R}, \ z \in \rho_i(V)\}.$$

The following lemma can be found in [7].

LEMMA 2.3. *If $I-H$ is invertible, the following statements are equivalent:*

(i) $(U; \mathbf{K}, \mathbf{F})$ *is closely connected.*

(ii) V *is* $\operatorname{ran} F$*- minimal.*

(iii) V^* *is* $\operatorname{ran} G^*$*-minimal.*

Proof. The equivalence of (i) and (ii) follows from the formulas

$$(I-zV)^{-1}F(I-H)^{-1} = (I-zT)^{-1}F(I-\Theta(z))^{-1},$$

which is valid for small values of $|z|$, and

$$\begin{aligned}(I-zV)^{-1}F(I-H)^{-1} &= \\ &= -(1/z)(I-(1/z)V^*)^{-1}V^*F(I-H)^{-1} \\ &= -(1/z)(I-(1/z)T^*)^{-1}G^*(I-\Theta(1/\bar{z})^*)^{-1},\end{aligned}$$

which is valid for large values of $|z|$. The equivalence of (ii) and (iii) follows from Lemma 2.2 (ii).

If $I-H$ is invertible, then the following expressions are all equal:

(*a*) $(I-H)^{-*}+(I-H)^{-1}-I$,

(*b*) $(I-H)^{-*}(I-H^*H)(I-H)^{-1}$,

(*c*) $(I-H)^{-1}(I-HH^*)(I-H)^{-*}$.

In particular, $I-H^*H$ is invertible if and only if $I-HH^*$ is invertible. It is straightforward to verify the following observation.

LEMMA 2.4. *Let $H \in \mathbf{L}(\mathbf{F})$ be such that $I-H$ is invertible, and let $\Omega \in \mathbf{L}(\mathbf{F}, \mathbf{K})$ be such that $\Omega^*\Omega = (I-H)^{-*}+(I-H)^{-1}-I$. Then*

$$I-\begin{pmatrix}\Omega \\ -I\end{pmatrix}(I-H)\begin{pmatrix}\Omega \\ -I\end{pmatrix}^* = \begin{pmatrix} I-\Omega(I-H)\Omega^* & \Omega(I-H) \\ (I-H)\Omega^* & H\end{pmatrix}$$

is unitary. If V is a unitary operator in $\mathbf{K}$, then

$$\begin{pmatrix}(I-\Omega(I-H)\Omega^*)V & \Omega(I-H) \\ (I-H)\Omega^*V & H\end{pmatrix} : \begin{pmatrix}\mathbf{K} \\ \mathbf{F}\end{pmatrix} \to \begin{pmatrix}\mathbf{K} \\ \mathbf{F}\end{pmatrix},$$

and

$$\begin{pmatrix}V(I-\Omega(I-H)\Omega^*) & V\Omega(I-H) \\ (I-H)\Omega^* & H\end{pmatrix} : \begin{pmatrix}\mathbf{K} \\ \mathbf{F}\end{pmatrix} \to \begin{pmatrix}\mathbf{K} \\ \mathbf{F}\end{pmatrix},$$

are unitary colligations. When these colligations are written in the form

$$U = \begin{pmatrix} T & F \\ G & H\end{pmatrix},$$

then in each case $T+F(I-H)^{-1}G$ is equal to the given unitary operator V. These colligations are closely connected if and only if V is $\operatorname{ran}\Omega$*- minimal.*

3. The isometric operator

We assume that $\mathbf{K}$ is a Pontryagin space, $\mathbf{F}$ is a Hilbert space and that $(U; \mathbf{K}, \mathbf{F}) = (T, F, G, H; \mathbf{K}, \mathbf{F})$ is an operator colligation.

LEMMA 3.1. *If U^* is isometric, the following statements are equivalent:*

(i) $I - HH^*$ *is invertible.*

(ii) $\operatorname{ran} G^*$ *is nondegenerate and closed,* $\ker G^* = \{0\}$.

If U is isometric, the following statements are equivalent:

(iii) $I - H^*H$ *is invertible.*

(iv) $\operatorname{ran} F$ *is nondegenerate and closed,* $\ker F = \{0\}$.

Proof. Since U^* is isometric, (i) is equivalent to GG^* is invertible. Assume that GG^* is invertible. Then $\ker G^* \subset \ker GG^* = \{0\}$, and $[G^*x, G^*y] = 0$ for all $y \in \mathbf{F}$ implies $x = 0$, which proves that $\operatorname{ran} G^*$ is nondegenerate. Since $G^*(GG^*)^{-1}G = I$ on $\operatorname{ran} G^*$, $\operatorname{ran} G^*$ is closed. Indeed, if $y \in \overline{\operatorname{ran} G^*}$ and $G^*x_n \to y$ for some sequence $x_n \in \mathbf{F}$, then also $G^*x_n = G^*(GG^*)^{-1}GG^*x_n \to G^*(GG^*)^{-1}Gy$, which implies that $y = G^*(GG^*)^{-1}Gy \in \operatorname{ran} G^*$. Therefore (ii) follows. Conversely, if $\operatorname{ran} G^*$ is nondegenerate and closed, then $\operatorname{ran} G^*$ is a regular subspace of $\mathbf{K}$, i.e., $\operatorname{ran} G^*$ is a Pontryagin space by itself, and therefore $\mathbf{K} = \operatorname{ran} G^* \oplus \ker G$, see [6, IX, Theorem 2.2]. Therefore $\operatorname{ran} G = \operatorname{ran} GG^*$, and hence $\ker GG^* = \ker G^* = \{0\}$, if we also assume that $\ker G^* = \{0\}$. Since $\operatorname{ran} G^*$ is closed, we obtain by [6, VI, Theorem 2.9], that $\operatorname{ran} G$ and also $\operatorname{ran} GG^*$ is closed, and thus $\operatorname{ran} GG^* = \mathbf{K}$. Hence GG^* is invertible and (i) follows. The equivalence of (iii) and (iv) follows from (i) and (ii) applied to U^*.

COROLLARY 3.2. *Assume that U is unitary and that $I - H$ is invertible. Then the following statements are equivalent:*

(i) $(I - H)^{-*} + (I - H)^{-1} - I$ *is invertible.*

(ii) $\operatorname{ran} G^*$ *is nondegenerate and closed,* $\ker G^* = \{0\}$.

(iii) $\operatorname{ran} F$ *is nondegenerate and closed,* $\ker F = \{0\}$.

