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A Lecture on Interpolation and Approximation of Matrices and Operators

ABSTRACT

We give a didactical account of how classical interpolation theory in the sense of Nevanlinna-Pick, Hermite-Fejer and Schur-Takagi can be generalized to a purely algebraic context. J-unitary or symplectic matrices and their state space realizations play hereby a major role. The theory leads to the solution of matrix and operator approximation problems of rather great generality. The exposition is informal and tries to indicate why the theory works. We refer the reader to the literature for precise mathematical treatments.

1. Introduction

This paper is a write-up of the lecture that I gave at the KNAW-workshop to which the present book of notes is devoted. I have tried to give an informal account on how I look at interpolation and approximation theory, rather than give a rehash of papers that are already published or on the way to the printer. The topic is interpolation and approximation of matrices and operators rather than complex functions. To me, it is amazing that theories like Nevanlinna-Pick interpolation and model reduction in the sense of 'AAK' [1] generalize to the context of arbitrary matrices and operators, once their algebraic structure is understood. Complex function theory is not needed anymore, and the results are rather more general than the classical ones. This is in line with the recent interest in time-varying system theory to which the present workshop is dedicated.

The main player in interpolation theory is the J-unitary or symplectic matrix. Its state space is strongly determined by the interpolation data. This was recognized by researchers working on Reproducing Kernel Hilbert Spaces or Reproducing Kernel Krein Spaces[3, 4, 2]. Here, I shall not pursue the theory from that point of view, but find it interesting to notice that a J-unitary matrix or operator (henceforth I shall use the term operator for both) determines such a reproducing kernel space $\mathcal{H}_\Theta$ (up to a constant factor). My approach will be based on system theory and more precisely on the properties of the Hankel operator and the spaces related to it. The natural linkage interpolation, controllability and J-unitarity turns out to provide the construct needed for strong approximation theory (i.e. in a strong norm).

Precise accounts of the results that I will cover intuitively here are now available [5, 6]. An interested reader should consult those papers rather than rely on my somewhat vague and intuitive treatment. The method that I will follow is to retrace the path from classical problems to the modern treatment, and to do that in an easygoing way provided by hindsight. I shall start out with the classical Nevanlinna-Pick problem, connect up with system theory, move to time varying systems, and finally show how a time varying version of interpolation can be solved and can also lead to a strong model order reduction theory.

The classical Nevanlinna-Pick interpolation theory

Let $\{z_i\}_{i=1..n}$ be a set of points in the unit disc of the complex plane $\mathbf{D} = \{\mathbf{z} \in \mathbf{C} : |\mathbf{z}| < 1\}$ and let $\{s_i\}_{i=1..n}$ be a set of values. The Nevanlinna-Pick problem is to find a complex function $S(z)$ with the following properties:
(1) $S(z)$ is analytic in $\mathbf{D}$;
(2) $\|S(z)\| < (\leq) 1$ in $\mathbf{D}$
(3) $S(z_i) = v_i$.

The *classical Schur problem* starts out with somewhat different data. Now $S(z)$ must satisfy (1), (2) above, and:
(3a) $S(z) = s_0 + s_1 z + \ldots + s_{n-1} z^{n-1} + O(z^n)$
i.e. the first n terms of the Maclaurin series of $S(z)$ are specified. The mixed case is often called the *Hermite-Fejer* case. It can be handled as easily as the two previous provided it is approached from the right angle. The data may be 'directional' as well (see further). When we'll treat generalized interpolation, we shall see how to handle all these cases in one big breadth.

The solution of the classical Nevanlinna-Pick Problem can be given in terms of a J-unitary or symplectic matrix function $\Theta(z)$ of the complex variable z. I'll give the solution here but it turns out to be a special case of a rather more general proof which I will describe later. We need a number of basic notions which I quickly review here.

1. Put:

$$U(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - z\bar{z}_i}$$

(in the Nevanlinna-Pick case) or $U(z) = z^n$ (in the Schur case). We shall say that two functions $f(z)$ and $g(z)$ which are analytic in the unit disc $\mathbf{D}$ are equivalent modulo U: $f \equiv g \bmod \mathrm{U}$ if $f(z) - g(z) = U(z)h(z)$ with $h(z)$ analytic in $\mathbf{D}$.

2. A function $S(z)$ which is analytic and contractive in $\mathbf{D}$ (meaning: for all $z \in \mathbf{D}$, $\|S(z)\| \leq 1$)

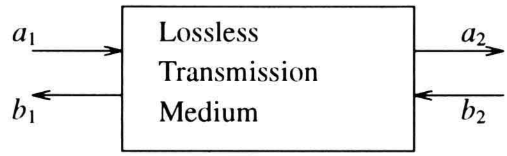

Figure 1: Circuit Diagram for a Transmission Medium

can be viewed as a transfer function representing a 'lossy load' which maps an 'incident wave' - say a with z-transform $\hat{a}(z)$ - to a reflected wave - say b with z-transform $\hat{b}(z)$, via the relation $\hat{b}(z) = \hat{a}(z)S(z)$. Underlying the z-transform formalism there is a Hilbert-space isomorphism between time-series bound in square norm representing energy, and the Hardy space H_2 of the unit disc $\mathbf{D}$ of the complex plane. The fact that $S(z) \in H_\infty$ stands for 'causality' of the reflection, and the norm constraint stands for loss of energy in the reflection process (it is lossy). Let us call the abstract scattering operator which maps a to b, $\mathcal{S}$. It is a contrative Hilbert space operator of the Töplitz type. If the total energy in the reflected wave is equal to the total energy in the incident wave, then $\mathcal{S}$ will be isometric. We say that $\mathcal{S}$ is *lossless*. Often we shall require a lossless $\mathcal{S}$ to be unitary as well.

3. A lossless *transmission medium* will connect a pair of incident/reflected waves at its 'first' port to a pair of incident/reflected waves at its 'second' port in such a way that the global operator mapping incident waves to reflected waves is unitary - see fig.1.

4. *Notation*: If nothing else is indicated we use a prefix notation. E.g. $u\mathcal{S}$ means $\mathcal{S}$ applied to u. This notation makes circuit diagrams (drawn from left to right) consistent with the application of functions. If u has a vector representation, it will typically be a 'row-vector'. Sometimes we have to deviate from the notation, in which case we use brackets to indicate application of a function. Unfortunately, brackets may also have a different meaning, namely the reordering of the formula.

5. With respect to fig. 1, the operator mapping $a_1, b_2 \mapsto a_2, b_1$ is the scattering operator Σ, while the operator mapping the waves of the left port a_1, b_1 to those of the right port a_2, b_2 is called the Chain Scattering Operator (CSM) Θ, if these operators exist, of course. A lossless transmission system can be used to manufacture lossy operators, or to transfer one such operator into another. This we do by connecting an arbitrary lossy load at port 2: $\hat{b}_2 = \hat{a}_2 S_L(z)$ - see fig. 2.

6. As anounced, we shall take Σ to be unitary. Corresponding to this property, the chain scattering matrix Θ will be J-unitary, where J is a sign matrix corresponding to the splitting of the port in incident and reflected waves. E.g. if m is the (instantaneous) dimension of a_1 and n of b_1 then the corresponding splitting is given by

$$J = \begin{bmatrix} I_m & \\ & -I_n \end{bmatrix}$$

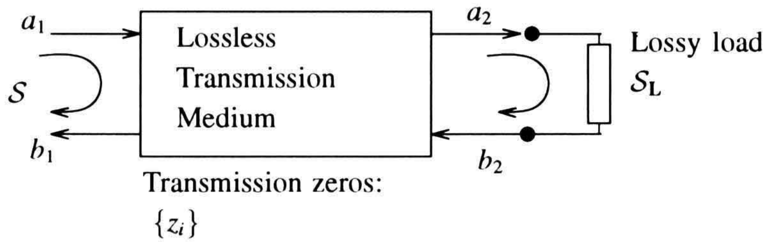

Figure 2: A lossless medium with CSM Θ transfers a lossy load S_L to a lossy input scattering function $S = T_\Theta(S_L)$.

