Organized Vortices as Maximum Entropy Structures

Abstract

A theory of equilibrium statistical mechanics for two-dimensional perfect fluids is summarized. It predicts the organization of a turbulent flow into a steady final structure. In the limit of low energy, the final structure is explicitly obtained as an expansion in the successive moments of the probability distribution function for the initial vorticity field. The previous results of point vortex statistics and selective decay into minimum enstrophy structures appear as special limits of this theory.

A good agreement is obtained with direct numerical computations and some experimental results. Applications to isolated vortex structures (monopoles and modons) are particularly developed here.

1. Introduction

The formation of coherent structures in strongly turbulent regimes is a remarkable property of two-dimensional turbulence. Such organization is observed in large scale oceanic or atmospheric flows, and can be reproduced in laboratory experiments. A general explanation of this long time organization has been proposed by Onsager (1949), in terms of equilibrium statistical mechanics for a set of point vortices. This is a remarkable anticipation since observations were very scarce at that time. This idea has been developed then by Montgomery & Joyce (1974), who have obtained explicit predictions by a mean field approximation. The result appears as a relationship between locally averaged vorticity and stream function, which characterizes a steady solution of the Euler equations (possibly in a rotating or translating frame of reference). However, the modeling of a continuous vorticity field by a set of point vortices can lead to inconsistencies: in particular the maximum vorticity is not bounded by the initial maximum, as it should for the two-dimensional Euler equations.

This contradiction is resolved by a new theory proposed by Robert (1990), Robert & Sommeria (1991), and independently by Miller (1990). This equilibrium statistical theory is performed directly on the continuous Euler equations. Then, the standard procedure for Hamiltonian systems of particles is not available, but still the method is justified (on a weaker basis) by a set of rigorous properties (Robert 1990, Robert 1991). The result is again a steady solution of the Euler equation, on which fine scale vorticity fluctuations are superimposed. The relationship between vorticity and stream function is different than in Montgomery & Joyce (1974), and it is now quite consistent with the properties of the continuous Euler equations. Nevertheless, the relationship of Montgomery & Joyce is recovered in what we call the dilute limit, for which non-zero vorticity occupies only a small area of the domain. Another interesting limit is the case of low energy, in which the minimum enstrophy states (Leith, 1984) can be obtained.

A brief overview of the theory is given in section 2, referring to the original papers for a more thorough discussion and justification. The general properties of the resulting equilibrium structures are discussed in section 3, with emphasis on the important limit of low energy, for which the results can be linearized and expanded as a function of the energy. The application to isolated vorticity structures is emphasized in this paper: the case of monopole is studied in section 5, while the case of modons is discussed in section 6.

2. Principle of the statistical theory

The Euler equations are known to develop very complex vorticity filaments, and a deterministic description of the flow would require a rapidly increasing amount of information as time goes on. However, the conservation laws of the system bring important constraints to the evolution, and an essential property of the present theory is to take into account all the known conservation laws for the Euler equation (unlike the truncated spectral models, Kraichnan, 1975). In addition to the energy, the vorticity of each fluid particle is conserved, which results in the conservation of any function of the vorticity. It is often convenient (but not necessary) to approximate the vorticity field by patches with uniform vorticity level. Then the area of each level is conserved. Additional conservation laws for angular or linear mementum are also obtained in specific geometries.

In spite of these constraints, the vorticity field can still take many configurations, especially as the filaments become finer and finer. The idea of the statistical description is to give the same weight to all these possible configurations, called the microscopic states. Nevertheless we have to consider only the possible configurations which have the right values of the conserved quantities. Then it is remarkable that, if we introduce a coarse grain (macroscopic) description, most of the microscopic state will be close to a given macroscopic state. Thus, this optimal state is very likely to be reached if the vorticity configuration is chosen at random.