Also the following statements are equivalent:

(iv) $(I - H)^{-*} + (I - H)^{-1} - I$ *is positive definite.*

(v) $\operatorname{ran} G^*$ *is a Hilbert space.*

(vi) $\operatorname{ran} F$ *is a Hilbert space.*

From now on in this section we assume that $(T, F, G, H; \mathbf{K}, \mathbf{F})$ is a unitary colligation. It follows from $T^*T + G^*G = I$, that the operator $T \in \mathbf{L}(\mathbf{K})$ is a contraction, and from similar identities that its restriction $T|_{\ker G}$ is a bounded isometric operator, mapping $\ker G$ onto $\ker F^*$. If $I - H$ is invertible, then the identity $V = T + F(I - H)^{-1}G$ shows that $V|_{\ker G} = T|_{\ker G}$, so that V is a unitary extension in $\mathbf{K}$ of the isometric operator $T|_{\ker G}$. If, in addition, $(I - H)^{-*} + (I - H)^{-1} - I$ is invertible, then $\ker G$ and $\ker F^*$ are regular subspaces of $\mathbf{K}$. Hence, in this case, also the defect spaces $(\ker G)^{\perp} = \operatorname{ran} G^*$ and $(\ker F^*)^{\perp} = \operatorname{ran} F$ of $T|_{\ker G}$ are regular subspaces of $\mathbf{K}$. Clearly, $\Gamma_{G^*} = G^*(I - H)^{-*}$ is a bounded invertible mapping from $\mathbf{F}$ onto $\operatorname{ran} G^*$ and $\Gamma_F = F(I - H)^{-1}$ is a bounded invertible mapping from $\mathbf{F}$ onto $\operatorname{ran} F$. Associated with $T|_{\ker G}$ is the unitary colligation

$$\begin{pmatrix} TP_{\ker G} & I \\ P_{\operatorname{ran} G^*} & 0 \end{pmatrix} : \begin{pmatrix} \mathbf{K} \\ \operatorname{ran} F \end{pmatrix} \to \begin{pmatrix} \mathbf{K} \\ \operatorname{ran} G^* \end{pmatrix}.$$

The characteristic function $X : \mathbf{D} \to \mathbf{L}(\operatorname{ran} F, \operatorname{ran} G^*)$ of this colligation is given by

$$X(z) = zP_{\operatorname{ran} G^*}(I - zTP_{\ker G})^{-1}|_{\operatorname{ran} F}.$$

It is holomorphic in a neighborhood of 0. It follows from the general formulas (2.1) and (2.2) that

$$\frac{I - X(w)^* X(z)}{1 - z\bar{w}} = P_{\operatorname{ran} F}(I - wTP_{\ker G})^{-*}(I - zTP_{\ker G})^{-1}|_{\operatorname{ran} F},$$

$$\frac{I - X(z)X(w)^*}{1 - z\bar{w}} = P_{\operatorname{ran} G^*}(I - zTP_{\ker G})^{-1}(I - wTP_{\ker G})^{-*}|_{\operatorname{ran} G^*},$$

for $z, w \in \rho_i(T)$. If, in addition, $\operatorname{ran} F$, or equivalently, $\operatorname{ran} G^*$ is a Hilbert space, then $TP_{\ker G}$ is a contraction in the Pontryagin space $\mathbf{K}$ and, hence, the function X is meromorphic on $\mathbf{D}$.

THEOREM 3.3. *Assume that $(U; \mathbf{K}, \mathbf{F}) = (T, F, G, H; \mathbf{K}, \mathbf{F})$ is a closely connected unitary colligation and that the operators $I - H$ and $(I - H)^{-*} + (I - H)^{-1} - I$ are invertible. Then*

$$(3.1)\quad (\Gamma_{G^*})^{-1}X(z)\Gamma_F = z(\Gamma_{G^*}^*(I - zV)^{-1}\Gamma_{G^*})^{-1}\Gamma_{G^*}^*(I - zV)^{-1}\Gamma_F,$$

for z in a small neighborhood of 0. If, in addition, $(I - H)^{-*} + (I - H)^{-1} - I$ is positive definite, then the characteristic function X belongs to $\mathbf{S}_\kappa(\operatorname{ran} F, \operatorname{ran} G^*)$.

Proof. From Lemma 2.2 (ii) and $V^*TP_{\ker G} = P_{\ker G}$ (as $T|_{\ker G} = V|_{\ker G}$), it follows that

$$\begin{aligned}
&zV(I - zV)^{-1}(I - X(z)V)\Gamma_{G^*} = \\
&= z(V^* - z)^{-1}(V^* - X(z))V\Gamma_{G^*} \\
&= z(V^* - z)^{-1}(V^*(I - zTP_{\ker G}) - zP_{\operatorname{ran} G^*})(I - zTP_{\ker G})^{-1}\Gamma_F \\
&= z(V^* - z)^{-1}(V^* - zP_{\ker G} - zP_{\operatorname{ran} G^*})(I - zTP_{\ker G})^{-1}\Gamma_F \\
&= z(I - zTP_{\ker G})^{-1}\Gamma_F \\
&= z(P_{\operatorname{ran} G^*} + P_{\ker G})(I - zTP_{\ker G})^{-1}\Gamma_F \\
&= X(z)\Gamma_F + zP_{\ker G}(I - zTP_{\ker G})^{-1}\Gamma_F.
\end{aligned}$$

From the definition of Γ_{G^*}, it follows that $\Gamma_{G^*}^* P_{\ker G} = (I - H)^{-1}GP_{\ker G} = 0$, and therefore the above identity leads to

$$(3.2)\quad \Gamma_{G^*}^* zV(I - zV)^{-1}(I - X(z)V)\Gamma_{G^*} = \Gamma_{G^*}^* X(z)\Gamma_F.$$

This leads to

$$z\Gamma_{G^*}^*(I - zV)^{-1}\Gamma_F = \Gamma_{G^*}^*(I - zV)^{-1}X(z)\Gamma_F,$$

and so (3.1) follows.

Next we assume that $(I-H)^{-*} + (I-H)^{-1} - I$ is positive definite, and we prove that the unitary colligation of which X is the characteristic function is closely connected. For this it suffices to prove that the operator $TP_{\ker G}$ is completely nonunitary. Let $\mathbf{K}_0$ be a closed subspace of $\mathbf{K}$, such that

$$TP_{\ker G}\mathbf{K}_0 = \mathbf{K}_0, \quad TP_{\ker G}|_{\mathbf{K}_0} \text{ is isometric.}$$

Then for all $x, y \in \mathbf{K}_0$ we have

$$[P_{\ker G}x, P_{\ker G}y] = [TP_{\ker G}x, TP_{\ker G}y] = [x, y].$$

Therefore $[(I-P_{\ker G})x, (I-P_{\ker G})y] = 0$ for all $x, y \in \mathbf{K}_0$. This leads to $(I-P_{\ker G})x = 0$ for all $x \in \mathbf{K}_0$, since $\operatorname{ran} G^*$ is a Hilbert space. Hence $\mathbf{K}_0 \subset \ker G$. Therefore $\mathbf{K}_0$ has the properties

$$T\mathbf{K}_0 = \mathbf{K}_0, \quad T|_{\mathbf{K}_0} \text{ is isometric.}$$

As we assume that the colligation $(U; \mathbf{K}, \mathbf{F})$ is closely connected, the operator $T \in \mathbf{L}(\mathbf{K})$ is completely nonunitary and we conclude that $\mathbf{K}_0$ is the trivial subspace. In particular, we obtain $X \in \mathbf{S}_\kappa(\operatorname{ran} F, \operatorname{ran} G^*)$. This completes the proof.

In the case that the operators $I-H$ and $(I-H)^{-*} + (I-H)^{-1} - I$ are invertible, it also follows from the relation (3.2) that

$$\Gamma^*_{G^*}(I+zV)(I-zV)^{-1}(I-X(z)V)\Gamma_{G^*} = \Gamma^*_{G^*}(I+X(z)V)\Gamma_{G^*}.$$

Note that in a small neighborhood of 0, the operator $\Gamma^{-1}_{G^*}(I-X(z)V)\Gamma_{G^*}$ is invertible, and that

$$(\Gamma^{-1}_{G^*}(I-X(z)V)\Gamma_{G^*})^{-1} = \Gamma^{-1}_{G^*}(I-X(z)V|_{\operatorname{ran} G^*})^{-1}\Gamma_{G^*}.$$

Therefore, we obtain

$$\text{(3.3)} \quad \Gamma^*_{G^*}(I+X(z)V)(I-X(z)V|_{\operatorname{ran} G^*})^{-1}\Gamma_{G^*} = \Gamma^*_{G^*}(I+zV)(I-zV)^{-1}\Gamma_{G^*},$$

for z in a small neighborhood of 0.

