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Orthogonal rational functions on the unit circle and the real line

ABSTRACT

We give the recurrence relations, interpolating properties and convergence results for a sequence of orthogonal rational functions of increasing order that have fixed poles in some region **O** of the complex plane. They are orthogonalized with respect to a measure supported on the boundary **S** of **O**. We give formulations of the results which are valid for **S** equal to the unit circle or the real line. The corresponding region **O** is then the exterior of the unit disk or the lower half plane.

1 Introduction

In [9], a three term recurrence relation and in [2] a Szegő type of recurrence was obtained for the functions orthogonal with respect to a linear functional and where the poles were cyclically repeated. Also, the converse, a Favard type theorem was formulated. In [3, 1] we derived recurrence relations and interpolating properties for rational functions orthogonal with respect to a measure supported on the complex unit circle **T** and in [4] a Favard theorem was obtained without the repetition of the poles. In this paper we shall reformulate the results of [3] so that they are valid for both measures on the unit circle and on the real line. We follow an approach similar to [8]. Most of the proofs for the unit circle need only minor adaptations, so that a careful formulation of the results is sufficient to adapt the proofs of the circle case by yourself. Due to page limitations we do not include any proof. Some of them can be found in [5].

We use the notational conventions given in the table below. it will help you to specify the general formulations to the circle or real line case. We quickly skim through this table.

It can be easily checked that **S** can be described by the equation $\varpi_z(z) = 0$.

If we set for some $\alpha_i \in \mathbf{C}$ $\varpi_i(z) = \varpi_{\alpha_i}(z)$, then we have for the exceptional point α_0

$$\varpi_0(\alpha_0) = \begin{cases} 1 & \text{for } \mathbf{T} \\ 2\mathbf{i} & \text{for } \mathbf{R} \end{cases}$$

The sets $\mathbf{I}$ and $\mathbf{O}$ (inside and outside) are defined by

$$\mathbf{I} = \left\{ z \in \mathbf{C} : \frac{\varpi_z(z)}{\varpi_0(\alpha_0)} > 0 \right\} \quad \text{and} \quad \mathbf{O} = \left\{ z \in \mathbf{C} : \frac{\varpi_z(z)}{\varpi_0(\alpha_0)} < 0 \right\}.$$

$\mathbf{S}$	$\mathbf{T}$	$\mathbf{R}$	
$\hat{z}$	$1/\overline{z}$	$\overline{z}$	reflection in $\mathbf{S}$
$\varpi_\alpha(z)$	$1-\overline{\alpha}z$	$z-\overline{\alpha}$	
α_0	0	$\mathbf{i}$	exceptional point
$\mathbf{I}$	$\mathbf{D}$	$\mathbf{U}$	inside
$\mathbf{O}$	$\mathbf{E}$	$\mathbf{L}$	outside
$d\lambda(t)$	$\frac{dt}{2\pi \mathbf{i} t}$	$\frac{dt}{\pi}$	Lebesgue measure
z_k	$-\frac{\overline{\alpha}_k}{\lvert\alpha_k\rvert}$	$\frac{\lvert 1+\alpha_k^2\rvert}{1+\alpha_k^2}$	convergence factor
$\zeta_k(z)$	$z_k\frac{z-\alpha_k}{1-\overline{\alpha}_k z}$	$z_k\frac{z-\alpha_k}{z-\overline{\alpha}_k}$	Blaschke factor
$D(t,z)$	$\frac{t+z}{t-z}$	$\frac{1}{\mathbf{i}}\frac{1+tz}{t-z}$	Carathéodory kernel
$P(t,z)$	$\frac{1-\lvert z\rvert^2}{(t-z)(\frac{1}{t}-\overline{z})}$	$\frac{\Im z}{(t-z)(t-\overline{z})}$	Poisson kernel

So

$$\mathbf{I} = \begin{cases} \mathbf{D} & \text{for } \mathbf{T} \\ \mathbf{U} & \text{for } \mathbf{R} \end{cases} \quad \text{and} \quad \mathbf{O} = \begin{cases} \mathbf{E} & \text{for } \mathbf{T} \\ \mathbf{L} & \text{for } \mathbf{R} \end{cases}$$