More specifically, a macroscopic state is defined as the field of probability $p_i(\mathbf{r})$ of finding the level a_i in a small neighbourhood of the point $\mathbf{r}$. The most probable macroscopic state is obtained by maximizing the entropy

$$S = \int_D s(p)\, d^2\mathbf{r}, \qquad \text{where } s(p) = -\sum_i p_i \, Log\, p_i, \tag{1}$$

with the constraints brought by the conserved quantities. These are the energy, the total area

$$F_i = \int_D p_i d^2\mathbf{r}$$

of each vorticity level i, and the linear or angular momentum if these quantities are conserved (due to specific symmetries of the fluid domain). The energy E is expressed in terms of the vorticity field $\omega(\mathbf{r})$ and the associated stream function $\psi(\mathbf{r})$ defined by

$$-\Delta\psi = \omega, \qquad \psi = \text{const. on boundaries,} \tag{2}$$

$$E\{\omega\} = \frac{1}{2}\int_D \psi\omega d^2\mathbf{r}. \tag{3}$$

The energy can be also expressed in terms of the macroscopic state by replacing $\omega(\mathbf{r})$ in (3) by the locally averaged vorticity $\bar{\omega}(\mathbf{r})$, and using the corresponding stream function Ψ, defined by

$$\bar{\omega}(\mathbf{r}) = \sum a_i p_i(\mathbf{r}), \qquad -\Delta\Psi = \bar{\omega} \tag{4}$$

(the vorticity has fine-scale fluctuations, but these fluctuations are smoothed out for the quantities obtained as integral of the vorticity, like the stream function or the energy).

The variational problem is treated by introducing the Lagrange parameters corresponding to the conserved quantities, so that the first variations satisfy

$$\delta S - \sum \alpha_i \delta F_i - \beta\delta E = 0 \tag{5}$$

As a consequence, the optimal probability $p_i(\mathbf{r})$ is related to the equilibrium stream function Ψ by the relationship (see Robert & Sommeria, 1991).

$$p_i(\mathbf{r}) = \frac{e^{-\alpha_i - \beta a_i \Psi}}{Z(\Psi)}, \qquad \text{with } Z(\Psi) = -\sum_i e^{-\alpha_i - \beta a_i \Psi}. \tag{6}$$

Then the locally averaged vorticity $\bar{\omega}(\mathbf{r})$ is expressed by (4) as a function $f_{\alpha,\beta}$ of the stream function Ψ. The resulting flow can be calculated in a self-consistent way by solving the corresponding partial differential equation. A steady Euler flow is characterized in general by the existence of a relationship between vorticity and stream function, so the function $f_{\alpha,\beta}$ selects a particular steady inviscid flow. We notice that in the absence of the energy constraint, i.e. $\beta = 0$, p_i is uniform so that the mixing is complete. In general, the energy constraint prevents complete mixing and a flow structure remains.

This structure depends on the Lagrange parameters, which are not directly given. (We do not know about the possibility of a thermal bath that could set the temperature of the system, like in usual thermodynamics). Generally the available information is rather integral quantities, like the energy and the other conserved quantities known from the initial condition (but other kinds of integral constraints can be obtained in forced cases). In summary, the stream function Ψ, together with the Lagrange parameters, are obtained by solving the system of equations resulting from (4), (6) and the integral constraints

$$-\Delta\Psi = f_{\alpha,\beta}(\Psi) = \frac{\sum\limits_{i=1} a_i e^{\alpha_i - \beta a_i \Psi}}{Z(\Psi)} \tag{7a}$$

$$\Psi = \text{const. at the boundaries} \tag{7b}$$

$$\int_D \frac{e^{-\alpha_i - \beta a_i \Psi}}{Z(\Psi)} d^2\mathbf{r} = F_i, \qquad i = 1, n-1 \tag{7c}$$

$$1/2 \int_D (-\Delta\Psi)\Psi\, d^2\mathbf{r} = E. \tag{7d}$$

The general case of a continuous distribution of vorticity levels is a straightforward generalization of (7). The final state depends then on the energy and the continuous probability distribution function g(a) of the vorticity levels, given by the initial condition: g(a) d(a) is the fraction of the domain with vorticity between a and $a + \mathrm{d}a$. The terms $\mathrm{e}^{-\alpha_i}$ are then replaced by a continuous function $\lambda(a) \geqslant 0$, 0, so that

$$p(a, \mathbf{r}) = \lambda(a) \frac{e^{-\beta a \Psi}}{Z(\Psi)}, \qquad \text{with } Z(\Psi) = \int_{-\infty}^{+\infty} \lambda(a)\, e^{-\beta a \Psi}\, da$$