4. Carathéodory functions

In this section we show that each (generalized) Carathéodory function of class $\mathbf{C}_\kappa(\mathbf{F})$ has a representation in terms of a unitary operator in a Pontryagin space $\mathbf{K}$ with index κ. We derive this representation by proving that there is a one-to-one correspondence between $\mathbf{C}_\kappa(\mathbf{F})$ and all (generalized) Schur functions Θ of class $\mathbf{S}_\kappa(\mathbf{F})$ for which $I-\Theta(0)$ is invertible.

We first prove a preliminary result concerning generalized Carathéodory functions. A corresponding result for generalized Schur functions can be found in [11]. Recall that if T is a bounded linear operator in a Banach space $\mathbf{X}$, a number $z \in \mathbb{C}$ is a *normal* eigenvalue of T, if its algebraic multiplicity is finite and if the space $\mathbf{X}$ is the direct sum of the root subspace of T corresponding to z and an invariant subspace of T, in which $T-z$ is invertible.

LEMMA 4.1. *Let the function Φ belong to the class $\mathbf{C}_\kappa(\mathbf{F})$. Then for each $z \in \mathcal{D}(\Phi)$ the set $\sigma(\Phi(z)) \cap \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < 0\}$ consists of an at most discrete set of normal eigenvalues.*

Proof. Let $z \in \mathcal{D}(\Phi)$. As the Carathéodory kernel of Φ has κ negative squares, it follows that the dimension of the subspace $\mathbf{H}(z) \subset \mathbf{F}$ on which the selfadjoint operator $\Phi(z) + \Phi(z)^*$ is negative is at most κ. Therefore $\Phi(z)$ has the decomposition

$$\Phi(z) = A(z) - B(z),$$

where $A(z) \in \mathbf{L}(\mathbf{F})$ with $\operatorname{Re} A(z) \geq 0$, and $B(z) \in \mathbf{L}(\mathbf{F})$ is finite dimensional with $B(z) \geq 0$. A result from perturbation theory (cf. [15, Chapter 1, Lemma 5.2]) shows that for each $z \in \mathcal{D}(\Phi)$ the open lefthand plane consists of points in the resolvent set of $\Phi(z)$ or of normal eigenvalues, whose accumulation points are on the imaginary axis. This completes the proof.

THEOREM 4.2. *Let $\Theta \in \mathbf{S}_\kappa(\mathbf{F})$ have the property that $I-\Theta(0)$ is invertible. Let $(U;\mathbf{K},\mathbf{F}) = (T,F,G,H;\mathbf{K},\mathbf{F})$ with $\mathbf{K}$ a Pontryagin space of index κ, be a closely connected unitary colligation whose characteristic function is Θ. Then $I-\Theta(z)$ is invertible for all $z \in \rho_i(V) \cap \rho_i(T)$. For any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ the function*

$$(4.1)\quad \Phi(z) = (I-\Theta(z))^{-1}(\bar{\alpha} + \alpha\Theta(z)), \quad z \in \rho_i(V) \cap \rho_i(T),$$

belongs to $\mathbf{C}_\kappa(\mathbf{F})$ and $\Phi(z)+\alpha$ is invertible for all $z \in \rho_i(V) \cap \rho_i(T)$. Moreover, V is a $\operatorname{ran}\Lambda$*-minimal unitary operator, and Φ has the representation*

$$(4.2)\quad \Phi(z) = -\Phi(0)^* + \Lambda^*(I-zV)^{-1}\Lambda, \quad z \in \rho_i(V),$$

when Λ is given by $\Lambda = (2\operatorname{Re}\alpha)^{\frac{1}{2}}F(I-H)^{-1}$ or *by* $\Lambda = (2\operatorname{Re}\alpha)^{\frac{1}{2}}G^*(I-H)^{-*}$.

Conversely, let $\Phi : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ be a function with the representation (4.2), where $\mathbf{F}$ is a Hilbert space, $\mathbf{K}$ is a Pontryagin space of index κ, $\Lambda \in \mathbf{L}(\mathbf{F},\mathbf{K})$ and V is a unitary operator in the Pontryagin space $\mathbf{K}$, which is $\operatorname{ran}\Lambda$*-minimal. Then $\Phi \in \mathbf{C}_\kappa(\mathbf{F})$, and there exists an $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, such that*

$$\begin{pmatrix} (I-\Lambda(\Phi(0)+\alpha)^{-1}\Lambda^*)V & (2\operatorname{Re}\alpha)^{\frac{1}{2}}\Lambda(\Phi(0)+\alpha)^{-1} \\ (2\operatorname{Re}\alpha)^{\frac{1}{2}}(\Phi(0)+\alpha)^{-*}\Lambda^*V & (\Phi(0)-\bar{\alpha})(\Phi(0)+\alpha)^{-1} \end{pmatrix},$$

$$\begin{pmatrix} V(I-\Lambda(\Phi(0)+\alpha)^{-1}\Lambda^*) & V(2\operatorname{Re}\alpha)^{\frac{1}{2}}\Lambda(\Phi(0)+\alpha)^{-1} \\ (2\operatorname{Re}\alpha)^{\frac{1}{2}}(\Phi(0)+\alpha)^{-*}\Lambda^* & (\Phi(0)-\bar{\alpha})(\Phi(0)+\alpha)^{-1} \end{pmatrix},$$

are closely connected unitary colligations in $\mathbf{F} \oplus \mathbf{K}$, whose characteristic function Θ belong to $\mathbf{S}_\kappa(\mathbf{F})$ and give rise to the function Φ via (4.1).

Proof. For the closely connected unitary colligation $(T,F,G,H;\mathbf{K},\mathbf{F})$ with $I-H$ invertible and any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, we put $\overset{\circ}{\Gamma}_F = (2\operatorname{Re}\alpha)^{\frac{1}{2}}F(I-H)^{-1}$ and $\overset{\circ}{\Gamma}G^* = (2\operatorname{Re}\alpha)^{\frac{1}{2}}G^*(I-H)^{-*}$. We define the function $\Phi : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ by

$$(4.3)\quad \Phi(z) = -(I-H)^{-*}(\alpha + \bar{\alpha}H^*) + (\overset{\circ}{\Gamma}_F)^*(I-zV)^{-1}\overset{\circ}{\Gamma}_F, \quad z \in \rho_i(V).$$

It follows that

$$\Phi(z) + \alpha = (2\operatorname{Re}\alpha)(I-H)^{-*}(-H^* + F^*(I-zV)^{-1}F(I-H)^{-1}), \quad z \in \rho_i(V),$$

and hence from $H^*G = -F^*T$ and Lemma 2.2 (iii) we obtain

$$\begin{aligned}
(4.4)\quad (2\operatorname{Re}\alpha)^{-1}&(I-H)^*(\Phi(z)+\alpha)(I-\Theta(z)) = \\
&= (-H^* + F^*(I-zV)^{-1}F(I-H)^{-1})(I-\Theta(z)) \\
&= -H^*(I-H-zG(I-zT)^{-1}F) + F^*(I-zV)^{-1}F(I-H)^{-1})(I-\Theta(z)) \\
&= -H^*(I-H) - zF^*T(I-zT)^{-1}F + F^*(I-zT)^{-1}F \\
&= (I-H)^*, \quad z \in \rho_i(V) \cap \rho_i(T).
\end{aligned}$$