The notation $d\lambda$ refers to the normalized Lebesgue measure on $\mathbf{S}$. We shall also use the notation

$$d\tilde{\lambda}(t) = \frac{d\lambda(t)}{|\varpi_0(t)|^2} = \begin{cases} d\lambda(t) & \text{for } \mathbf{T} \\ \frac{d\lambda(t)}{1+t^2} & \text{for } \mathbf{R} \end{cases}$$

More generally, for any measure μ on $\mathbf{S}$, (not necessarily finite) we set $d\tilde{\mu}(t) = |\varpi_0(t)|^{-2} d\mu(t)$. By $\tilde{H}_p := \tilde{H}_p(\mathbf{I})$ we denote the classical Hardy spaces $H_p(\mathbf{D})$ for the unit disk when $\mathbf{I} = \mathbf{D}$. For $\mathbf{I} = \mathbf{U}$, these spaces consist of the functions f for which $f \circ \tau \in H_p(\mathbf{D})$ where τ is the Cayley transform of the upper half plane $\mathbf{U}$ to the unit disk $\mathbf{D}$: $\tau(z) = (\mathbf{i}-z)/(\mathbf{i}+z)$. Note that these are not the Hardy spaces for $\mathbf{U}$ (except for $p = \infty$). The latter can be defined for $0 < p < \infty$ as (see e.g., [7, 10])

$$H_p(\mathbf{U}) = \{(z+\mathbf{i})^{-2/p} f(z) : f \in \tilde{H}_p(\mathbf{U})\}.$$

We consider the inner product in the space $L_2(\tilde{\mu})$ $\langle f, g\rangle_{\tilde{\mu}} = \int f(z)\overline{g(z)}d\tilde{\mu}(z)$ where $\tilde{\mu}$ is a finite positive measure on $\mathbf{S}$. Without loss of generality, we shall usually assume that $\int d\tilde{\mu} = 1$.

We also define the substar conjugate of a function to mean $f_*(z) = \overline{f(\hat{z})}$. It is clear that for $z \in \mathbf{S} : f_*(z) = \overline{f(z)}$, so that we have immediately $\langle f, g\rangle_{\tilde{\mu}} = \langle g_*, f_*\rangle_{\tilde{\mu}} = \int f g_* d\tilde{\mu}$. The (generalized) Poisson kernel is defined as

$$P(t, z) = \frac{\varpi_z(z)/\varpi_0(\alpha_0)}{(t-z)(t-z)_*} \quad ; \quad \text{(substar w.r.t. } t\text{)}.$$

For $t \in \mathbf{S}$, this reduces to the classical Poisson kernels. Furthermore, we introduce the kernels

$$D(t, z) = \begin{cases} \frac{t+z}{t-z} & \text{for } \mathbf{T} \\ \frac{1}{\mathbf{i}}\frac{1+tz}{t-z} & \text{for } \mathbf{R}. \end{cases}$$

One can check that $\varpi_0(t)\varpi_{0*}(t)P(t, z) = \frac{1}{2}[D(t, z) + D(t, z)_*]$ (substar w.r.t. t).
Let $\tilde{\mu}$ be a probability measure on $\mathbf{S}$ ($\int d\tilde{\mu} = 1$), then

$$\Omega(z) = \mathbf{i}c + \int D(t, z)d\tilde{\mu}(t), \quad z \in \mathbf{I}, c \in \mathbf{R}$$

defines a function analytic in $\mathbf{I}$ ($\Omega \in H(\mathbf{I})$) and with a positive real part since the real part of Ω is given by $\Re\Omega(z) = \int \Re D(t, z)d\tilde{\mu}(t) = \int P(t, z)d\mu(t)$, $\mu = |\varpi_0|^2\tilde{\mu}$ (note that μ need not be finite for $\mathbf{S} = \mathbf{R}$). We say that Ω belongs to $\mathcal{P}$, the class of positive real functions for $\mathbf{I}$. Since $D(t, \alpha_0) = 1$, we find that $\Omega(\alpha_0) = 1 + \mathbf{i}c$. If we require $\Omega(\alpha_0) > 0$, then we should take $c = 0$. In that case $\Omega(\alpha_0) = 1$. The relation between μ with $\int |\varpi_0(t)|^{-2}d\mu(t) = 1$ and Ω with $\Omega(\alpha_0) = 1$ is one to one. We shall sometimes indicate this by writing Ω_μ for Ω. By Fatou's theorem, we know that if μ is finite on $\mathbf{S}$, and has the Lebesgue decomposition $d\mu(t) = \omega(t)d\lambda(t) + d\mu_s(t)$ into an absolutely continuous and a singular part, then $\Re\Omega_\mu(z)$ has a nontangential limit to $\mathbf{S}$ and it is equal to $\omega(t)$.