In this elementary case the splitting of the first port has to be equal to the splitting of the second port, and Θ has to be J-unitary in the sense that

$$\Theta J \Theta^* = J, \quad \Theta^* J \Theta = J.$$

In particular, Θ_{11} and Θ_{22} will be invertible. Θ need not be analytic in **D**, although we shall only use such Θ's. The fact that Θ is J-unitary guarantees the existence of the scattering operator Σ, but does not guarantee that it is analytic in **D**, even though Θ is. In section 5 we shall encounter a case where the lack of analyticity of Σ (which then corresponds to an unphysical system) is an essential ingredient of the solution of the problem.

7. A characteristic property of the lossless transmission medium is its collection of transmission zeros. They are zeros of both Θ and of Σ_{11}, which regulates the transmission from first port to second port in the medium. Let's investigate how these transmission zeros affect the signal propagation. An expression for the transfer of the incident wave a_1 at the first port to the propagating (reflected) wave at the second port in terms of Σ and S_L is given by:

$$\hat{a}_2 = \hat{a}_1 \Sigma_{11} [I - S_L \Sigma_{21}]^{-1} \tag{1}$$

We show (later) that Σ_{21} - the scattering function looking into the medium when the input port is perfectly matched, i.e. $\hat{a}_1 = 0$ - is strictly contractive. It follows that $\hat{a}_2$ inherits in a certain sense the zeros of Σ_{11}. Let us choose $\hat{a}_1$ as belonging to an H_2 space (i.e. the wave is zero for negative time points). If the function $\hat{a}_1$ is scalar then it is obvious that the resulting $\hat{a}_2$ will inherit the zeros of Σ_{11}. If not, then a slight complication arises due to the fact that $\hat{a}_1$ is a factor on the right hand side of Σ_{11}. This is dealt with as follows. Instead of considering only one input function $\hat{a}_1$ at a time, we consider a collection (as dictated by the problem at hand), and stack the individual waves into one global vector of functions $\hat{a}_1$. For example, if our interest goes to interpolation properties of the input scattering function S, we may take as input the identity $\hat{a}_1 = I$ with as reflected wave $\hat{b}_1 = S$. But this is not the only possibility, see later sections. With such a generalized input $\hat{a}_1$, let $U(z)$ be an inner matrix function (i.e. a function in an appropriate H_∞-space which is unitary on the unit circle) such that $\hat{a}_2 = U(z)\hat{a}'_2$, then we

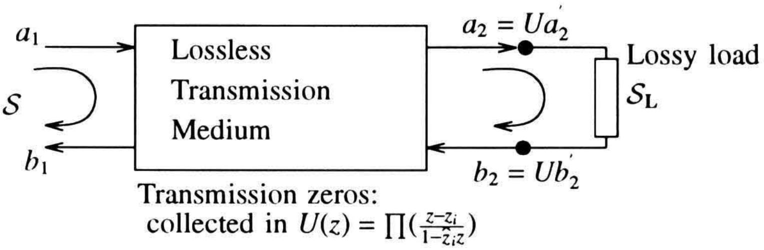

Figure 3: Propagating signals with transmission zeros collected in U

see that also $\hat{b}_2$ has a left inner factor $U(z)$, $\hat{b}_2(z) = U(z)\hat{b}'_2(z)$, and we obtain the factorization:

$$[\hat{a}_1 \quad \hat{b}_1]\Theta(z) = U(z)[\hat{a}'_2 \quad \hat{b}'_2] \tag{2}$$

Equation (2) can be interpreted as: Θ shifts $U^*(z)[\hat{a}_1 \quad \hat{b}_1]$ to analytic. We shall see that this has profound system-theoretic implications.

Let us look at the special case $\hat{a}_1 = I$, $\hat{b}_1 = S$, $U(z) = \prod_{i=1}^{n} \frac{z-z_i}{1-z\bar{z}_i}$. The waves are scalar and U is a Blaschke product. The value of S at the points z_i becomes independent of S_L, since at those frequenties, $\hat{a}_2 = 0$, $\hat{b}_2 = 0$ and S_L has no means to influence the input scattering function. This property is expressed as a *blocking property* of the transmission medium at the frequency z_i. It is an interpolation property on S: if for two loads, $S_1 = T_\Theta(S_{L_1})$ and $S_2 = T_\Theta(S_{L_2})$, we find:

$$S_1(z_i) = S_2(z_i)$$

The situation is depicted in fig. 3.

The figure helps us understand the connection between interpolation problems and *lossless inverse scattering*. We are given interpolation data on the input scattering function, and have to construct a lossless transmission medium which fits the data in the sense of the figure, namely which produces the correct transmission zeros. The process will produce a large family of interpolating input scattering functions. We shall be able to show that all interpolants can be obtained in this way, provided a minimal Θ is found that matches the given interpolation data. We now turn to the generalized context in which we shall make these ideas work.

2. The algebraic framework

The base spaces on top of which most of out objects are definid, consist of sequences of vectors which are finite in an l_2 norm. We shall need sequences with non-uniform member spaces, since our goal will be to find approximations and the behaviour of the system and its approximant might change from time point to time point.

Let $\mathcal{M}$ denote an indexed series of vector spaces:

$$\mathcal{M} = [\ldots, \mathcal{M}_{-1}, [\mathcal{M}_0], \mathcal{M}_1, \ldots]$$

in which the bracketed entry denotes the term of index zero, to identify the 'origin' of the sequence. $l_2^{\mathcal{M}}$ will consist of sequences of the type:

$$[\ldots,\ a_{-1},\ [a_0],\ a_1,\ \ldots]$$

where each $a_i \in \mathcal{M}_i$ and $\sum \|a_i\|_2^2$ is convergent. The space under consideration we denote as $\mathcal{X}$:

$$\mathcal{X} = \{T:\ l_2^{\mathcal{M}} \to l_2^{\mathcal{N}}\ : \text{T is bounded}\}$$

where $\mathcal{M}$ is the input sequence and $\mathcal{N}$ is the output sequence. $T \in \mathcal{X}$ has a representation:

$$T = \begin{bmatrix} & & \cdots & & & \\ & & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ & \cdots & T_{0,-1} & [T_{0,0}] & T_{0,1} & \cdots \\ & & \cdots & T_{1,0} & T_{1,1} & \\ & & & & \cdots & \end{bmatrix}$$

in which $T_{i,j}$ is the partial map of the ith entry in the input sequence to the jth entry in the output sequence (assuming that T works on row vectors). T has the appearance of a *tableau of matrix blocks*. Finite matrices are easily accomodated in this scheme. Take e.g. all $\mathcal{M}_j = \varnothing$ for $i \le 0$ and $i > m$ in the input sequence and $\mathcal{N}_j = \varnothing$ for $j \le 0$ and $j > n$ in the output sequence, and you obtain an $m \times n$ matrix. (*Remark:* we shall not use the vector space $\{0\}$ consisting of the sole element zero: it is irregular since it cannot have a dimension equal to either its row or column size).

Entries $T_{i,i}$ in the tableau are called *diagonal*. If T is an operator such that its entries below the diagonal are zero, then we will say that T is *upper*. The collection of upper operators (whether bounded or not will depend on the context) we shall call $\mathcal{U}$. Similarly for the lower operators $\mathcal{L}$ and the diagonal operators $\mathcal{D}$. One should be careful with projecting operators on lower and upper parts. If T is bounded, then the tableau consisting of its upper part is not necessarily bounded. The classical example of this phenomenon is obtained by taking

$$T_{k,j} = \frac{\sin \pi(k-j)/2}{\pi(k-j)},\quad (T_{k,k} = \frac{1}{2}$$

In this case T is Töplitz and corresponds to the spectral function $W(\theta) = \{1$ if $-\pi/2 \le \theta \le \pi/2$ else $0\}$. The upper part would then correspond to the transfer function $\frac{1}{2} + \frac{1}{\pi}\arctan z$ which does not belong to H_∞ and whose corresponding Toeplitz operator is not bounded, since the norm of the latter is equal to that H_∞ norm.