and the system (7) becomes

$$-\Delta\Psi = f_{\alpha,\beta}(\Psi) = \int_{-\infty}^{+\infty} a\lambda(a) \frac{e^{-\beta a \Psi}}{Z(\Psi)} da \tag{8a}$$

$$\Psi = \text{const. at the boundaries} \tag{8b}$$

$$\int_D \lambda(a) \frac{e^{-\beta a \Psi}}{Z(\Psi)} d^2\mathbf{r} = \frac{g(a)}{|D|} \tag{8c}$$

$$1/2 \int_D \Psi(-\Delta\Psi)\, d^2\mathbf{r} = E. \tag{8d}$$

3. Methods of determination and general properties of the equilibrium states

The solution of the problem (7) must be generally obtained by a numerical method. It requires much less spatial resolution than solving the initial Euler equations, since the fine-scale vorticity structures have been filtered out by the statistical averaging. Nevertheless, the problem is still difficult because it generally involves bifurcations with several parameters, and the Lagrange parameters α_i, β are only indirectly determined by the n integral constraints.

A direct attack to this problem is presented by Thess & Sommeria (1993), and applied to the organization of a jet in a channel: (7) is discretized on N grid points (by a finite difference method), which yields a set of N + n algebraic equations, that can be solved by the Newton's method. The different branches of solution are then followed by continuity: Starting from a known solution, the parameters are varied by small steps, and the previous solution is used as the initial guess for the Newton's method. Equilibrium states can be obtained also as the final state of relaxation equations (Robert & Sommeria, 1992) which conserve all the constants of the motion, while entropy monotically increases. Notice finally that a new method of relaxation has been proposed by Turkington & Whitaker (1993), and applied to a shear layer with great efficiency.

Useful information can be also obtained from the theory without solving the problem (7) or (8). Indeed the function $f_{\alpha,\beta}(\Psi)$ has general properties discussed in Robert & Sommeria (1991). It is always a monotonous function bounded by the minimum and maximum initial vorticity levels a_i (this function is strictly increasing when $\beta < 0$, and strictly decreasing when $\beta > 0$). The mean field approximation of Joyce & Montgomery (1974) for point vortices can be recovered from (7) in the limit $Z(\Psi) \rightarrow 1$. This limit is obtained when most of the domain is filled by the zero vorticity level, and vorticity patches occupy a very small area, which is reasonable for a point vortex model.

There is another interesting special limit when the argument $\beta a \Psi$ is small and the results can be expanded in powers of this quantity. This will appear to correspond to the limit of low energy. We shall restrict the analysis to vorticity distributions which are symmetric with respect to a, and remain so in the equilibrium state, i.e. the function $\lambda(a)$ is also symmetric. In this restricted case, the expansion of $p(a, \mathbf{r})$ and the set of equations (7) yields

$$p(a, \mathbf{r}) = \lambda(a)\left[1 - a\beta\Psi + \frac{a^2 - A_2}{2}(\beta\Psi)^2 - \frac{3a^3 - A_2 a}{2}(\beta\Psi)^3 + \ldots\right] \tag{9}$$

$$-\Delta\Psi = -A_2\beta\Psi - \frac{3A_4 - A_2^2}{6}(\beta\Psi)^3 - \ldots) \tag{10}$$

$$\lambda(a) = g(a)\left[1 + \beta a\langle\Psi\rangle - \frac{a^2 - \langle a^2\rangle}{2}\beta^2\langle\Psi^2\rangle + \ldots\right] \tag{11}$$

$$E = -|D|\frac{\langle a^2\rangle\beta}{2}(\langle\Psi^2\rangle - \langle\Psi\rangle^2) + \ldots \tag{12}$$

The integral constraints associated with the conservation of the global probability distribution of vorticity have been inverted to give explicitly $\lambda(a)$ by the expansion (11), as a function of the spatial averages $\langle\Psi^n\rangle$. The successive momenta of the function $\lambda(a)$ are denoted

$$A_n = \int_{-\infty}^{+\infty} a^n \lambda(a)\, da$$

At the lowest order, $\lambda(a) = g(a)$, so that A_n is equal to the moment

$$\langle a^n \rangle = \int_{-\infty}^{+\infty} a^n g(a)\, da$$

of the global vorticity distribution. However, differences appear at higher orders as indicated by (11). The partial differential equation (10) can be solved by successive approximations, together with the conditions (11) and (12).