2 The spaces $\mathcal{L}_n$

We define here the spaces of rational functions of degree n for which we shall construct an orthogonal basis later on.

We define a Blaschke factor for $\alpha_i \in \mathbf{I}$ as $\zeta_i(z) = z_i\varpi_i^*(z)/\varpi_i(z)$. The denominator is

$$\varpi_i(z) = \varpi_{\alpha_i}(z) = \begin{cases} 1 - \bar{\alpha}_i z & \text{for } \mathbf{T} \\ z - \bar{\alpha}_i & \text{for } \mathbf{R} \end{cases}$$

The numerator introduces the superstar for a polynomial. In general, the superstar of a polynomial $p_n \in \Pi_n$ of degree n is a transformation of a polynomial into another polynomial:

$$p_n^*(z) = \begin{cases} z^n p_{n*}(z) & \text{for } \mathbf{T} \\ p_{n*}(z) & \text{for } \mathbf{R}. \end{cases}$$

Thus here $\varpi_i^*(z) = z - \alpha_i$ in both cases. The constant factors $z_i \in \mathbf{T}$ are equal to 1 if $\alpha_i = \alpha_0$ and otherwise they are given by

$$z_i = \begin{cases} -\bar{\alpha}_i/|\alpha_i| & \text{for } \mathbf{T} \\ |\alpha_i^2 + 1|/(\alpha_i^2 + 1) & \text{for } \mathbf{R}. \end{cases}$$

These factors are needed to make $\prod_{i=1}^{\infty} \zeta_i(z)$ converge iff the Blaschke condition is satisfied. That is iff

$$\begin{cases} \sum_{k\geq 1}(1 - |\alpha_k|) < \infty & \text{for } \mathbf{T} \\ \sum_{k\geq 1} \frac{\Im \alpha_k}{1+|\alpha_k|^2} < \infty & \text{for } \mathbf{R}. \end{cases}$$

With the Blaschke factors, we define the finite Blaschke products $B_0 = 1, \quad B_n = B_{n-1}\zeta_n, \quad n \geq 1$. The spaces $\mathcal{L}_n$ are now defined by

$$\begin{aligned} \mathcal{L}_n &= \operatorname{span}\{B_k : k = 0, 1, \ldots, n\} \\ &= \left\{ \frac{p_n}{\pi_n}, p_n \in \Pi_n, \pi_n(z) = \prod_{j=1}^{n} \varpi_i(z) \right\}. \end{aligned}$$

For functions in $\mathcal{L}_n$, we also define a superstar conjugate namely $f_n^* = B_n f_*$. One can easily check that for $f_n \in \mathcal{L}_n$

$$f_n = \frac{p_n}{\pi_n} \Rightarrow f_n^* = \eta_n \frac{p_n^*}{\pi_n}, \quad \eta_n = \prod_{j=1}^{n} z_j \in \mathbf{T}.$$

Also, because $|B_n| = 1$ on $\mathbf{S}$ and $B_{n*} = 1/B_n$, we have $\langle f, g\rangle_{\tilde{\mu}} = \langle f_*, g_*\rangle_{\tilde{\mu}} = \langle f^*, g^*\rangle_{\tilde{\mu}}, \quad f, g \in \mathcal{L}_n$. If $f_n \in \mathcal{L}_n$ equals $f_n = a_0 + a_1 B_1 + \ldots + a_k B_k + \ldots + a_n B_n$, then $f_n^* = \bar{a}_0 B_n + \bar{a}_1 B_{n\setminus 1} + \ldots + \bar{a}_k B_{n\setminus k} + \ldots + \bar{a}_n$ where $B_{n\setminus k} = B_n/B_k$ and therefore $a_n = \overline{f_n^*(\alpha_n)}$. We call a_n the leading coefficient of f_n (w.r.t. the basis B_k).

Using the Gram-Schmidt procedure, we can orthogonalize the basis $B_0, \ldots, B_n$ in $L_2(\tilde{\mu})$ to find the orthonormal basis functions $\phi_0, \ldots, \phi_n$. We fix them uniquely by choosing their leading coefficient positive: $\kappa_n = \phi_n^*(\alpha_n) > 0$. Throughout this paper we shall assume that the measure guarantees the existence of all orthogonal functions $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}, n = 0, 1, 2, \ldots$.