Even in the matrix case, the inverse of an upper matrix does not have to be upper. Take e.g.

$\mathcal{M} = \{\mathbf{C}, [\mathbf{C}], \mathbf{C}\}$ and $\mathcal{N} = \{[\mathbf{C}], \mathbf{C}, \mathbf{C}\}$, and consider the upper matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ [1] & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(where the 0, 0th element is indicated with square braces). Its inverse is:

$$T^{-1} = \begin{bmatrix} 1 & [0] & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

which is a lower triangular matrix! The example may seem contrived, but a further study of the approximation theory will convince you that it is not: the indexing scheme is essential in determining properties like upper/lower, therefore, instead of using the term 'matrix', we use the term 'tableau'.

Upper operators correspond to causal operators in the sense of system theory, when input signals act on the left hand side of the tableau, as in

$$[.., 0, .., 0, [a_0], a_1, ..] \begin{bmatrix} .. & & & & \\ & T_{-1,-1} & T_{-1,0} & .. & \\ & & [T_{0,0}] & T_{0,1} & .. \\ & & & T_{1,1} & \\ & & & & .. \end{bmatrix}$$

$$= [.., 0, [a_0 T_{0,0}], a_0 T_{0,1} + a_1 T_{1,1}, ..]$$

Our approximation/interpolation theory will be based on a so-called 'time-varying' system theory. We shall need an analysis of the system theoretic situation from the input-output point of view at each point in the sequence. In particular, we shall determine a canonical state space model from input-output 'measurements'. This is best achieved by extending the input space (and in a similar way the output space): for each point in the (time-) sequence we utilize a test input and stack all these inputs in a large input tableau. We shall require some boundedness of such input-tableaus. Not only will each row in the tableau be in an l_2-space, we shall request that the whole tableau itself has bounded quadratic norm, namely that it is Hilbert-Schmidt. Let us denote by $\mathcal{X}_2^{\mathcal{M}}$ or $\mathcal{X}_2^{\mathbf{T}\times\mathcal{M}}$ the Hilbert-Schmidt space obtained by taking a series from $\mathcal{M}$ for each point in $\mathbf{T}$ in such a way that the global sum of squared norms of entries is finite. Typically, $\mathbf{T} = \mathbf{Z}$, but it could also be a finite sequence, depending on the needs. An element

$u \in \mathcal{X}_2^{\mathcal{M}}$ will look as follows:

$$u = \begin{bmatrix} & & \cdot\cdot & & \\ & u_{-1,-1} & u_{-1,0} & u_{-1,1} & \cdot\cdot \\ \cdot\cdot & u_{0,-1} & [u_{0,0}] & u_{0,1} & \cdot\cdot \\ & \cdot\cdot & u_{1,0} & u_{1,1} & \cdot\cdot \\ & \underbrace{\cdot\cdot}_{} & \underbrace{\cdot\cdot}_{} & \underbrace{\cdot\cdot}_{} & \end{bmatrix}$$
$$\cdot\cdot \quad \mathcal{M}_{-1} \quad \mathcal{M}_0 \quad \mathcal{M}_1 \quad \cdot\cdot$$

where, for each k, $u_{k,i} \in \mathcal{M}_i$, and $\sum_{i,j} \|u_{i,j}\|^2$ is bounded. As before, we may consider upper, diagonal and lower tableaus in $\mathcal{X}_2^{\mathcal{M}}$, and we shall denote these by $\mathcal{U}_2^{\mathcal{M}}$, $\mathcal{D}_2^{\mathcal{M}}$, $\mathcal{L}_2^{\mathcal{M}}$ respect. All these spaces are Hilbert spaces for the classical Hilbert-Schmidt inner product:

$$(u, v)_{HS} = \text{trace}(uv^*)$$

where the tableau product uv^* has to be interpreted as inner products of constituent rows-with-columns. The action of an operator $T : l_2^{\mathcal{M}} \to l_2^{\mathcal{N}}$ can be extended to $T : \mathcal{X}_2^{\mathcal{M}} \to \mathcal{X}_2^{\mathcal{N}}$ just by stacking multiple applications: for every point in the time series there is a map. A state space model for T, if one exists, can be obtained from what we shall call the *Hankel operator* in the present situation. Let $\Pi_{\mathcal{A}}$ indicate the Hilbert space projection operator on the subspace $\mathcal{A}$, then we define, for an operator T, the Hankel operator

$$H_T = \Pi_{\mathcal{U}_2^{\mathcal{N}}}(\cdot T)\,|_{\mathcal{L}_2^{\mathcal{M}} Z^{-1}}$$

Let

$$\begin{cases} K_c &= \text{kernel}(H_T) &\subset \mathcal{L}_2^{\mathcal{M}} Z^{-1} \\ H_c &= \overline{\text{range}}(H_T^*) &\subset \mathcal{L}_2^{\mathcal{M}} Z^{-1} \\ K_o &= \text{kernel}(H_T^*) &\subset \mathcal{U}_2^{\mathcal{N}} \\ H_o &= \overline{\text{range}}(H_T) &\subset \mathcal{U}_2^{\mathcal{N}} \end{cases}$$

define the canonical spaces related to the operator H_T. They have the following basic properties:
(1) K_c, H_c, K_o, H_o are invariant for left multiplication with diagonal operators (of appropriate dimensions).
(2) K_o is invariant for left multiplication with the shift operator

$$Z = \begin{bmatrix} \ddots & & & & \\ 0 & I & & & \\ & [0] & I & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}$$

K_c is invariant for left multiplication with the shift operator Z^*.

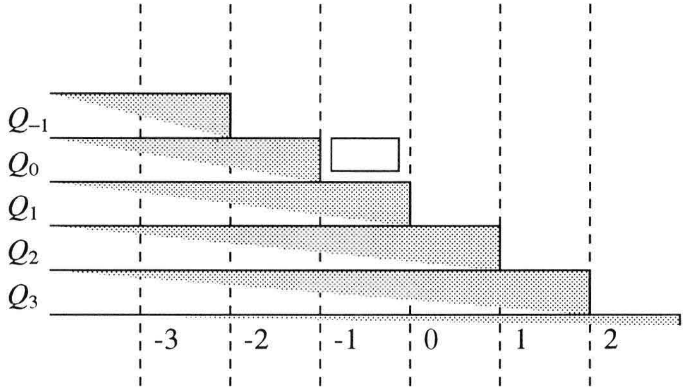

Figure 4: Stacking base vectors in the sliced space H_c

(3) H_c is invariant for the restricted shift defined by

$$\Pi_{\mathcal{L}_2 Z^{-1}}(Z\cdot)$$

while H_o is invariant for the restricted shift defined by

$$\Pi_{\mathcal{U}_2}(Z^*\cdot)$$

Because of the first property, H_c and H_o are what we could call *sliced spaces*. A basis for these spaces can be constituted of local bases, one for each point in the base sequence. Let $\Pi_i H_c$ denote the projection on the ith slice. We have that $\Pi_i H_c \subset H_c$. We shall say that H_c is locally finite if $\Pi_i H_c$ is finite dimensional for all i.