As $(I-zV)^{-1} = V(I-zV)^{-1}V^*$ we can also rewrite (4.3) as

$$(4.5)\quad \Phi(z) = -(I-H)^{-*}(\alpha + \bar{\alpha}H^*) + (\overset{\circ}{\Gamma}G^*)^*(I-zV)^{-1}\,\overset{\circ}{\Gamma}G^*, \quad z \in \rho_i(V).$$

It follows that

$$\Phi(z) + \alpha = (2\operatorname{Re}\alpha)(-H^* + (I-H)^{-1}G(I-zV)^{-1}G^*)(I-H)^{-*}, \quad z \in \rho_i(V),$$

so that from $FH^* = -TG^*$ and Lemma 2.2 (iv) we obtain

(4.6) $$(2\mathrm{Re}\,\alpha)^{-1}(I-\Theta(z))(\Phi(z)+\alpha)(I-H)^* =$$
$$\begin{aligned} &= (I-\Theta(z))(-H^* + (I-H)^{-1}G(I-zV)^{-1}G^*) \\ &= -(I-H-zG(I-zT)^{-1}F)H^* + (I-\Theta(z))(I-H)^{-1}G(I-zV)^{-1}G^* \\ &= -(I-H)H^* - zG(I-zT)^{-1}TG^* + G(I-zT)^{-1}G^* \\ &= (I-H)^*, \quad z \in \rho_i(V) \cap \rho_i(T). \end{aligned}$$

The identities (4.4) and (4.6) show that $\Phi(z)+\alpha$ is invertible and

$$\Phi(z)+\alpha = (2\mathrm{Re}\,\alpha)(I-\Theta(z))^{-1},$$

which gives (4.1). From this it follows that

$$(2\mathrm{Re}\,\alpha)^{-1}\,\frac{\Phi(z)+\Phi(w)^*}{1-z\bar{w}} = (I-\Theta(w))^{-*}\,\frac{I-\Theta(w)^*\Theta(z)}{1-z\bar{w}}\,(I-\Theta(z))^{-1},$$

for all $z, w \in \rho_i(V) \cap \rho_i(T)$, which shows that $\Phi \in \mathbf{C}_\kappa(\mathbf{F})$. From the definition of Φ it follows that $\Phi(0) = (\bar{\alpha} + \alpha H)(I-H)^{-1}$. Hence (4.3) and (4.5) lead to (4.2) with Λ given by $\Lambda = \mathring{\Gamma}_F$ or by $\Lambda = \mathring{\Gamma}_{G^*}$, respectively.

Conversely, assume that Φ is given by (4.2). It is clear from this representation that $\Phi(0)+\Phi(0)^* = \Lambda^*\Lambda$, and therefore that

$$\frac{\Phi(z)+\Phi(w)^*}{1-z\bar{w}} = \Lambda^*(I-zV)^{-1}(I-wV)^{-*}\Lambda,$$

for all $z, w \in \rho_i(V)$. Hence, the function Φ belongs to the class $\mathbf{C}_\kappa(\mathbf{F})$. According to Lemma 4.1, there exists an $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$, such that $\Phi(0)+\alpha$ is invertible. We put

$$H = (\Phi(0)-\bar{\alpha})(\Phi(0)+\alpha)^{-1}, \quad \Omega = (2\mathrm{Re}\,\alpha)^{-1/2}\Lambda.$$

Then $I-H$ is invertible and $\Omega^*\Omega = (I-H)^{-*}(I-H^*H)(I-H)^{-1}$. Therefore, by Lemma 2.4, the colligations in the proposition have all the indicated properties.

THEOREM 4.3. *Let Φ in $\mathbf{C}_\kappa(\mathbf{F})$. There exists an $\alpha \in \mathbb{C}$ with* $\mathrm{Re}\,\alpha > 0$ *such that* $(\Phi(z)+\alpha)^{-1} \in \mathbf{L}(\mathbf{F})$ *for all $z \in \mathcal{D}_\alpha$, where the set $\mathcal{D}_\alpha$ is equal to $\mathcal{D}(\Phi)$, the domain of holomorphy in $\mathbf{D}$, with the exception of at most κ nonzero points. The function $\Theta : \mathbf{D} \to \mathbf{L}(\mathbf{F})$ defined by*

(4.7) $$\Theta(z) = (\Phi(z)-\bar{\alpha})(\Phi(z)+\alpha)^{-1}, \quad z \in \mathcal{D}_\alpha,$$

is a Schur function of class $\mathbf{S}_\kappa(\mathbf{K})$ with the additional property that $I-\Theta(z)$ is invertible for all $z \in \mathcal{D}_\alpha$. Consequently, there exist a Pontryagin space $\mathbf{K}$ of index κ, a mapping $\Lambda \in \mathbf{L}(\mathbf{F},\mathbf{K})$ and a unitary operator V in $\mathbf{K}$, which is ran *Λ-minimal, such that (4.2) holds.*

Proof. According to Lemma 4.1 there exists an $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$, such that $(\Phi(0)+\alpha)^{-1} \in \mathbf{L}(\mathbf{F})$. A Neumann series argument shows that $(\Phi(z)+\alpha)^{-1} \in \mathbf{L}(\mathbf{F})$ for all z in a small disk around 0. We claim that there are at most κ non-zero points z in $\mathcal{D}(\Phi)$ for which $(\Phi(z)+\alpha)^{-1} \notin \mathbf{L}(\mathbf{F})$. To prove this claim, we assume that there are $\kappa+1$ non-zero points $z_1, \ldots, z_{\kappa+1} \in \mathcal{D}(\Phi)$ for which $(\Phi(z_i)+\alpha)^{-1} \notin \mathbf{L}(\mathbf{F})$, $i = 1, \ldots, \kappa+1$. Then, according to Lemma 4.1, $-\alpha$ is necessarily a normal eigenvalue of $\Phi(z_i)$, and hence there are non-zero vectors $f_i \in \mathbf{F}$ such that $\Phi(z_i)f_i = -\alpha f_i$, $i = 1, \ldots, \kappa+1$, and therefore

$$\left(\left[\frac{\Phi(z_i)+\Phi(z_j)^*}{1-z_i\bar{z}_j}f_i, f_j\right]\right)_{i,j=1,2,\ldots,\kappa+1} = -2\alpha\left(\frac{[f_i, f_j]}{1-z_i\bar{z}_j}\right)_{i,j=1,2,\ldots,\kappa+1}$$

As the Schur product of two positive matrices is again positive (cf. [2]), the matrix on the lefthand side has $\kappa+1$ negative eigenvalues, a contradiction, which substantiates our claim. The function $\Theta : \mathbf{D} \to \mathbf{L}(\mathbf{F})$ in (4.3) is well-defined. It is clear that $I-\Theta(z) = (2\mathrm{Re}\,\alpha)(\Phi(z)+\alpha)^{-1}$ for $z \in \mathcal{D}_\alpha$. From the identity

$$\frac{I-\Theta(w)^*\Theta(z)}{1-z\bar{w}} = (2\mathrm{Re}\,\alpha)(\Phi(w)+\alpha)^{-*}\ \frac{\Phi(z)+\Phi(w)^*}{1-z\bar{w}}\ (\Phi(z)+\alpha)^{-1},$$

for z, w in a neighborhood of 0, it follows that Θ is a Schur function of class $\mathbf{S}_\kappa(\mathbf{K})$. Let $U = (T, F, G, H; \mathbf{K}, \mathbf{F})$ be a closely connected unitary colligation whose characteristic function is Θ. As $I-\Theta(0)$ is invertible, the function Φ has the representation (4.2).