Note that if $\phi_n \in \mathcal{L}_n$ is orthogonal to $\mathcal{L}_{n-1}$, then ϕ_n^* will be orthogonal to $\zeta_n \mathcal{L}_n$, that is $\langle \phi_n^*, f\rangle_{\tilde{\mu}} = 0$ for all $f \in \mathcal{L}_n$ which vanish at $z = \alpha_n$.

The kernels

$$k_n(z, w) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)} \tag{2.1}$$

are reproducing in $\mathcal{L}_n$ in the sense that $\langle f_n(z), k_n(z, w)\rangle_{\tilde{\mu}} = f_n(w), \quad \forall f_n \in \mathcal{L}_n, w \in \mathbf{I}$. For these kernels, the following properties hold:

Theorem 2.1 *If $k_n(z, w)$ is the reproducing kernel for $\mathcal{L}_n$ and $\phi_k(z)$, $k = 0, \ldots, n$ represent the orthonormal basis functions, then*

1. $k_n(z, w) = \overline{k_n(w, z)}$ *(sesqui-analytic)*

2. $k_n^*(z, w) = k_n^*(w, z)$ *(superstar w.r.t. first argument)*

3. $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ $(\kappa_n = \overline{\phi_n^*(\alpha_n)} > 0)$

4. $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof. (1) follows from the formula (2.1). For (2) we refer to [5]. (3) follows from (2) and (4) follows from (3), by setting $w = \alpha_n$. □

Corollary 2.2 *Setting* $\mu_{ij} = \langle B_i, B_j \rangle_{\tilde{\mu}}$, *we get the following determinant expressions:* $\kappa_n^2 = \det G_{n-1}/\det G_n$ *and*

$$\phi_n(z) = (\det G_{n-1} \det G_n)^{-1/2} \det \begin{bmatrix} \mu_{00} & \cdots & \mu_{0n} \\ \vdots & & \vdots \\ \mu_{n-1,0} & \cdots & \mu_{n-1,n} \\ B_0(z) & \cdots & B_n(z) \end{bmatrix}.$$

Proof. This is a classical result. The proof is the same as in [3]. □

We have the Christoffel-Darboux type of relations.

Theorem 2.3 *Let* $k_n(z, w)$ *be the reproducing kernel for* α_n *and* ϕ_k, $k = 0, 1, \ldots, n$ *the orthonormal basis functions, then*

$$\begin{aligned} k_n(z, w) &= \frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1 - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}} \\ &= \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}} \end{aligned}$$

Proof. The result is as in the case $\mathbf{S} = \mathbf{T}$. The proof of [3] needs only minor adaptations. □

An immediate consequence of these relations is that (note $|\zeta_{n+1}| < 1$ in $\mathbf{I}$)

$$0 < k_n(z, z) = \sum_{k=0}^{n} |\phi_k(z)|^2 \Rightarrow 0 < |\phi_{n+1}^*(z)|^2 - |\phi_{n+1}(z)|^2, \quad z \in \mathbf{I}$$

Thus $\phi_{n+1}^*(z)$ is not zero in $\mathbf{I}$. Consequently we also have that

$$\frac{\phi_{n+1}(z)}{\phi_{n+1}^*(z)} \in \mathbf{D}\,(\mathbf{T}, \mathbf{E}) \quad \text{when} \quad z \in \mathbf{I}(\mathbf{S}, \mathbf{O})$$

because the same holds for $\zeta_{n+1}(z)$.

3 Recurrence for the orthonormal basis functions

The basis functions satisfy the following recurrence relation.

Theorem 3.1 *Let $\phi_k, k = 0, \ldots, n$ be the orthonormal basis functions with positive leading coefficient. Then*

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = N_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{bmatrix} 1 & \bar{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix}.$$

The matrix N_n has the form $N_n = e_n \operatorname{diag}(\eta_n^1, \eta_n^2)$, $\eta_n^1, \eta_n^2 \in \mathbf{T}$ where η_n^1 is chosen to make $\kappa_n = \phi_n^*(\alpha_n) > 0$ and η_n^2 is related to η_n^1 by $\eta_n^2 = \bar{\eta}_n^1 \bar{z}_{n-1} z_n$. The parameter λ_n is given by

$$\lambda_n = \eta \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}} \in \mathbf{D} \quad \text{with} \quad \eta = \frac{\varpi_{n-1}(\alpha_n)}{\varpi_n(\alpha_{n-1})} z_n \bar{z}_{n-1} \in \mathbf{T}.$$

Proof. This proof is much like in the case of the unit circle, see [3]. For a uniform approach to the circle and the line, see [5]. □

The orthogonal functions in the previous recurrence were computed using the parameters λ_n. However, these λ_n were defined in terms of the ϕ_n themselves. Therefore we give now another expression for these, as well as for the e_n.