Our next goal is to produce a 'concrete' tableau representation for the spaces H_c and H_o. Let's consider the first, the reasoning for the second will be dual. Let Q_i be a tableau consisting of δ_i rows forming an orthonormal basis for $\Pi_i H_c$. Each has support on the interval $(-\infty, \ldots, i-1]$. We stack these Q_i to form a large tableau of the form of fig. 4. Let $\mathcal{B} = \{\mathbf{C}^{\delta_i} : i = -\infty \ldots \infty\}$, then Q is a tableau of dimensions $\mathcal{B}\times\mathcal{M}$. We shall soon see under which conditions it belongs to $\mathcal{X}^{\mathcal{B}\times\mathcal{M}}$. Even though it may not be bounded in the operator norm, operations like multiplication with a diagonal and shifting with Z or Z^* are meaningful on Q. The restricted shift invariance of the previous paragraph says that if a slice, e.g. Q_i, is shifted one notch upwards, and its head $[Q_i]_{i-1}$ is subsequently chopped off, the remainder (its tail) $[Q_i]_p$ lays in the space generated by Q_{i-1}. Hence there will exist a $\delta_{i-1}\times\delta_i$ matrix A_{i-1} such that

$$[Q_i]_p = A_{i-1}^* Q_{i-1}$$

Using a global notation, let us represent Q by the diagonal series

$$Q = Z^* Q_{[-1]} + Z^{*2} Q_{[-2]} + ..,$$

let us collect the $\{A_{[i]}\}$ in a diagonal matrix $A = \mathrm{diag}(.., A_{-1}, A_0, A_1, ..)$, and let $B = Q_{[-1]}^*$, then we have, by shift invariance,

$$\Pi_{\mathcal{L}_2Z^*}(ZQ) = A^*Q$$

and

$$Q = Z^*B^* + Z^*A^*Q. \qquad (3)$$

Assume that $l_A < 1$. Then (3) can be solved for Q and we obtain

$$Q = (I - Z^*A^*)^{-1}Z^*B^*. \qquad (4)$$

(4) is an efficient 'state representation' for Q, which will belong to $\mathcal{X}^{\mathcal{B}\times\mathcal{M}}$. Since the basis choosen in $\mathcal{B}$ was orthonormal, it will follow from (3) that

$$A^*A + B^*B = I$$

and the pair A, B is in 'input normal form'. The realization can be completed by computing matrices C and D from the application of the transfer operator $T = T_{[0]} + ZT_{[1]} + ..$:

$$D = T_{[0]}, \;\; C = (QT)_{[0]}$$

and D is the map from instantaneous input to instantaneous output, while C maps the actual state to the instantaneous output. It is now easy to see that the model for T in input normal form has become:

$$\begin{cases} XZ^{-1} & = & XA + UB \\ Y & = & XC + UD \end{cases}$$

$$T = D + BZ(I - AZ)^{-1}C$$

the state being given by projection of the (strict) past input U_p on the space H_o:

$$X = \Pi_{H_o}(U_p) = U_pQ^*Q.$$

A similar reasoning can be performed on H_c to yield a realization in output normal form, i.e. a realization for which

$$AA^* + CC^* = I.$$

We see also that 'non-minimal' realizations in input/output normal form can be obtained, when shift-invariant spaces larger than H_c or H_o are choosen and the canonical realization procedure is exercised on them. Finally, we see that reachability and observability spaces (either minimal or not) have the following concrete representations, when $l_A < 1$:

$$H_c = \mathcal{D}_2^{\mathcal{B}}(I - Z^*A^*)^{-1}Z^*B^*$$

$$H_o = \mathcal{D}_2^{\mathcal{B}}(I - AZ)^{-1}C$$

expressions which will be useful in the sequel.

External factorizations

Suppose that $[A, B, C, D]$ is a realization in output normal form for $T \in \mathcal{U}$ (minimal or not), then we can define a unitary transfer operator U_r via the realization $[A, B_u, C, D_u]$ in which

$$\begin{bmatrix} A & C \\ B_u & D_u \end{bmatrix}$$

is a block diagonal matrix which is unitary at each point i. Then T has the representation

$$T = \Delta_r^* U_r (= \Delta^* (U_r^*)^{-1}) \tag{5}$$

with $\Delta_r \in \mathcal{U}$. We call (5) a *(right) external factorization* since it represents T as the (right) ratio of two upper operators. Likewise, a *(left) external factorization* can be obtained from a realization in input normal form. When the realizations are minimal, then the external factorizations are actually coprime in the sense that they will have no common non-trivial factor between them.

The computation of U_r amounts to a kind of 'Beurling-Lax' theorem for subspaces with the invariance properties of H_c or H_o.

State space transformations

Suppose that the diagonal operators $[A, B, C, D]$ form a realization for T and that $l_A < 1$, then another realization with the same boundedness property can be obtained by applying a state transformation

$$X = \hat{X}R$$

in which R is a diagonal operator which is bounded and has bounded inverse. The state transformation is actually an isometry on $\mathcal{X}_2^B$. The transformation induced on the state space representation follows easily:

$$\begin{cases} \hat{X}Z^{-1} & = & XR^{-1}Z^{-1} = XZ^{-1}(R^{(-1)})^{-1} \\ & = & \hat{X}[RAR^{(-1)})^{-1}] + UB(R^{(-1)})^{-1} \\ Y & = & \hat{X}RC + UD \end{cases}$$

so that

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} \mapsto \begin{bmatrix} RA(R^{(-1)})^{-1} & RC \\ B(R^{(-1)})^{-1} & D \end{bmatrix}.$$

Suppose that our original realization is not in input normal form and that we wish to find a transformation R that will bring it there, then R would have to satisfy the equation

$$A^*(R^*R)A + B^*B = (R^*R)^{(-1)}$$

or, with $R^*R = M$,

$$A^*MA + B^*B = M^{(-1)}. \tag{6}$$

(6) is called a discrete-time Lyapunov equation (in the mathematical literature sometimes a Lyapunov-Stein equation). It has a unique solution, when $l_A < 1$, which can be expressed in terms of the continuous products of shifts of A. Let $A^{\{k\}} = A^{(k-1)}A^{(k-2)}..A$, with $A^{\{0\}} = I$ and $A^{\{1\}} = A$, then the solution is given by

$$M = \left[\sum_{k=0}^{\infty}(A^{\{k\}})^*(B^*B)^{(k)}A^{\{k\}}\right]^{(+1)} = \mathcal{C}^*\mathcal{C}$$

in which $\mathcal{C}$ is the operator

$$\mathcal{C} = \begin{bmatrix} B^{(+1)} \\ B^{(+2)}A^{(+1)} \\ B^{(+3)}A^{(+2)}A^{(+1)} \\ .. \end{bmatrix}$$

(it maps l_2-sequences of diagonals to a diagonal). An alternative expression for M in terms of Q has been given earlier:

$$M = (Q^*Q)_{[0]} = \left\{(I - Z^*A^*)^{-1}Z^*B^*BZ(I - AZ)^{-1}\right\}_{[0]}$$

M is the (Hilbert-Schmidt) Gramian is of the basis for H_c, which is implicitely choosen by the realization pair $[A, B]$. We say that the realization is *strictly controllable* if M is boundedly invertible. In that case, M can be factored as $M = R^*R$ with R boundedly invertible. Applying R as state transformation, we obtain an input normal form for which $M = I$.

Inner-outer factorizations

We shall say that an operator $F \in \mathcal{U}$ is outer if $\mathcal{U}_2F$ is dense in $\mathcal{U}_2$. Consider the transfer operator $T \in \mathcal{U}$ and let $\mathcal{H} = \mathcal{U}_2 \ominus \mathcal{U}_2T$. Then we can check easily that $\mathcal{H}$ is left D-invariant, and also invariant for the restricted shift $\Pi_{\mathcal{U}_2}(Z^*\cdot)$. $\mathcal{H}$ is thus a sliced space. If it is of local finite dimension, then it may be considered to be the observability space $\mathcal{H}_o$ of an inner operator U. If U has an orthogonal realization $\{A_u, B_u, C_u, D_u\}$, then $\mathcal{H}_o$ determines an isometric pair $\{A_u, C_u\}$ by a dual form of the state space realization theory detailed above. The pair $\{B_u, D_u\}$ can be found by completing the isometry to a unitary operator, and this is a local, finite dimensional matrix operation.