The two previous theorems show in a precise sense that the formulas (4.1) and (4.7) give a one-to-one correspondence between $\Theta \in \mathbf{S}_\kappa(\mathbf{F})$ with $I-\Theta(0)$ invertible, and $\Phi \in \mathbf{C}_\kappa(\mathbf{F})$ with $\Phi(0)+\alpha$ invertible. Now let $\Phi : \mathbf{D} \to \mathbf{L}(\mathbf{F})$ be a holomorphic function, such that for all $z \in \mathbf{D}$, $\mathrm{Re}\,\Phi(z) \geq 0$. It is clear that for any α with $\mathrm{Re}\,\alpha > 0$ the function $\Theta : \mathbf{D} \to \mathbf{F}$, defined by (4.7) is holomorphic on $\mathbf{D}$ and that $\Theta(z)$ is a contraction for each $z \in \mathbf{D}$. Hence Θ is a Schur function of class $\mathbf{S}_0(\mathbf{F})$, and therefore Φ is a Carathéodory function of class $\mathbf{C}_0(\mathbf{F})$.

COROLLARY 4.4. *Let $\Theta \in \mathbf{S}_\kappa(\mathbf{F})$ and $\Phi \in \mathbf{C}_\kappa(\mathbf{F})$, be connected via (4.1) and (4.7). Let $\{U; \mathbf{F}, \mathbf{K}\} = \{T, F, G, H; \mathbf{F}, \mathbf{K}\}$ be the corresponding closely connected unitary colligation. Then the compressed resolvent of U is given by*

(4.8) $$P_{\mathbf{F}}(I-zU)^{-1}|_{\mathbf{F}} = \frac{1}{1-z}\left(\Phi(z)+\frac{\alpha+\bar{\alpha}z}{1-z}\right)^{-1}(\Phi(z)+\alpha),$$

for $z \in \rho_i(V) \cap \rho_i(T) \cap \rho_i(U)$.

Proof. By means of the bilinear transformation (4.7) we obtain

$$(I-z\Theta(z)) = (1-z)\left(\Phi(z)+\frac{\alpha+\bar{\alpha}z}{1-z}\right)(\Phi(z)+\alpha)^{-1}, \quad z \in \rho_i(V) \cap \rho_i(T).$$

On account of (2.3) this leads to (4.8).

For a (generalized) Carathéodory function Φ of class $\mathbf{C}_\kappa(\mathbf{F})$ with corresponding closely connected unitary colligation $(T, F, G, H; \mathbf{K}, \mathbf{F})$ we introduce the meromorphic functions Δ_{G^*} and Δ_F: $\mathbf{D} \to \mathbf{L}(\mathbf{F}, \mathbf{K})$ given by

$$\begin{aligned}\Delta_{G^*}(z) &= (2\mathrm{Re}\,\alpha)^{\frac{1}{2}}(I-\bar{z}V)^{-*}G^*(I-H)^{-*}, \quad z \in \rho_i(V^*),\\ \Delta_F(z) &= (2\mathrm{Re}\,\alpha)^{\frac{1}{2}}(I-zV)^{-1}F(I-H)^{-1}, \quad z \in \rho_i(V).\end{aligned}$$

Clearly, the function Φ satisfies,

$$\frac{\Phi(z)+\Phi(w)^*}{1-z\bar{w}} = \Delta_{G^*}(\bar{z})^*\Delta_{G^*}(\bar{w}) = \Delta_F(w)^*\Delta_F(z),$$

for $z, w \in \rho_i(V)$. Conversely, this equation uniquely determines Φ up to i times a constant selfadjoint operator in $\mathbf{L}(\mathbf{F})$, and if the colligation $(U; \mathbf{K}, \mathbf{F})$ is closely connected, then Φ belongs to $\mathbf{C}_\kappa(\mathbf{F})$.

When $\Phi(0)+\Phi(0)^*$ or, equivalently, $(I-H)^{-*}+(I-H)^{-1}-I$ is invertible, then Φ is called the Q-function of the closed, isometric operator $T|_{\ker G}$ with respect to the unitary extension V of $T|_{\ker G}$ and the bounded, invertible mapping $G^*(I-H)^{-*}$ from $\mathbf{F}$ onto the regular subspace $\operatorname{dom}(T|_{\ker G})^\perp = \operatorname{ran} G^*$. Similarly, under the same condition Φ could be called the Q-function of the closed, isometric operator $T^*|_{\ker F^*}$ with respect to the unitary extension V^* of $T^*|_{\ker F^*}$ and the bounded, invertible mapping $F(I-H)^{-1}$ from $\mathbf{F}$ onto $\operatorname{dom}(T^*|_{\ker F^*})^\perp = \operatorname{ran} F$.

THEOREM 4.5. *Let $\Phi \in \mathbf{C}_\kappa(\mathbf{F})$, and assume that $\Phi(0)+\Phi(0)^*$ is invertible. Then*

(4.9) $(G^*(I-H)^{-*})^{-1}X(z)F(I-H)^{-1} = (\Phi(z)+\Phi(0)^*)^{-1}(\Phi(z)-\Phi(0)),$

for all z in a small neighborhood of 0. If $\Phi(0)+\Phi(0)^*$ is positive, then (4.9) holds for all $z \in \mathcal{D}(\Phi)$ with the exception of at most κ non-zero points.

Proof. We assume that $\Phi(0)+\Phi(0)^*$ is invertible. We recall the definition (4.3) of the function Φ:

$$\Phi(z)+\Phi(0)^* = (\mathring{\Gamma}_F)^*(I-zV)^{-1}\mathring{\Gamma}_F, \quad z \in \rho_i(V),$$

which leads to

$$\Phi(z)-\Phi(0) = z(\mathring{\Gamma}_F)^*(I-zV)^{-1}\mathring{\Gamma}_{G^*}, \quad z \in \rho_i(V).$$

Hence (4.9) follows from Theorem 3.3.

Now assume that $\Phi(0)+\Phi(0)^*$ is positive. As we have seen in the proof of Lemma 4.1, for each $z \in \mathcal{D}(\Phi)$ the operator $\Phi(z)$ has the decomposition

$$\Phi(z) = A(z) - B(z),$$

where $A(z) \in \mathbf{L}(\mathbf{F})$ with $\operatorname{Re} A(z) \geq 0$, and $B(z) \in \mathbf{L}(\mathbf{F})$ is finite dimensional with $B(z) \geq 0$. Hence the operator $\Phi(z)+\Phi(0)^*$ has the decomposition

$$\Phi(z)+\Phi(0)^* = A(z)+A(0)^* - (B(z)+B(0)^*),$$

where $\operatorname{Re}(A(z)+A(0)^*) \geq 0$, and $B(z)+B(0)^*$ is finite dimensional with $(B(z)+B(0)^*) \geq 0$. A result from perturbation theory (cf. [15, Chapter 1, Lemma 5.2]) shows that for each $z \in \mathcal{D}(\Phi)$ the open lefthand plane consists of points in the resolvent set of $\Phi(z)$ or of an at most discrete set of normal eigenvalues. Since $\Phi(0)+\Phi(0)^*$ is invertible, a Neumann series argument shows that $\Phi(z)+\Phi(0)^*$ is invertible for all z in a small disk around 0. Now assume that $\Phi(0)+\Phi(0)^*$ is positive. We claim that there are at most κ non-zero points z in $\mathcal{D}(\Phi)$ for which $\Phi(z)+\Phi(0)^*$ is not invertible. To prove this claim, we assume that there are $\kappa+1$ non-zero points $z_1, \ldots, z_{\kappa+1} \in \mathcal{D}(\Phi)$ for which $\Phi(z_i)+\Phi(0)^*$ is not invertible, $i = 1, \ldots, \kappa+1$. Then 0 is necessarily a normal eigenvalue of $\Phi(z_i)+\Phi(0)^*$, and hence there are non-zero vectors $f_i \in \mathbf{F}$ such that $\Phi(z_i)f_i = -\Phi(0)^*f_i$, $i = 1, \ldots, \kappa+1$, and therefore

$$\left(\left[\frac{\Phi(z_i)+\Phi(z_j)^*}{1-z_i\bar{z}_j}f_i, f_j\right]\right)_{i,j=1,\ldots,\kappa+1} = -\left(\left[\frac{\Phi(0)+\Phi(0)^*}{1-z_i\bar{z}_j}f_i, f_j\right]\right)_{i,j=1,\ldots,\kappa+1}.$$