Theorem 3.2 *The parameter λ_n in the previous recurrence relation is also given by*

$$\lambda_n = \bar{z}_{n-1} \frac{\langle \phi_k, \frac{z-\alpha_{n-1}}{\varpi_n(z)} \phi_{n-1} \rangle_{\tilde{\mu}}}{\langle \phi_k, \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \phi_{n-1}^* \rangle_{\tilde{\mu}}} \quad \text{for any } k, 0 \le k \le n-1$$

and the values of $e_n > 0$ are given by

$$e_n^2 = \frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1 - |\lambda_n|^2}.$$

Proof. The expression for λ_n follows easily from $\phi_n \perp \phi_k$ for any $k = 0, 1, \ldots, n-1$. So replacing ϕ_n by the recurrence in $\langle \phi_n, \phi_k \rangle_{\tilde{\mu}} = 0$ gives the result.

For the derivation of the expression for $e_n^2 = |d_n|^2$ we refer to [5]. □

As for the circle case, we can avoid the factor N_n in the recurrence by rotating the orthonormal functions. We have to define in general

$$\varepsilon_0 = 1 \quad \text{and} \quad \varepsilon_n = \varepsilon_{n-1} z_n \frac{\bar{d}_n}{|d_n|} = \varepsilon_n \eta_n^2 \quad \text{for} \quad n \ge 1 \tag{3.2}$$

and then we get for $\Phi_n = \varepsilon_n \phi_n$ the following recurrence relation

$$\begin{bmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} = e_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{bmatrix} 1 & \bar{\Lambda}_n \\ \Lambda_n & 1 \end{bmatrix} \begin{bmatrix} Z_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(z) \\ \Phi_{n-1}^*(z) \end{bmatrix}$$

with

$$\Lambda_n = \bar{\varepsilon}_{n-1}^2 \bar{z}_n z_{n-1} \lambda_n = \bar{\varepsilon}_{n-1}^2 \frac{\varpi_{n-1}(\alpha_n)}{\varpi_n(\alpha_{n-1})} \overline{\frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})}} \quad \text{and} \quad Z_n = z_n \bar{z}_{n-1} \zeta_{n-1}.$$

4 Functions of the second kind and interpolation

In this section we define the functions of the second kind, prove their recurrence relation and derive some interpolating properties.

Let the kernel $D(t, z)$ be as defined in section 1 and introduce $E(t, z) = D(t, z) + 1$. Note that when $t \in \mathbf{S}$, $D(z, t) = -D(t, z)$ and thus also $E(z, t) = 1 - D(t, z)$. We now give some equivalent definitions of the functions of the second kind:

$$\begin{aligned}\psi_n(z) &= \int [E(t, z)\phi_n(t) - D(t, z)\phi_n(z)]d\tilde{\mu}(t) \\ &= \int D(t, z)[\phi_n(t) - \phi_n(z)]d\tilde{\mu}(t) + \int \phi_n(t)d\tilde{\mu}(t) \\ &= \begin{cases} 1 & \text{if } n = 0 \\ \int D(t, z)[\phi_n(t) - \phi_n(z)]d\tilde{\mu}(t) & \text{if } n \geq 1. \end{cases}\end{aligned}$$

The following properties were proved in [3] for the unit circle, but the proofs are independent of the choice of $\mathbf{S}$, so we state without proof the following results.

Theorem 4.1 *For the functions of the second kind ψ_n, are in $\mathcal{L}_n$. Moreover*

1. *we have for $0 \leq k < n$*

$$\begin{aligned}\frac{\psi_n(z)}{B_k(z)} &= \int D(t, z)[\frac{\phi_n(t)}{B_k(t)} - \frac{\phi_n(z)}{B_k(z)}]d\tilde{\mu}(t) \\ &= \int [E(t, z)\frac{\phi_n(t)}{B_k(t)} - D(t, z)\frac{\phi_n(z)}{B_k(z)}]d\tilde{\mu}(t).\end{aligned}$$

   This also holds for $n = 0$ if you set then $B_k = 1$.