We obtain:

$$\mathcal{U}_2F = \mathcal{U}_2U$$

and $F_1 = FU^*$ is an outer operator. We have found an outer-inner factorization $F = F_1U$. An inner-outer factorization will follow from a dual construction. A (backward) recursive formula for $\{A_u, C_u\}$ is as follows. The recursion will be defined in terms of a diagonal operator N, to be defined now. Suppose N_i known, then $\{A_{ui}, C_{ui}\}$ is the largest isometry such that

$$[B_{i-1}N_i \;\; D_i]\begin{bmatrix} C_{ui}^* \\ D_{ui}^* \end{bmatrix} = 0$$

and

$$N_{i-1} = [A_{i-1}N_i \;\; C_{i-1}]\begin{bmatrix} C_{ui}^* \\ D_{ui}^* \end{bmatrix}$$

The recursion converges much as the Lyapunov recursion does, and it is computable in practical situations, i.e. when the behaviour of the system for times going to infinity is known. Outer-inner factorizations play an important role in robust controller design for time-varying systems, see the contribution of Michel Verhaegen in this issue.

3. J-unitary operators

The state space representation theory for J-unitary operators to be used in the next section necessitates special attention. Let us introduce first a splitting of the input and output spaces as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$. We call a signal $a_1 \in l_2^{\mathcal{M}_1}$ a *wave incident on the system*, and a signal $b_1 \in l_2^{\mathcal{M}_2}$, a wave *reflected* from the system. Likewise, $a_2 \in l_2^{\mathcal{N}_1}$ will be a reflected wave and $b_2 \in l_2^{\mathcal{N}_2}$ will be incident - see fig. 1. The left ports carry an indefinite inner product governed by the signature matrix

$$J_1 = \begin{bmatrix} I_{\mathcal{M}_1} & \\ & -I_{\mathcal{M}_2} \end{bmatrix}$$

and defined by

$$\begin{aligned} \|[a_1 \; b_1]\|_{J_1} &= [a_1 \; b_1]J_1\begin{bmatrix} a_1^* \\ b_1^* \end{bmatrix} \\ &= a_1a_1^* - b_1b_1^* \end{aligned}$$

It has the connotation of total absorbed energy by the system at the left port. Likewise, the output splitting goes with an indefinite inner product of signature matrix

$$J_2 = \begin{bmatrix} I_{\mathcal{N}_1} & \\ & -I_{\mathcal{N}_2} \end{bmatrix}$$

and whose meaning is the total energy delivered to the environment connected to the right ports. In the sequel, we shall also have to assign energy to the state, so that a complete energy balance can be set up in space and time. For that purpose we introduce a state space splitting as well. Let $\mathcal{B}$ be the sequence of state spaces, and let $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ be a decomposition in a direct sum of two spaces, and let

$$J_{\mathcal{B}} = \begin{bmatrix} I_{\mathcal{B}_+} & \\ & -I_{\mathcal{B}_-} \end{bmatrix}$$

then the energy assigned to the state will be the inner product

$$\begin{aligned} \|[x_+ \ x_-]\|_{J_{\mathcal{B}}} &= [x_+ \ x_-] J_{\mathcal{B}} \begin{bmatrix} x_+^* \\ x_-^* \end{bmatrix} \\ &= x_+ x_+^* - x_- x_-^* \end{aligned}$$

The realization theory that we shall need uses realizations that conserve energy, as follows:

Theorem 3.1

Let

$$\theta = \begin{bmatrix} A & | & C_1 & C_2 \\ \text{—} & & \text{—} & \text{—} \\ B_1 & | & D_{11} & D_{12} \\ B_2 & | & D_{21} & D_{22} \end{bmatrix}$$

be a realization of a locally finite system with input/output and state space splittings as given above, let $l_A < 1$, and assume further that each θ_i is a square matrix such that

$$\theta^* \begin{bmatrix} J_{\mathcal{B}} & \\ & J_1 \end{bmatrix} \theta = \begin{bmatrix} J_{\mathcal{B}}^{(-1)} & \\ & J_2 \end{bmatrix}$$

is satisfied, then the corresponding transfer map Θ is block-upper and satisfies

$$\Theta^* J_1 \Theta = J_2, \quad \Theta J_2 \Theta^* = J_1.$$

The proof is by brute force calculation on the expression for Θ in terms of its given realization. The property that Θ is block upper is, of course a direct consequence of the assumption $l_A < 1$. An interesting observation is that in the given circumstances, the reachability operator is $J_{\mathcal{B}}$-isometric in the sense that (P_0 indicating projection on the main diagonal)

$$P_0 \left[(I - Z^* A^*)^{-1} Z^* (B_1^* B_1 - B_2^* B_2) Z (I - AZ)^{-1} \right] = J_{\mathcal{B}}.$$

The strategy that we shall use to solve our interpolation and approximation problems will consist in constructing the appropriate θ matrix. The energy relations that it satisfies will determine the properties of the result. Let us therefore construct the corresponding scattering operator.

Since Θ is J-unitary, we have that both $\Theta_{22}^*\Theta_{22} \geq I$ and $\Theta_{22}\Theta_{22}^* \geq I$. It follows, see e.g. [7], that Θ_{22} has a bounded inverse. Hence we can define

$$\Sigma = \begin{bmatrix} I & -\Theta_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Theta_{11} & \\ & \Theta_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \Theta_{21} & I \end{bmatrix}.$$

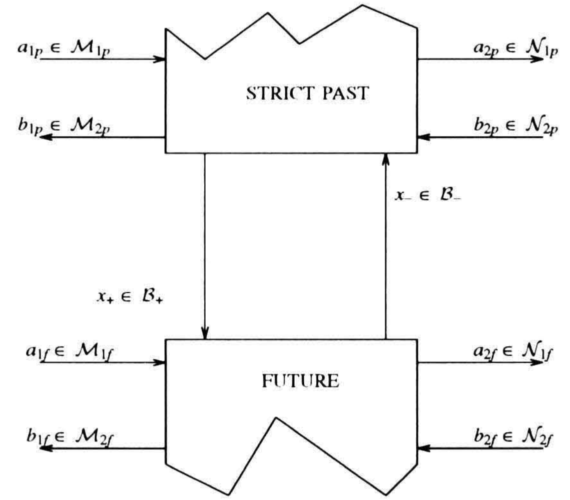

Figure 5: The energy flow defined by a $J_{\mathcal{B}}$ splitting of the chain scattering operator Θ

Σ stands for the map $[a_2 \;\; b_1] = [a_1 \;\; b_2]\Sigma$ which transfers incident waves to reflected waves. The J-unitarity of Θ reflects conservation of energy and translates to the unitarity of Σ. We write the block entries of Σ out as follows:

$$\Sigma = \begin{bmatrix} T_{in} & S_{in} \\ S_{out} & T_{out} \end{bmatrix}$$

There is no reason why Σ should be block upper. However, if Θ is block upper (and all the $\Theta's$ that we shall construct will have that property), then Σ will be block upper when the same is true for Θ_{22}^{-1}. We shall soon see that the lower part of this operator is closely related to the state space splitting that we introduced earlier.

Be this as it may, the crucial property which the entries of Σ bring with them is that they are all contractive. In the present situation a little more is true. The (2,1)-entry of Σ is the zero load input scattering operator. It satisfies $S_{in} = -\Theta_{12}\Theta_{22}^{-1}$ and inherits from the boundedness of both Θ_{22} and its inverse the property

$$I - S_{in}^* S_{in} > 0$$

and is thus *strictly* contractive. In addition, it has important interpolation properties which we shall exploit in due time.