As the Schur product of positive matrices is again positive (cf. [2]), the matrix on the lefthand side has $\kappa+1$ negative eigenvalues, a contradiction, which substantiates our claim.

5. Nevanlinna functions

In this section we derive a representation for a (generalized) Nevanlinna function of class $\mathbf{N}_\kappa(\mathbf{F})$ in terms of a selfadjoint relation in a Pontryagin space $\mathbf{K}$ with index κ. In order to formulate the following results we define for any $\mu \in \mathbb{C}^+$ the bijective mapping $z : \mathbb{C}^+ \to \mathbb{D}$ by $z(\ell) = (\ell - \mu)/(\ell - \bar{\mu})$, $\ell \in \mathbb{C}^+$, and for any $\beta \in \mathbb{C}^+$ the mapping $w : \mathbb{C}^+ \to \mathbb{C}$ by

$$w(\ell) = \frac{\beta(\ell - \bar{\mu}) - \bar{\beta}(\ell - \mu)}{\mu - \bar{\mu}}, \quad \ell \in \mathbb{C}^+.$$

THEOREM 5.1. *Let $N \in \mathbf{N}_\kappa(\mathbf{F})$, and let N be holomorphic at $\mu \in \mathbb{C}^+$. Then there exists a $\beta \in \mathbb{C}^+$ such that $N(\ell)+\beta$ is invertible for all $\ell \in \mathcal{D}_\beta$, where the set $\mathcal{D}_\beta$ is equal to $\mathcal{D}(N)$, the domain of holomorphy of N in $\mathbb{C}^+$, with the exception of at most κ points, different from μ. The function $\Theta : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ defined by*

$$\Theta(z(\ell)) = (N(\ell) + \bar{\beta})(N(\ell) + \beta)^{-1}, \quad \ell \in \mathcal{D}_\beta,$$

is of class $\mathbf{S}_\kappa(\mathbf{F})$ and has the additional property that $I - \Theta(z(\ell))$ is invertible for all $\ell \in \mathcal{D}_\beta$. Consequently, depending on β, there exist a Pontryagin space $\mathbf{K}$ of index κ, a selfadjoint relation A in $\mathbf{K} \oplus \mathbf{F}$ with nonempty resolvent set $\rho(A)$ and a mapping $\Lambda \in \mathbf{L}(\mathbf{F}, \mathbf{K})$ such that

(5.1) $B = \{ \{f, P_{\mathbf{K}}g\} \mid \{f, g\} \in A, \;\; f \in \mathbf{K} \}$

is a ran Λ*-minimal selfadjoint relation in $\mathbf{K}$ with nonempty resolvent set $\rho(B)$ and*

(5.2) $C = \{ \{P_{\mathbf{K}}f, P_{\mathbf{K}}g\} \mid \{f, g\} \in A, \;\; g - \mu f \in \mathbf{K} \}, \quad \mu \in \mathbb{C}\backslash\mathbf{R},$

is a maximal dissipative relation with nonempty resolvent set $\rho(C)$. The function N has the following representations

(5.3) $N(\ell) = N(\mu)^* + (\ell - \bar{\mu})\Lambda^*(I + (\ell - \mu)(B - \ell)^{-1})\Lambda$

for all $\ell \in \rho(B) \cap \mathbb{C}^+$, and

(5.4) $\dfrac{\beta - \bar{\beta}}{\mu - \bar{\mu}}(N(\ell) + w(\ell))^{-1} = -P_{\mathbf{F}}(A - \ell)^{-1}|_{\mathbf{F}}$

for all $\ell \in \mathbb{C}^+$ in $\rho(A) \cap \rho(B) \cap \rho(C)$.

When we may take $\beta = \mu$ (which is the case, for instance, when $\kappa = 0$), then $w(\ell) = \ell$ and (5.4) reduces to

$$(N(\ell) + \ell)^{-1} = -P_{\mathbf{F}}(A - \ell)^{-1}|_{\mathbf{F}}.$$

Proof of Theorem 5.1. Define the function $\Phi : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ by

$$\Phi(z(\ell)) = N(\ell)/(\mu - \bar{\mu}), \quad \ell \in \mathcal{D}(N).$$

Then

$$\frac{\Phi(z(\ell)) + \Phi(z(\lambda))^*}{1 - z(\ell)\overline{z(\lambda)}} = \left(\frac{\lambda - \bar{\mu}}{\mu - \bar{\mu}}\right)^* \frac{N(\ell) - N(\lambda)^*}{\ell - \bar{\lambda}} \left(\frac{\ell - \bar{\mu}}{\mu - \bar{\mu}}\right), \quad \ell, \lambda \in \mathcal{D}(N),$$

which implies that Φ belongs to the class $\mathbf{C}_\kappa(\mathbf{F})$. Hence, by Theorem 4.3, there is an $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$ such that $\Phi(z) + \alpha$ is invertible for all $z \in \mathcal{D}_\alpha$. By defining $\beta = (\mu - \bar{\mu})\alpha$, we obtain $\beta \in \mathbb{C}^+$ such that $N(\ell) + \beta$ is invertible for all $\ell \in \mathcal{D}_\beta$. By Theorem 4.3 the function $\Theta : \mathbb{D} \to \mathbf{L}(\mathbf{F})$ defined by $\Theta(z) = (\Phi(z) - \bar{\alpha})(\Phi(z) + \alpha)^{-1}$ belongs to the class $\mathbf{S}_\kappa(\mathbf{F})$. The function Θ can be directly expressed in terms of the function N:

$$\Theta(z(\ell)) = (N(\ell) + \bar{\beta})(N(\ell) + \beta)^{-1}$$

and

$$I - \Theta(z(\ell)) = (\beta - \bar{\beta})(N(\ell) + \beta)^{-1}$$

for $\ell \in \mathcal{D}_\beta$. Note that

$$\frac{I - \Theta(z(\lambda))^*\Theta(z(\ell))}{1 - z(\ell)\overline{z(\lambda)}} =$$
$$= \frac{\beta - \bar{\beta}}{\mu - \bar{\mu}} \left(\frac{N(\lambda) + \beta}{\lambda - \bar{\mu}}\right)^{-*} \frac{N(\ell) - N(\lambda)^*}{\ell - \bar{\lambda}} \left(\frac{N(\ell) + \beta}{\ell - \bar{\mu}}\right)^{-1},$$