At the lowest order, the vorticity is an eigenfunction of the Laplacian with eigenvalue βA_2, which must be negative (i.e. $\beta < 0$), as a consequence of the positivity of energy in (12). Then each eigenfunction of the Laplacian initiates a branch of solutions, but we can check that the branch corresponding to the lowest eigenvalue has always the maximum entropy, and must be selected. The result of the statistical theory then corresponds to the minimum enstrophy structure in this limit. The eigenvalue of the Laplacian $\langle a^2 \rangle\, \beta$ is of the order $1/|D|$, so that, using the energy value (12), $(\beta a \Psi)^2$ is of order

$$\frac{E}{(a^2|D|)\,|D|}.$$

Therefore the expansion is indeed valid for low energy. More precisely, the ratio of the energy over the initial enstrophy defines an initial scale of motion, and the expansion is valid when this initial scale is sufficiently small in comparison with the size of the domain.

The deviation from this linear approximation can be obtained by solving (10) by a perturbation method. The corrective term in Ψ^3 has the sign of $(\mathrm{Ku} - 3)$, where Ku is the Kurtosis of $\lambda(a)$, i.e. $\mathrm{Ku} = A_4/A_2^2$, which is equal to the Kurtosis of the vorticity distribution $g(a)$ in first approximation. Therefore, in the domain of validity of the expansion (i.e. low energy), we can predict the behaviour of the function $f_{\alpha,\beta}(\Psi)$ from the successive momenta of the initial probability distribution function of vorticity. For instance, in the case of two non-zero vorticity levels a and $-a$ which occupy a total area F, $\mathrm{Ku} = |D|/F/F$. If $F > |D|/3$, the coefficient of the cubic term Ψ^3 in (10) is negative ($f_{\alpha,\beta}(\Psi)$ behaves like a tanh), while if $F < |D|/3$, the coefficient is positive ($f_{\alpha,\beta}(\Psi)$ behaves like a sinh). Notice that a similar expansion can be obtained from the results of Joyce & Montgomery (1974), but the coefficient of the cubic term is then only $A_4/2$, which is always positive. This result is recovered in the new theory when the Kurtosis is very large, which is indeed obtained in the dilute case, for which the area of the initial vorticity patches is small. [1] Notice finally that when $\lambda(a)$ is a

[1] This expansion should apply to typical computations of two-dimensional turbulence, with initial state at small scale, predicting a nearly linear relationship between vorticity and stream function. However the time of evolution is long before reaching a final state, so that small viscosity effects can modify the distribution of vorticity levels, leading to a high Kurtosis. Therefore the behaviour in sinh obtained by Montgomery *et al.* (1992) can be explained, at least qualitatively.

gaussian, Ku = 3, so the cubic term vanishes. Moreover, it is then easily shown by the exact calculation of $f_{\alpha,\beta}(\Psi)$, that all the higher order terms also vanish: the relation between vorticity and stream function is linear (however, the corresponding vorticity distribution function $g(a)$ depends on the energy and is generally not gaussian).

Coming back to the general case, $f_{\alpha,\beta}$ appears obviously as a single-valued function. However the method of Langrange multipliers leading to (7) or (8) allows only to find solutions p_i of extremal entropy which are continuous. Discontinuous solutions may also exist, as in the case of the modons discussed in section 6. Then, if the optimal functions p_i are continuous and strictly positive on some open connected subset $D' \subset D$, the same argument on Lagrange multipliers works on D′ and shows that the representation (7) or (8) of this solution holds on D′. Therefore the function $f_{\alpha,\beta}$ could take several values, on different subsets D′, separated by discontinuities of at least one of the p_i . Moreover, it can be proved that if one of the optimal functions p_i is strictly equal to zero at some point, then the solution p_i is discontinuous.

4. Tests and applications of the theory

The result of the theory has been found in good agreement with direct numerical computations of the Navier-Stokes equations at high Reynolds number in several configurations. One of the simplest geometries is a channel with periodic boundary conditions in x and impermeable transverse boundaries. The case of a shear layer in a channel with stream-wise periodic boundary conditions has been investigated by Sommeria *et al.* (1991). Good agreement is also obtained for a jet in the same domain by Thess & Sommeria (1993). We shall present similar comparisons in the next section, for the problem of vortex merging in a large domain.