Let us now explore the properties of Θ_{22}^{-1}. A formal treatment is to be found in [5], here we will suffice with a 'physical' argument, which, however, closely parallels the formal one. The energy picture connected to the $J_{\mathcal{B}}$ splitting of the J-unitary operator (the chain scattering operator) is shown in fig. 5. The picture holds w.r. to any point in the sequence. Θ_{22} is the map $\mathcal{M}_2 \to \mathcal{N}_2 : b_1 \mapsto b_2$, under the condition $a_1 \equiv 0$. The (strictly) upper part of Θ_{22}^{-*} has as its reachability space, the range of Θ_{22}^{-1} restricted to the strict past, i.e. the b_{1p} which originates from b_{2f} under the condition $a_1 \equiv 0$ and $b_{2p} = 0$. Conservation of energy in this situation says that

$$\|x_-\|^2 = \|b_{1p}\|^2 + \|a_{2p}\|^2.$$

A careful study of this relation shows that the dimension of the range cannot be larger than the dimension of the local $\mathcal{B}_-$ space.

If $\mathcal{M}$ is a sequence of spaces, let $\#\mathcal{M}$ denote the sequence of their dimensions, and let $N_- = \#\mathcal{B}_-$. We find that the dimension of the state space of a realization of the strictly lower part of Θ_{22}^{-1} is at most N_-. In particular, Θ_{22} is invertible in $\mathcal{U}$ when N_- is the zero sequence, i.e. when $J_{\mathcal{B}} = I$ so that the reachability operator is positive definite for the $I \oplus J_1$ metric.

In the more general case, if $J_{\mathcal{B}} \oplus J_1$ is the signature of the reachability operator, then the degree of the lower part of Θ_{22}^{-1} will be equal to the number of negative signs.

4. Interpolation and the LIS principle

With reference to fig. 3, suppose that we are given an inner function U (i.e. U is upper and unitary), and suppose that for some given, upper pair of waves $[a_1\, b_1]$ we have found a lossless transmission medium characterized by a block upper J-unitary chain scattering matrix Θ such that

$$[a_1\, b_1]\Theta = U[a_2'\, b_2'] \tag{7}$$

with a_2', b_2' upper also. Taking the second entries in this expression, we find

$$a_1\Theta_{12} + b_1\Theta_{22} = Ub_2'. \tag{8}$$

Since Θ_{22} is invertible, let $S_{in} = -\Theta_{12}\Theta_{22}^{-1}$, and

$$-a_1 S_{in} + b_1 = UB_2'\Theta_{22}^{-1} \tag{9}$$

¿From the properties of the Θ matrix, we have that

$$\|S_{in}\| < 1. \tag{10}$$

Let's first assume that Θ_{22}^{-1} is upper, then S_{in} will be upper also, and (9) says that

$$a_1 S_{in} = b_1 \bmod U \tag{11}$$

if we mean by $a = b \bmod U$ that the difference $a - b$ has a left factor U. In the case of scalar functions of the complex plane, when $\mathcal{U}$ is represented by $H_2(\mathbf{C})$, U will typically be a Blaschke product $U = \prod_{i=1}^{k} \left(\frac{z-v_i}{1-\bar{v}_i z}\right)$ and we see that (11) implies

$$a_1(v_i)S_{in}(v_i) = b_1(v_i)$$

If a_1 has been choosen $\equiv 1$ and $b_1(v_i) = s_i$ for some set of values $\{s_i\}$, then S_{in} solves an interpolation problem of the Nevanlinna-Pick type. In our more general algebraic setting similar properties hold. If V_i is a left zero of U, i.e. if we can write $U = (Z - V_i)U'$ with U' again upper, then the difference $b_1 - a_1 S_{in}$ will have the same left zero.

This fact can be expressed in terms of the *diagonal evaluation* or *W-transform* originally introduced in [8]. Let us define, for any diagonal operator W

$$W^{[k]} = WW^{(1)} .. W^{(k-1)}$$

Let A be any upper operator with diagonal expansion

$$A = \sum_{i=0}^{\infty} Z^{[i]} A_{[i]}$$

the we define the *W-transform* of A, if it exists, by the expression

$$A^{\wedge}(W) = \sum_{i=0}^{\infty} W^{[i]} A_{[i]} \tag{12}$$

The sum (12) will converge when A is bounded and the spectral radius of WZ^*, $l_W = \lim_{i\to\infty} \|W^{[i]}\|^{1/i}$ is less than one. We have that $U = (Z - V_i)U'$ if and only if $U^{\wedge}(V_i) = 0$ in which case the interpolation

$$(a_i S_{in})^{\wedge}(V_i) = s_i$$

with $s_i = b_1^{\wedge}(V_i)$ will hold in (9).

The expression (7) says that Θ maps $U^*[a_1 \ b_1]$ to upper. It is a fundamental property of unitary and J-unitary matrices that *they map their reachability space to upper.* As a result, the solution of the interpolation problem will consists in translating the interpolation data to a form which specifies a minimal reachability space of a chain scattering operator.

The case where Θ_{22}^{-1} is not upper is more delicate. It turns out that under certain conditions there will be no 'cancellations' in the product on the right hand side of (9), and a modified form of interpolation remains valid. This topic will be of great importance to the solution of the generalized AAK problem which we shall consider in the next section.

A further remark is that S_{in} is not the only contractive interpolant in $\mathcal{U}$. We may 'load' Θ with any contractive upper S_L and we will still obtain an interpolant:

$$[a_1 \ \ b_1]\Theta \begin{bmatrix} -S_L \\ I \end{bmatrix} = U(-a_2' S_L + b_2')$$

$$a_1(-\Theta_{11} S_L + \Theta_{12}) + b_1(-\Theta_{21} S_L + \Theta_{22}) = U(-a_2' S_L + b_2')$$

If S_L is contractive, then $(-\Theta_{21}S_L + \Theta_{22})$ will be invertible in $\mathcal{X}$. It will, in addition, lay in U if $\Theta_{22}^{-1} \in \mathcal{U}$, and

$$S_{in} = T_\Theta(S_L) \stackrel{\Delta}{=} -(-\Theta_{11}S_L + \Theta_{22})(-\Theta_{21}S_L + \Theta_{22})^{-1} \tag{13}$$

is a new candidate interpolant.

Back to our main theme: the interpolation problem can be reduced to the construction of a reachability space of a chain scattering operator Θ. Let us see how a few classical cases generalize to the present algebraic setting.

The Nevanlinna-Pick case

Suppose that a set of k *diagonals* $\{V_i\}$ are given as interpolation points, and a set of k *diagonals* s_i as interpolation values, and that we wish to find a contractive operator S such that

$$S^\wedge(V_i) = s_i.$$

Let us suppose that we have succeeded in finding a Θ such that, for each i,

$$(Z - V_i)^{-1}[1 \;\; s_i] \in H_c(\Theta)$$

the reachability space of Θ. Let $S_{in} = -\Theta_{12}\Theta_{22}^{-1}$. Then

$$(Z - V_i)^{-1}[I \;\; -s_i]\Theta = [a_i' \;\; b_i']$$

with a_i', b_i' upper and application of the previous theory gives

$$S_{in} - s_i = (Z - V_i)b_i'\Theta_{22}^{-1}.$$

We shall have $S_{in} \in \mathcal{U}$ when $\Theta_{22}^{-1} \in \mathcal{U}$ and then also $S_{in}^\wedge(V_i) = s_i$. Since $H_c(\Theta)$ has to be left invariant for multiplication with diagonals and for the restricted shift (see earlier), it is necessary that $H_c(\Theta)$ contains at least

$$\sum_{i=1}^{k} D_i(Z - V_i)^{-1}[I \;\; s_i]$$

with $\{D_i\}$ an arbitrary set of diagonals. Under the assumption that the $\{(Z-V_i)^{-1}[I \;\; s_i]$ with $\{D_i\}$ form a strongly diagonally independent set of operators, i.e. $\sum D_i(Z-V_i)^{-1} = 0 \Leftrightarrow (\forall i : D_i = 0)$, we have that the $\{(Z - V_i)^{-1}[I \;\; s_i]\}$ form a basis for the reachability space of a chain scattering matrix Θ with $\Theta_{22}^{-1} \in \mathcal{U}$ iff they form a strictly J-positive space, i.e. iff the block operator matrix Λ with

$$\Lambda_{i,j} = \left\{(Z - V_i)^{-1}(I - s_i s_j^*)(Z^* - V_j^*)^{-1}\right\}_{[0]} \tag{14}$$

satisfies $\Lambda > \varepsilon I$ for some $\varepsilon > 0$. It turns out that this condition is equivalent and reduces to the classical Pick condition. For good reason, (14) may be called the Pick operator [7].