which again shows that Θ belongs to the class $\mathbf{S}_\kappa(\mathbf{F})$. The function Θ is the characteristic function of a unitary colligation $(T, F, G, H; \mathbf{K}, \mathbf{F})$ as in Theorem 4.2. With $\mu \in \mathbb{C}^+$ we consider the inverse Cayley transforms $A = F_\mu(U)$, $B = F_\mu(V)$ and $C = F_\mu(T)$ of the unitary operators U, V and the contraction T, respectively. In $\mathbf{K} \oplus \mathbf{F}$ we define the relation A by

$$A = \{ \{(U - I)h, (\mu U - \bar{\mu})h\} \mid h \in \mathbf{K} \oplus \mathbf{F} \},$$

and in $\mathbf{K}$ we define the relations B and C by

$$B = \{ \{(V - I)h, (\mu V - \bar{\mu})h\} \mid h \in \mathbf{K} \}$$

and

$$C = \{ \{(T - I)h, (\mu T - \bar{\mu})h\} \mid h \in \mathbf{K} \}.$$

It turns out that A and B are selfadjoint relations, and that their resolvent sets are non-empty (see [9],[10]) and that the relation C is maximal dissipative ($\mathrm{Im}\, C \leq 0$) with non-empty resolvent set. From the definition of V it follows that

$$B = \{ \{P_\mathbf{K}(U - I)h, P_\mathbf{K}(\mu U - \bar{\mu})h\} \mid h \in \mathbf{K} \oplus \mathbf{F}, (I - U)h \in \mathbf{K} \},$$

which implies that (5.1) is valid. From the definition of T, i.e., $T = P_\mathbf{K} U|_\mathbf{K}$, it follows that

$$C = \{ \{P_\mathbf{K}(U - I)h, P_\mathbf{K}(\mu U - \bar{\mu})h\} \mid h \in \mathbf{K} \},$$

which implies that (5.2) is valid. According to Theorem 4.3 the function Φ has the representation:

$$\Phi(z) = -\Phi(0)^* + \Lambda^*(I - zV)^{-1}\Lambda, \quad z \in \rho_i(V),$$

Now we use the identity

$$(I - z(\ell)V)^{-1} = \frac{\ell - \bar{\mu}}{\mu - \bar{\mu}}(I + (\ell - \mu)(B - \ell)^{-1})$$

to obtain (5.3). We use the identity (4.8) valid for $z \in \rho_i(V) \cap \rho_i(T) \cap \rho_i(U)$ to obtain

$$P_\mathbf{F}(I - zU)^{-1}|_\mathbf{F} = \frac{\ell - \bar{\mu}}{\mu - \bar{\mu}}(I - \frac{\beta - \bar{\beta}}{\mu - \bar{\mu}}(\ell - \mu)(N(\ell) + w(\ell))^{-1}).$$

Therefore

$$\frac{\beta - \bar{\beta}}{\mu - \bar{\mu}}(N(\ell) + w(\ell))^{-1} = \frac{1}{\ell - \mu} - \frac{\mu - \bar{\mu}}{(\ell - \mu)(\ell - \bar{\mu})} P_\mathbf{F}(I - zU)^{-1}|_\mathbf{F},$$

which shows (5.4). This completes the proof.

A consequence of the representation (5.3) in Theorem 5.1 is that

$$(5.5)\quad \frac{N(\mu)-N(\mu)^*}{\mu-\bar{\mu}}=\Lambda^*\Lambda.$$

The representation (5.3) can be found in [18], [20] under additional conditions on the asymptotic behaviour of N in order to make sure that the selfadjoint relation in the representation is an operator. If $\kappa=0$ these asymptotic conditions were dropped in [22]. In all these papers there is the extra condition that the operator $\Lambda^*\Lambda$ in (5.5) is positive. Note that if $N:\mathbf{C}^+\to\mathbf{L}(\mathbf{F})$ is a holomorphic function, such that $\operatorname{Im} N(\ell)\geq 0$ when $\ell\in\mathbf{C}^+$, then N belongs to the class $\mathbf{N}_0(\mathbf{F})$.

To facilitate the statement of the next theorem we introduce the following notation. For $\beta\in\mathbf{C}^+$ we define $r(\beta)$ by

$$r(\beta)=\frac{-\bar{\beta}\mu+\beta\bar{\mu}-(\mu-\bar{\mu})N(\mu)}{\beta-\bar{\beta}}.$$

THEOREM 5.2. *Let* $\mathbf{F}$ *be a Hilbert space and let* $\mathbf{K}$ *be a Pontryagin space of index* κ. *Let* $\Lambda\in\mathbf{L}(\mathbf{F},\mathbf{K})$, *and let* B *be a selfadjoint relation in the Pontryagin space* $\mathbf{K}$, *which is* ran Λ-*minimal. Then the function* $N:\mathbf{C}^+\to\mathbf{L}(\mathbf{F})$ *with representation (5.3) belongs to* $\mathbf{N}_\kappa(\mathbf{F})$, *and is holomorphic at* μ. *Moreover, there exists a* $\beta\in\mathbf{C}^+$, *such that*

$$(5.6)\quad A=\left\{\left\{\begin{pmatrix} f-\Lambda k\\ k\end{pmatrix},\begin{pmatrix} g-\mu\Lambda k\\ \Lambda^*(g-\bar{\mu}f)+r(\beta)k\end{pmatrix}\right\}\mid \{f,g\}\in B,\ k\in\mathbf{F}\right\}$$

is a selfadjoint relation in $\mathbf{F}\oplus\mathbf{K}$ *as in Theorem 5.1, for which (5.4) holds.*

Again, when we may take $\beta=\mu$, then $w(\ell)=\ell$, $r(\beta)=-N(\mu)$ and (5.6) reduces to

$$A=\left\{\left\{\begin{pmatrix} f-\Lambda k\\ k\end{pmatrix},\begin{pmatrix} g-\mu\Lambda k\\ \Lambda^*(g-\bar{\mu}f)-N(\mu)k\end{pmatrix}\right\}\mid \{f,g\}\in B,\ k\in\mathbf{F}\right\}.$$

Any $N\in\mathbf{N}_\kappa(\mathbf{F})$ which is holomorphic at $\mu\in\mathbf{C}^+$ has a representation (5.3). Using (5.5), we obtain for $\ell,\lambda\in\rho(B)\cap\mathbf{C}^+$

$$\begin{aligned}&N(\ell)-N(\lambda)^*=\\ &\quad=\Lambda^*[\,\ell-\bar{\lambda}+(\ell-\mu)(\ell-\bar{\mu})(B-\ell)^{-1}-(\bar{\lambda}-\mu)(\bar{\lambda}-\bar{\mu})(B-\bar{\lambda})^{-1}\,]\Lambda.\end{aligned}$$

For $\mu\in\mathbf{C}$ we define the function $\Delta_\mu:\mathbf{C}^+\to\mathbf{L}(\mathbf{K})$ by

$$\Delta_\mu(\ell)=(I+(\ell-\mu)(B-\ell)^{-1})\Lambda.$$

Then, by means of the resolvent identity

$$(B-\ell)^{-1}-(B-\bar{\lambda})^{-1}=(\ell-\bar{\lambda})(B-\ell)^{-1}(B-\bar{\lambda})^{-1},\quad \ell,\lambda\in\rho(B),$$

we obtain

$$\begin{aligned}&(\ell-\bar{\lambda})\Delta_\mu(\lambda)^*\Delta_\mu(\ell)=\\ &\quad=\Lambda^*[\,\ell-\bar{\lambda}+(\ell-\mu)(\ell-\bar{\mu})(B-\ell)^{-1}-(\bar{\lambda}-\mu)(\bar{\lambda}-\bar{\mu})(B-\bar{\lambda})^{-1}\,]\Lambda.\end{aligned}$$

Hence we have shown that

$$\frac{N(\ell) - N(\lambda)^*}{\ell - \bar{\lambda}} = \Delta_\mu(\lambda)^* \Delta_\mu(\ell),$$

for $\ell, \lambda \in \rho(B) \cap \mathbb{C}^+$.