Different kinds of fluid systems can be described by a two-dimensional dynamics, mostly in Geophysical Fluid Dynamics and plasma physics. The extension of the theory to quasi-geostrophic systems is straightforward, even in multilayer cases, by replacing the vorticity by the potential vorticity. Therefore, a new way of understanding and predicting the organization of atmospheric and oceanic systems can be developed. For instance, it is possible to predict the formation of coherent jets and vortices like the Great Red Spot of Jupiter, as shown by Sommeria *et al.* (1991) and Michel *et al.* (1994).

The dynamics of an electron plasma in a magnetic field can be described by the 2D Euler equations, as shown by the beautiful experiments of Fine *et al.* (1991), and Peurrung & Fajans (1993). In this context the point vortex statistics has been applied to a circular domain by Pointin & Lundgren (1976) and Smith & O'Neill (1990), and the difference with the present theory remains to be tested. The similar flow of a liquid metal in a uniform magnetic field has been experimentally studied by Denoix *et al.* (1993), and a good fit with the present theory is obtained. However, such flows are strongly confined by the fluid domain, and possible boundary layer detachments make the problem more complex. I shall discuss here the case of vorticity structures in a very large domain, for which boundary effects are not important (but the ergodicity of such an open system and its possibility to really reach a statistical equilibrium are more questionable).

5. Application to the vortex merging and the organization of a monopole

To apply the statistical theory, we need first to note the conserved quantities associated with the specific symmetries. In an unbounded domain, the two components of the linear momentum **P** are conserved, as well as the angular momentum M:

$$\mathbf{P} = \int_D \omega \times \mathbf{r} d^2\mathbf{r}$$

$$M = \int_D \omega \mathbf{r}^2 d^2\mathbf{r}$$

(these quantities depend on the choice of the origin of the position vector **r**). The corresponding Lagrange multipliers have to be introduced in (5). The modification is straightforward and leads to equilibrium states of the form

$$\omega = f(\Psi - \mathbf{U} \cdot \mathbf{r} - \gamma \mathbf{r}^2),$$

i.e. steady Euler flows in a translating or rotating frame of reference. There are two cases, depending on the total circulation

$$\Gamma = \int_D \omega d^2\mathbf{r}.$$

For the case of monopoles that we consider in this section, $\Gamma \neq 0$, while the case $\Gamma = 0$ (dipoles) is postponed to next section. If $\gamma \neq 0$, the term $\mathbf{U}.\mathbf{r}$ can always be suppressed by a change of the coordinate origin, taking $\mathbf{r}' = \mathbf{r} + \mathbf{U}/2\gamma$. Then a purely rotating structure is obtained, and the linear momentum, calculated with the new origin, must vanish: otherwise it would rotate with the structure and would not be conserved. This means that the centre of the rotation must be the centre of mass of the vorticity distribution[2] (If $\gamma = 0$, the structure is translating with velocity **U**, but since the centre of mass cannot move, we must have $\mathbf{U} = 0$, this is just a regular limit of the case $\gamma \neq 0$).

The equilibrium state has been obtained by Delbende *et al.* (1994) as the final evolution of the relaxation equations proposed by Robert & Sommeria (1992). A turbulent diffusion term increases the entropy, while keeping constant all the

[2] However, this is only possible if the centre of mass is defined, i.e. if $\Gamma \neq 0$. Otherwise, **P** does not depend on the origin of the coordinates: then if $\mathbf{P} \neq 0$, we must have $\gamma = 0$, and get a translating structure.

conserved quantities, until a maximum entropy state is reached. From a numerical point of view, it is necessary to have boundaries, and a circular domain, centred at the centre of mass of the vorticity field, would be appropriate to keep the conservation laws. However, for practical reasons, we have used instead a square domain while still imposing the conservation of angular momentum. Since the domain is much wider than the size of the vorticity distribution, this brings only weak perturbations.