2. *Denote $B_{n\backslash k} = B_n/B_k$ $(k \leq n)$, then*

$$\begin{aligned}\frac{\psi_n}{B_{n\backslash k}(z)} &= \int D(z, t)\left[\frac{\phi_n^*(t)}{B_{n\backslash k}(t)} - \frac{\phi_n^*(z)}{B_{n\backslash k}(z)}\right]d\tilde{\mu}(t) \\ &= \int \left[E(z, t)\frac{\phi_n^*(t)}{B_{n\backslash k}(t)} - D(z, t)\frac{\phi_n^*(z)}{B_{n\backslash k}(z)}\right]d\tilde{\mu}(t)\end{aligned}$$

   for $0 \leq k < n$. The second relation also holds for $n = 0$ if you set $B_{n\backslash k} = 1$.

3. *Define $\tilde{B}_{-1} = 1$, $\tilde{B}_0 = \zeta_0$, $\tilde{B}_k = \zeta_0 B_k$, $k \geq 1$. If $\Omega = \int D(t, z)d\tilde{\mu}$, then*

$$\begin{aligned}\frac{\phi_n\Omega + \psi_n}{\tilde{B}_{n-1}} &= g \in H(\mathbf{I}),\ n \geq 0 \\ \frac{\phi_n^*\Omega - \psi_n^*}{\tilde{B}_n} &= h \in H(\mathbf{I}),\ n \geq 0\end{aligned}$$

Also the recurrence relation for these functions can be found to be

$$\begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix} = N_n\frac{\varpi_{n-1}(z)}{\varpi_n(z)}\begin{bmatrix} 1 & \bar{\lambda}_n \\ \lambda_n & 1 \end{bmatrix}\begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \psi_{n-1}(z) \\ -\psi_{n-1}^*(z) \end{bmatrix}$$

where all the quantities are defined before. Its proof is an easy adaptation of the corresponding proof in [3]. The adaptations needed are similar to the ones used in the proof of the recurrence for the orthogonal functions.

We conclude this section with the following:

Theorem 4.2 *Let ϕ_k be the orthonormal basis functions for $\mathcal{L}_n$ with respect to $\tilde{\mu}$. Define the absolutely continuous measure μ_n by*

$$d\mu_n(t) = \frac{|\varpi_0(t)|^2 P(t, \alpha_n)}{|\phi_n(t)|^2} d\tilde{\lambda}(t) = \frac{P(t, \alpha_n)}{|\phi_n(t)|^2} d\lambda(t)$$

where P is the Poisson kernel. Then on $\mathcal{L}_n$, the inner product with respect to μ_n and $\tilde{\mu}$ is the same: $\langle \cdot, \cdot \rangle_{\mu_n} = \langle \cdot, \cdot \rangle_{\tilde{\mu}}$.

Proof. One first proves that $\|\phi_n\|^2_{\mu_n} = \|\phi_n\|^2_{\tilde{\mu}} = 1$, which is obvious. Next, show that $\langle \phi_n, \phi_k \rangle_{\mu_n} = \langle \phi_n, \phi_k \rangle_{\tilde{\mu}} = 0$ for $0 \leq k < n$. Thus ϕ_n is an orthonormal function both w.r.t. $\tilde{\mu}$ and w.r.t. μ_n. Since λ_n is completely defined by ϕ_n, and thus also ϕ_{n-1} by inverting the recurrence formula etc. (recall that all ϕ_k had a positive leading coefficient). Thus because ϕ_n is orthonormal w.r.t. μ_n, also all the previous ϕ_k, $0 \leq k < n$ will be orthonormal w.r.t. μ_n. Hence $\langle \cdot, \cdot \rangle_{\mu_n} = \langle \cdot, \cdot \rangle_{\tilde{\mu}}$ in $\mathcal{L}_n$. □

5 *J*-inner matrices and determinant formula

To derive the determinant formula, we use the notation of a J-inner matrix [6, 8].