Directional Nevanlinna-Pick

The directional Nevanlinna-Pick problem is but a slight extension of the previous case. If, for each V_i, directions ξ_i and η_i are assigned, and we succeed in finding a chain scattering matrix Θ such that

$$(Z - V_i)^{-1}[\xi_i \ \eta_i]\Theta \in \text{upper}$$

then, with $S_{in} = -\Theta_{12}\Theta_{22}^{-1}$, and some upper β_i,

$$(\xi_i S_{in} - \eta_i) = (Z - V_i)\beta_i$$

and the interpolation

$$(\xi_i S_{in})^{\wedge}(V_i) = \eta_i \tag{15}$$

holds. Conversely, when the interpolation data (15) is given, and under the additional assumption that the $(Z - V_i)^{-1}\xi_i$ are left diagonally independent, we shall have that the set $\{(Z - V_i)^{-1}[\xi_i \ \eta_i]\}$ forms a basis for the reachability space $H_c(\Theta)$ of a Θ-matrix with $\Theta_{22}^{-1} \in \mathcal{U}$ iff the appropriate Pick operator matrix Λ with

$$\Lambda_{i,j} = \left\{(Z - V_i)^{-1}(\xi_i\xi_j^* - \eta_i\eta_j^*)(Z^* - V_j^*)^{-1}\right\}_{[0]}$$

is strictly positive definite.

Hermite-Fejer problems

How can we discover further possibilities? In the two previous examples we saw that Θ would always move interpolation data into an invariant subspace characterized by an inner operator U. We may, of course, reverse the reasoning: to every inner U there will correspond a meaningful interpolation problem. It turns out that this principle helps us to formulate the more general Hermite-Fejer interpolation problems. They seem hard to comprehend if approached in a purely algebraic fashion [9]. I adapt the method of [2] which is both direct and elegant in the present context. For a more exhaustive treatment, see [10].

Let V be some diagonal operator of dimensions $\mathcal{N} \times \mathcal{N}^{(+1)}$ where $\mathcal{N}$ is some input sequence, and let ξ be a diagonal operator of dimensions $\mathcal{N} \times \mathcal{M}$. Define the product

$$\pi_k(V, \xi) = (Z - V^{(k-1)})^{-1}(..)(Z - V)^{-1}\xi \tag{16}$$

It is an operator which maps $l_2^{\mathcal{N}^{(+k)}} \to l_2^{\mathcal{M}^{(+k)}}$. Let R denote the left restricted shift on $\mathcal{L}_2Z^{-1}$: $R = \Pi_{\mathcal{L}_2Z^{-1}}(Z\cdot)$, then we have the

Lemma 4.1
The products (16) satisfy the following restricted shift relations:
(1) $R[\pi_1(V, \xi)] = V\pi_1(V, \xi)$
(2) if $k > 1$ $R[\pi_k(V, \xi)] = \pi_{k-1}(V, \xi) + V^{(k-1)}\pi_k(V, \xi)$.

Proof By the expansion $(I - VZ^*)^{-1} = I + VZ^*(I - VZ^*)^{-1}$!

□

Let's drop the (V, ξ) qualification in $\pi_k(V, \xi)$. From the lemma it follows that the space

$$H_c(U) = \left\{ \sum_{i=1}^{k} D_i \pi_k \, : \, D_i \in \mathcal{D}_2 \right\}$$

is a left D,R-invariant subspace of $\mathcal{L}_2 Z^{-1}$.

Now, let

$$F = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_k \end{bmatrix}$$

and let's search for diagonal matrices $[A, B]$ such that

$$F = (I - Z^* A^*)^{-1} Z^* B^*.$$

(from our realization theory we know that they must exist since F generates a sliced space). A^* and B^* are defined by the recursive relation

$$ZF = B^* + A^* F$$

and can be determined by puting

$$B^* = [ZF]_{[0]}, A^* F = ZF - B^* = RF$$

For the case at hand we find, using the lemma:

$$B^* = \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix}, A^* F = \begin{bmatrix} V\pi_1 \\ \pi_1 + V^{(1)}\pi_2 \\ \vdots \\ \pi_{k-1} + V^{(k-1)}\pi_k \end{bmatrix}.$$

Going back to U we see that we must put:

$$A_U^* = \begin{bmatrix} V & & 0 & \\ I & V^{(1)} & & \\ & \ddots & \ddots & \\ 0 & & I & V^{(k-1)} \end{bmatrix}, B_U^* = \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The π_i's will form a left diagonally independent set if the corresponding reachability operator is strictly positive definite, in which case consistent interpolation data can be defined. If a contractive interpolant is desired that matches the first k coefficients in an expansion

$$\xi S = \eta_0 + (Z - V)\eta_1 + \ldots + [(Z - V)(Z - V)^{(1)} \ldots (Z - V)^{(k-1)}]\eta_{k-1} + ..$$

then a Θ has to be found for which $[A_\Theta, B_\Theta]$ defined as

$$A_\Theta^* = \begin{bmatrix} V & & 0 & \\ I & V^{(1)} & & \\ & \ddots & \ddots & \\ 0 & & I & V^{(k-1)} \end{bmatrix}, \quad B_\Theta^* = \begin{bmatrix} \xi & \eta_0 \\ 0 & \eta_1 \\ \vdots & \vdots \\ 0 & \eta_{k-1} \end{bmatrix}$$

form a reachability pair.

The necessary and sufficient condition for that turns out to be again the strict J-positive definiteness of the related Gramian.

This is not the only possible variant of the Hermite-Fejer problem in the algebraic setting. For an alternative, see [10] and some other contributions in this workshop. In all cases, however, the basic construction is the generation of a reachability space $H_c(U)$ and a reachability space $H_c(\Theta)$ which contains, for any upper interpolant S,

$$[x,\ R(xS)],\ x \in H_c(U). \tag{17}$$

The interpolation data uniquely specifies $R(xS)$ when $x \in H_c(U)$ so that (17) is actually independent of the choice of S.

The Hermite-Fejer problem specialises to the Schur case when $V = 0$. Various more general cases may be obtained by combining the previous. In the classical mathematical litterature, use is made of reproducing kernel Hilbert and Krein spaces. Our approach does not differ much from it: the reachability space turns out to be an RKHS for the indefinite metric, and Θ is closely related to the kernel.

5. The time-varying model order reduction problem

In this section we apply the chain-scattering or LIS machinery to solve the model order reduction problem in the algebraic or time-varying context.

Suppose that we are given a (high order) model $T \in \mathcal{U}$ and a hermitian diagonal operator Γ with Γ^{-1} bounded, to measure precision (Γ could be taken constant ε, but the theory works just as well with a more general, time-varying precision). Furthermore, assume that we dispose of a state-space description $[A, B, C, 0]$ for T in output normal form: $T = B(I - ZA)^{-1}ZC$ with $AA^* + BB^* = I$. An easy way to obtain such a description is by breaking off a diagonal expansion, much as what one would do breaking off a Maclaurin expansion at a far out point. If

$$T = T_1 Z + T_2 Z^{[2]} + .. + T_k Z^{[k]}$$

we can choose

$$A = \begin{bmatrix} 0 & & & 0 \\ I & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{bmatrix}, \quad C = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} T_1 & \dots & T_k \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

The realization is not necessarily minimal, but it is in output normal form as required.