The equilibrium state depends on the vorticity distribution function $g(a)$, the total vorticity area F, the angular momentum M, and the energy E. We restrict ourselves to the case of initial vorticity patches with a single non-zero vorticity level $\omega = a$ with total area F. The equilibrium vorticity structure still depends on the two non-dimensional parameters $M/(aF^2)$ and $E/(aF)^2$, while the area F and vorticity a determine the size and typical vorticity of the structure. We have still restricted the problem to an initial state with two circular vortices of radius R whose axis are separated by a distance d. This is the classical problem of symmetric vortex merging, and the final structure depends only on a single parameter d/R.

In this problem, the final equilibrium state appears to be always axisymmetric, with exponentially decreasing vorticity at large distance. The radial structure of the vorticity depends on the parameter d/R, but the central core has always a very similar vorticity profile, shown in fig. 1a. For high values of the angular momentum (large d), a small proportion of the total vorticity is expelled at fairly large distances, with little change in the central core. Therefore merging is not prevented by the conservation of angular momentum, and we have never obtained equilibrium states with two separated vortices. This is the case even for ratios $d/R > 3.3$, for which it is not supposed to occur (see for instance, Christiansen & Zabusky 1973). The two vortex system with large separation may be a particular set of stable states which cannot initiate the mixing process[3]. However, when a complex vortex distortion is initiated, we can explain the merging process and its final axisymmetric organization by the tendency of the system to increase its entropy, with the constraints of the conservation laws.

Quantitative agreement with direct numerical computations of the Navier-Stokes equations is indeed obtained. Although we are interested in the inviscid dynamics, a small viscosity must be introduced to avoid the development of fine-scale noise (and the vorticity field must be smoothed at the edge of the patches). For applying the theory to a real flow, *it is essential to assume that viscosity has no significant effect on the distribution of the vorticity levels, during the time of formation of the final equilibrium structure*: then the only effect of viscosity is to suppress the local fluctuations of the equilibrium state, leading to a vorticity field equal to the local average of the inviscid theory $\bar{\omega}(\mathbf{r})$. We use a pseudo-spectral code in a periodic domain with symmetry conditions in x and y, so that

[3] Nevertheless, further investigation of possible two vortex equilibrium states has still to be performed, with a rigorous test for bifurcations, and the search for possible discontinuous equilibrium states.