Let θ be a 2×2 matrix whose entries are functions in the Nevanlinna class N for **I** [10]. We consider the indefinite metric $J = \mathrm{diag}(1, -1)$. We shall say that θ is J-unitary for **S** iff $\theta^H J \theta = J$ on **S**. The substar conjugate of a matrix is the elementary substar conjugate of the transposed matrix. Thus

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}_* = \begin{bmatrix} \theta_{11*} & \theta_{21*} \\ \theta_{12*} & \theta_{22*} \end{bmatrix}.$$

Then the previous relation expressing J-unitarity on **S** can now be generalised to **C** by writing $\theta_* J \theta = J$ in **C** since these θ matrices have pseudomeromorphic extensions accross **S** [6].

A matrix of a previous form is J-contractive in **I**, iff $\theta^H J \theta \leq J \quad z \in \mathbf{I}$ where inequality means that $J - \theta^H J \theta$ is positive semi definite. We call θ J-inner in **I** iff it is both J-unitary and J-contractive in **I**. The class of J-inner matrices for **S** is denoted by $\mathcal{B}_J$.

The following properties for J-contractive matrices were given in [6] for the unit circle, but the same holds for the real line.

Theorem 5.1 *$\mathcal{B}_J$ is closed under multiplication and moreover, for $\theta \in \mathcal{B}_J$ it holds that*

1. $\theta^H \in \mathcal{B}_J$ *2. $\theta^H J \theta \geq J$ in* **O**

3. $(\theta_{11}+\theta_{12})_*^{-1}$ *and* $(\theta_{22}+\theta_{21})^{-1} \in \tilde{H}_2(\mathbf{I})$

4. $(\theta_{11}+\theta_{12})_*^{-1}(\theta_{11}-\theta_{12})_*$ *and* $(\theta_{22}+\theta_{21})^{-1}(\theta_{22}-\theta_{21}) \in \mathcal{P}$

J-inner matrices appear in the recurrence relation as shown in the following:

Lemma 5.2 *Define*

$$t_n = N_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{bmatrix} 1 & \bar{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$

with all the parameters defined in Theorem 3..1. Set $T_n = t_n t_{n-1} \ldots t_1$. *Then*

$$T_n = \frac{1}{2} \begin{bmatrix} \phi_n + \psi_n & \phi_n - \psi_n \\ \phi_n^* - \psi_n^* & \phi_n^* + \psi_n^* \end{bmatrix}.$$

Moreover, there exists a positive constant c_n *such that*

$$\Theta_n(z) = \frac{\varpi_n(z)}{c_n \varpi_0(z)} T_n(z)$$

is J-inner.

Proof. See [5]. □

This lemma now gives the following result.

Theorem 5.3 *With the notations of the previous lemma, we have*

1. *The determinant formula* $\frac{1}{2}[\psi_n \phi_{n*} + \psi_{n*}\phi_n] = \varpi_0(z)\varpi_{0*}(z)P(z, \alpha_n)$, *thus*

$$\frac{1}{2}\left[\frac{\psi_n}{\phi_n} + \frac{\psi_n^*}{\phi_n^*}\right] = \frac{P(z,\alpha_n)}{\phi_n(z)\phi_{n*}(z)} \varpi_0(z)\varpi_{0*}(z)$$

   with P the Poisson kernel.

2. $\psi_n^*/\phi_n^* = \psi_{n*}/\phi_{n*} \in \mathcal{P}$. *The measure corresponding to this positive real function is given in*

$$\frac{\psi_n^*(z)}{\phi_n^*(z)} = \int D(t,z)d\mu_n(t) \quad \text{with} \quad d\mu_n(t) = \frac{P(t,\alpha_n)}{|\phi_n^*(t)|^2} d\lambda(t).$$

Proof. See [5]. □

6 Interpolation algorithm for the orthogonal functions

In this section we give a Pick-Nevanlinna type of algorithm to compute the parameters λ_k. It is based on successive interpolation of a function that is in the Schur class $\mathcal{B}$. This class is the class of holomorphic functions in $\mathbf{I}$, which are bounded by 1. $\mathcal{B} = \{f \in H(\mathbf{I}) : f(z) \in \mathbf{D}, \forall z \in \mathbf{I}\}$. In fact, it will be difficult to compute the λ_k directly because of the factors η_n^1 and η_n^2 in the recurrence of the orthogonal functions ϕ_k and the functions of the second kind ψ_k. To avoid these factors, we shall work with the rotated functions $\Phi_k = \varepsilon_k \phi_k$ and $\Psi_k = \varepsilon_k \psi_k$, where ε_n is defined in (3.2).