Our goal will be to find an operator T' of possibly mixed type ($T' \in \mathcal{X}$) such that

$$\|\Gamma^{-1}(T - T')\| < 1 \tag{18}$$

i.e. T' approximates T within Γ, but it is not necessarily causal. If T_a is taken as the strictly upper part of T, then it turns out that T_a approximates T in so-called Hankel norm. Let, for $F \in \mathcal{U}$, the Hankel operator be defined as:

$$H_F : \mathcal{L}_2 Z^{-1} \to \mathcal{U}_2 : X \mapsto \Pi_{\mathcal{U}_2}(XF)|_{\mathcal{L}_2 Z^{-1}}$$

we call its norm the Hankel norm of T:

$$\|T\|_H = \|H_T\|.$$

It is easy to see that

$$\|H_{\Gamma^{-1}(T-T')}\| \le \|\Gamma^{-1}(T - T')\|$$

so that (18) insures that $\|H_{\Gamma^{-1}(T-T_a)}\| \le 1$. The Hankel norm is a pretty strong norm on U. In the case of finite matrices, one shows [6] that it dominates the l_2-norm on rows and columns.

The approximation procedure first converts the problem to an interpolation problem using an external factorization for T. Let

$$T = \Delta_r^* U_r$$

with U_r inner and $\Delta_r \in \mathcal{U}$ - see section 2, and assume that we have found a block-upper, J-unitary chain scattering matrix Θ such that

$$[U_r^* \;\; T^*\Gamma^{-1}]\Theta = [A' \;\; B'] \in \text{upper} \tag{19}$$

Let $S_{in} = -\Theta_{12}\Theta_{22}^{-1}$. We find for the second block column:

$$-U_r^* S_{in} + T^*\Gamma^{-1} = B'\Theta_{22}^{-1}$$

Define $T' = \Gamma\Theta_{22}^{-*}B'^*$, then

$$\Gamma^{-1}(T - T') = S_{in}^* U_r$$

and we find that $\|\Gamma^{-1}(T - T')\| \le 1$, since $\|S_{in}^*\| < 1$ (as an input scattering function of an energy conserving system) and $\|U_r\| = 1$ (as an inner function). This solves the problem provided T_a, the upper part of T' is of minimal degree. It is possible to show [5] that if Θ is choosen minimally, then T_a will be of the lowest possible degree and that all the approximants can be generated in this way using contractive loads.

What has all this to do with interpolation? Let's look at (19). Rewriting:

$$U_r^*[I \;\; \Delta_r\Gamma^{-1}]\Theta = [A' \;\; B']$$

we see that S_{in} interpolates $\Delta_r\Gamma^{-1} \bmod U_r$, much as was the case in the previous section. Here, however, the approximant itself is not the interpolant, but a quantity that is directly related to the approximation error.

The expression

$$T_a = \text{strictly upper part of } \Gamma\Theta_{22}^{-*}B'^*$$

is determined by three quantities: (trivially) Γ, the upper part of Θ_{22}^{-*} and B'^*. B' is by construction upper, hence B'^* will be lower. Γ and B'^* have a pondering effect on the strictly upper part of Θ_{22}^{-*}. Only the latter determines the state dimension of T_a. In section 3 we saw that this dimension is actually that of the part $\mathcal{B}_-$ in the state splitting $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$.

We conclude this section by making the state space construction of Θ explicit. We start out from the given realization $[A, B, C, 0]$ for T. It is assumed to be in output normal form. U_r has the state realization $[A, B_u, C, D_u]$ in which B_u and D_u are determined so as to make

$$\begin{bmatrix} A & C \\ B_u & D_u \end{bmatrix}$$

unitary. (19) indicates that $[U_r^* \;\; T^*\Gamma^{-1}]$ should lay in the reachability space of Θ. Since we know a state space realization for $\begin{bmatrix} U_r \\ \Gamma^{-1}T \end{bmatrix}$ namely

$$\begin{bmatrix} A & C \\ \begin{bmatrix} B_u \\ \Gamma^{-1}B \end{bmatrix} & \begin{bmatrix} D_u \\ 0 \end{bmatrix} \end{bmatrix}$$

we see that $(I - Z^*A^*)^{-1}Z^*[B_u^* \;\; B^*\Gamma^{-1}]$ should lay in the reachability space of Θ. A minimal Θ will be obtained if we let the latter space be generated by it. In view of the discussion of section 3 this will be possible if we can find a state space transformation R and a space splitting $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ such that

$$A^*R^*J_{\mathcal{B}}RA + B_u^*B_u - B^*\Gamma^{-2}B = (R^*J_{\mathcal{B}}R)^{(-1)}$$

or, with $M = R^*J_{\mathcal{B}}R$

$$A^*MA + B_u^*B_u - B^*\Gamma^{-2}B = M^{(-1)}$$

Let $\Lambda = I - M$ and remark that $A^*A + B_u^* B_u = I$ by construction, and we find

$$A^* \Lambda A + B^* \Gamma^{-2} B = \Lambda^{(-1)} \tag{20}$$

(20) contains only original data. It can be solved for Λ since we have assumed $l_A < 1$, see the solution to the Lyapunov-Krein equation presented earlier. Λ is the reachability Gramian of the given realization for $\Gamma^{-1}T$, and its inertia is closely related to a *dichotomy* on the Hankel operator of $\Gamma^{-1}T$: the sequence of positive signs corresponds to the number of singular values larger than 1, while the sequence of negative signs corresponds to the number of singular values smaller than 1. The state space splitting results from making the inertia of $M = I - \Lambda$ explicit: $M = R^* J_{\mathcal{B}} R$. Since M is a diagonal operator this is just a local operation. Our problem will have a solution if M is an invertible operator, and the solution will be obtained by completing the state space construction for Θ as explained in section 3.

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References

[1] V.M. Adamjan, D.Z. Arov and M.G. Krein, "Analytic Properties of Schmidt Pairs for a Hankel Operator and the Generalized Schur-Takagi Problem," *Mat. USSR Sbornik*, **15**(1):31-73, 1971 (trans. of *Iz. Akad. Nauk Armjan. SSR Ser. Mat.* 6 (1971)).

[2] H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, **71**, Conference Board of the Mathematical Sciences, American Mathematical Foundation, Providence, Rhode Island, 1989.

[3] L. de Branges, "Some Hilbert Spaces of analytic functions I,", *Transl. Amer. Math. Soc.* **106**(1963), 445-468.

[4] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt Rinehart and Winston, New York, 1966.

[5] P. Dewilde and A.-J. van der Veen, "On the Hankel-norm Approximation of Upper-Triangular Operators and Matrices," /it Integral Equations and Operator Theory, subm. 1992.

[6] A.-J. van der Veen and P. Dewilde, "On Low-Complexity Approximation of Matrices," /it subm. to Linear Algebra and its Applications, 1992.

[7] D. Alpay, P. Dewilde and H. Dym, "Lossless Inverse Scattering and Reproducing Kernels for Upper Triangular Operators," **47** *Operator Theory and its Applications*, pp. 61-135, Birkhäuser Verlag, Basel 1990.

[8] D. Alpay and P. Dewilde, "Time-varying Signal Approximation and Estimation", in *Signal Processing, Scattering and Operator Theory, and Numerical Methods*, vol. III, pp. 1-22, Birkäuser Verlag, Basel, 1990.

[9] .H. Sayed, T. Constantinescu and T. Kailath, "Lattice Structures for Time-variant Interpolation Problems," in *Proc. 31-st IEEE Conf. on Decision and Control*, Tucson (AZ), Dec. 1992.

[10] P. Dewilde and H. Dym, "Interpolation for Upper Triangular Operators," in *Operator Theory: Advances and Applications*, vol. OT56, pp. 153-260, Birkhäuser Verlag, Basel, 1992.

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