THEOREM 5.3. *Let $N \in \mathbf{N}_\kappa(\mathbf{F})$ and $\mu \in \mathbb{C}^+$. Assume that $\frac{N(\mu)-N(\mu)^*}{\mu-\bar{\mu}}$ is invertible. Then the selfadjoint relation B has a symmetric restriction B_0, such that the subspaces* $\operatorname{ran}(B_0 - \mu)$ *and* $\operatorname{ran}(B_0 - \bar{\mu})$ *are closed and nondegenerate. The corresponding defect spaces are given by*

$$\ker(B_0^* - \mu) = \operatorname{ran}\Lambda, \quad \ker(B_0^* - \bar{\mu}) = \operatorname{ran}(I + (\bar{\mu} - \mu)(B - \bar{\mu})^{-1})\Lambda.$$

The characteristic function of the closed symmetric relation B_0 and the selfadjoint extension B satisfies

$$(5.7) \quad Y(\ell)\Lambda = (I + (\bar{\mu} - \mu)(B - \bar{\mu})^{-1})\Lambda(N(\ell) - N(\mu))(N(\ell) - N(\bar{\mu}))^{-1},$$

for ℓ in a small neighborhood of μ. If $\frac{N(\mu)-N(\mu)^*}{\mu-\bar{\mu}}$ is positive, then (5.7) holds for all $\ell \in \mathcal{D}(N)$, with the exception of at most κ points, different from μ.

By the characteristic function $Y(\ell)$ we mean the function $X(z(\ell))$, the characteristic function of the Cayley transform $C_\mu(B_0)$, $\mu \in \mathbb{C}^+$; see [13].

Proof of Theorem 5.3. The selfadjoint relation B in $\mathbf{K}$ is the Cayley transform of the unitary operator V. It has a symmetric restriction B_0 corresponding to the isometric restriction $V_{\ker G}$ of V:

$$B_0 = \{ \{(V - I)h, (\mu V - \bar{\mu})h\} \mid h \in \ker G \}.$$

Clearly, $\operatorname{ran}(B_0 - \mu) = \ker G$ and $\operatorname{ran}(B_0 - \bar{\mu}) = V\ker G = \ker F^*$. If $\frac{N(\mu)-N(\mu)^*}{\mu-\bar{\mu}}$ is invertible, then these spaces are closed and nondegenerate, and hence the symmetric relation B_0 has defect spaces, given by

$$\ker(B_0^* - \mu) = (\ker F^*)^\perp = \operatorname{ran} F, \quad \ker(B_0^* - \bar{\mu}) = (\ker G)^\perp = \operatorname{ran} G^*,$$

which are equal to $\operatorname{ran}\Lambda$ and $\operatorname{ran}(I + (\bar{\mu} - \mu)(B - \bar{\mu})^{-1})\Lambda$, respectively.

References

[1] D. Alpay, *Reproducing kernel Kreĭn spaces of analytic functions and inverse scattering*, Doctoral dissertation, The Weizmann Institute of Science, Rehovot, Israel, 1985.

[2] D. Alpay, A. Dijksma, J. van der Ploeg and H.S.V. de Snoo, "Holomorphic operators between Kreĭn spaces and the number of squares of associated kernels", Operator Theory: Adv. Appl., 59 (1992), 11-29.

[3] D. Alpay, A. Dijksma and H.S.V. de Snoo, "Reproducing kernel Pontryagin spaces, operator colligations and operator models", in preparation.

[4] N. Aronszajn, "Theory of reproducing kernels", Trans. Amer. Math. Soc., 68 (1950), 337-404.

[5] T.Ya. Azizov and I.S. Iohvidov, *Linear operators in spaces with indefinite metric*, Nauka, Moscow, 1986 (Russian) (English translation: John Wiley, New York, 1989).

[6] J. Bognar, *Indefinite inner product spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[7] P. Bruinsma, *Interpolation problems for Schur and Nevanlinna pairs*, Doctoral dissertation, University of Groningen, 1991.

[8] K. Daho and H. Langer, "Matrix functions of the class N_κ", Math. Nachr., 120 (1985), 275-294.

[9] A. Dijksma and H.S.V. de Snoo, "Symmetric and selfadjoint relations in Kreĭn spaces I", Operator Theory: Adv. Appl., 24 (1987), 144-166.

[10] A. Dijksma and H.S.V. de Snoo, "Symmetric and selfadjoint relations in Kreĭn spaces II", Ann. Acad. Sci. Fenn., Ser. A I Math., 12 (1987), 199-216.

[11] A. Dijksma, H. Langer and H.S.V. de Snoo, "Selfadjoint Π_κ-extensions of symmetric subspaces: an abstract approach to boundary problems with spectral parameter in the boundary conditions", Integral Equations Operator Theory, 7 (1984), 459-515.

[12] A. Dijksma, H. Langer and H.S.V. de Snoo, "Characteristic functions of unitary operator colligations in Π_κ-spaces", Operator Theory: Adv. Appl., 19 (1986), 125-194.

[13] A. Dijksma, H. Langer and H.S.V. de Snoo, "Generalized coresolvents of standard isometric operators and generalized resolvents of standard symmetric relations in Kreĭn spaces", Operator Theory: Adv. Appl., 48 (1990), 261-274.

[14] A. Dijksma, H. Langer and H.S.V. de Snoo, "Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions", to appear in Math. Nachr.

[15] I.C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space*, Nauka, Moscow, 1965 (English translation: Amer. Math. Soc. Transl., Math. Monographs, 18 (1969)).

[16] I.S. Iohvidov, M.G. Kreĭn and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Akademie-Verlag, Berlin, 1982 (Reihe: Mathematical Research, 9).

[17] M.G. Kreĭn, "Hermitian-positive kernels on homogeneous spaces I", Ukrain. Math. Zh., 1(4) (1949), 64-98 (Russian) (English translation: Amer. Math. Soc. Transl., (2) 34 (1963), 69-108).

[18] M.G. Kreĭn and H. Langer, "Über die Q-Funktion eines π-hermiteschen Operators im Raume Π_κ, Acta Sci. Math. (Szeged), 34 (1973), 191-230.

[19] M.G. Kreĭn and H. Langer, "Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_κ", *Hilbert Space Operators and Operator Algebras* (Proc. Int. Conf., Tihany, 1970), Colloq. Math. Soc. Janos Bolyai, North-Holland, Amsterdam, 5 (1972), 353-399.

[20] M.G. Kreĭn and H. Langer, "Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhangen. I: Einige Funktionenklassen und ihre Darstellungen", Math. Nachr., 77 (1977), 187-236.

[21] M.G. Kreĭn and H. Langer, "Some propositions on analytic matrix functions related to the theory of operators in the space Π_κ", Acta Sci. Math. (Szeged), 43 (1981), 181-205.

[22] H. Langer and B. Textorius, "On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space", Pacific J. Math., 72 (1977), 135-165.

[23] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

AUTHORS' ADDRESS:

Instituut voor Wiskunde en Informatica
Rijksuniversiteit Groningen
Postbus 800
9700 AV Groningen, Nederland