We define remainder functions R_{n1} and R_{n2} by

$$\begin{bmatrix} \tilde{B}_{n-1} R_{n1}(z) \\ \tilde{B}_n R_{n2}(z) \end{bmatrix} = \begin{bmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} \Omega(z) + \begin{bmatrix} \Psi_n(z) \\ -\Psi_n^*(z) \end{bmatrix}$$

where $\Omega = \Omega_\mu$ is the function of class $\mathcal{P}$ corresponding to the measure μ, and the $\tilde{B}_k$ as in Theorem 4..1. The functions Φ_k are orthogonal with respect to $\tilde{\mu} = |\varpi_0|^{-2}\mu$. The functions R_{n1} and R_{n2} are rotated versions of the functions g and h appearing in Theorem 4..1. When applying the recurrence relation we get after some computations, the following recurrence.

Theorem 6.1 *The remainder functions $r_{n1} = z_n R_{n1}$ and $r_{n2} = R_{n2}$ satisfy*

$$\varpi_n \begin{bmatrix} r_{n1} \\ r_{n2} \end{bmatrix} = e_n \begin{bmatrix} 1 & 0 \\ 0 & 1/\zeta_n \end{bmatrix} \begin{bmatrix} 1 & \bar{L}_n \\ L_n & 1 \end{bmatrix} \varpi_{n-1} \begin{bmatrix} r_{n-1,1} \\ r_{n-1,2} \end{bmatrix}$$

with

$$L_n = z_n \Lambda_n = -\lim_{z \to \alpha_n} \frac{r_{n-1,2}(z)}{r_{n-1,1}(z)}; \quad e_n = \left[\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|L_n|^2} \right]^{1/2}.$$

Proof. See [5]. □

The previous theorem now gives directly the Pick-Nevanlinna type algorithm for the functions

$$\Gamma_n(z) = z_n \frac{R_{n2}(z)}{R_{n1}(z)} = \frac{r_{n2}(z)}{r_{n1}(z)}.$$

Note that $\Gamma_0(z) = \frac{1}{z}[\Omega(z) - 1]/[\Omega(z) + 1] \in \mathcal{B}$ is a Schur function. We have the following relation.

Theorem 6.2 *Define the function Γ_n as above, then*

$$\Gamma_n = \frac{1}{\zeta_n} \left(\frac{L_n + \Gamma_{n-1}}{1 + L_n \Gamma_{n-1}} \right)$$

with $L_n = -\Gamma_{n-1}(\alpha_n)$. They are all in the Schur class $\mathcal{B}$.

Proof. The recurrence follows immediately from the previous theorem. All the functions are in class $\mathcal{B}$ because Γ_0 is and the Moebius transformation between the brackets generates another Schur function (since $L_n \in \mathbf{D}$) which is zero in α_n. Therefore, dividing out ζ_n gives again a Schur function. □

7 Convergence results

The proofs of the convergence results in [3] do not make use of the fact that the functions are related to **T** specifically. They can be used without change when the appropriate domains are replaced. So we can restate them without proof.

Recall that the Blaschke products B_∞ will diverge iff the Blaschke condition is not satisfied, i.e. iff

$$\begin{cases} \sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty & \text{for } \mathbf{T} \\ \sum_{k=1}^{\infty} \dfrac{\Im \alpha_k}{1+|\alpha_k|^2} = \infty & \text{for } \mathbf{R} \end{cases}$$

Theorem 7.1 *Let ϕ_n and ψ_n be the orthonormal functions and the second kind functions for $\mathcal{L}_n$ with respect to the measure $\tilde{\mu}$. Define $\Omega_n = \psi_n^*/\phi_n^* \in \mathcal{P}$ and let $\Omega \in \mathcal{P}$ be the function associated with $\mu = |\varpi_0|^2\tilde{\mu}$: $\Omega(z) = \int D(t,z)d\tilde{\mu}(t)$ with $\int d\tilde{\mu} = 1$ and $\Omega(\alpha_0) = 1$. Then Ω_n converges to Ω uniformly on compact subsets of* **I** *if the Blaschke product diverges.*

8 Conclusion

We have studied orthogonal rational functions on the unit circle and on the real line. In [6], recurrence relations for the reproducing kernels were fully exploited and its connection with the Pick-Nevanlinna interpolation problem was established, while solving the prediction problem of a stationary stochastic process. In this paper we obtained similar results not for the kernels, but for the orthogonal functions themselves.

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