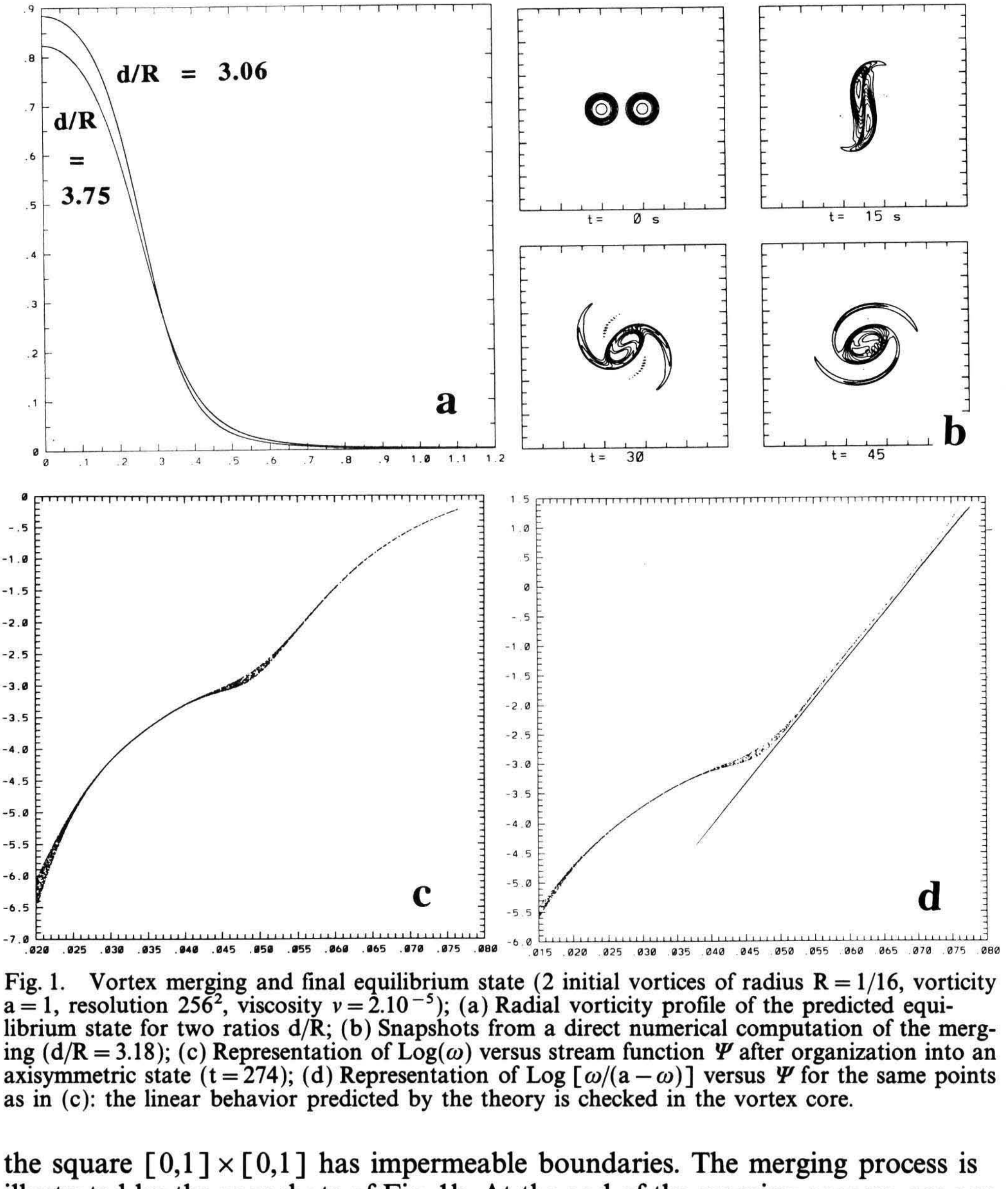

Fig. 1. Vortex merging and final equilibrium state (2 initial vortices of radius $R = 1/16$, vorticity $a = 1$, resolution 256^2, viscosity $\nu = 2.10^{-5}$); (a) Radial vorticity profile of the predicted equilibrium state for two ratios d/R; (b) Snapshots from a direct numerical computation of the merging ($d/R = 3.18$); (c) Representation of $\mathrm{Log}(\omega)$ versus stream function Ψ after organization into an axisymmetric state ($t = 274$); (d) Representation of $\mathrm{Log}\,[\omega/(a-\omega)]$ versus Ψ for the same points as in (c): the linear behavior predicted by the theory is checked in the vortex core.

the square $[0,1] \times [0,1]$ has impermeable boundaries. The merging process is illustrated by the snapshots of Fig. 1b. At the end of the merging process, we can distinguish two regions: a strongly mixed central core, and spiral arms which roll up around the core. A final axisymmetric vortex structure is obtained at $t = 200$ (with a very slow decay due to viscosity).

The quantitative agreement with theory is best checked by plotting vorticity versus stream function from the final state of the numerical computation. Two representations are given in Fig. 1c, d (at $t = 274$). We first observe that all the points nearly collapse on a single curve, an indication for a steady solution of the Euler equations. The point vortex statistics predicts an exponential function which should be represented by a straight line on Fig. 1c. There is clearly a discrepancy, especially near the vorticity maximum. By contrast, the present theory predicts a straight line for the representation of Fig. 1d, which is indeed well verified in the core of the vortex. However discrepancy appears in the spiral region. This result can be interpreted as the consequence of strong mixing in the core but only incomplete mixing outside. Indeed the filaments are in the first approximation stretched by the irrotational axisymmetric flow induced by the vortex core. This is an organized motion, rather than a complete mixing process. There is here a fundamental alternative: does the mixed core extend more and more in the limit of long times and vanishing velocity? Instead, does a barrier to mixing, or a complex set of barriers, remain in the inviscid limit? Limitations to mixing appear also in other cases, for instance in the partial merging of two unequal vortices (Dritschel & Waugh, 1992). In any case, the completely mixed state predicted by the theory is clearly a good starting point to study the merging process.

6. Maximum entropy solutions for a modon

Now let us consider the organization of an initial state with zero total vorticity $\Gamma = 0$, surrounded by an infinite irrotational fluid. The linear momentum **P** is now an important conserved quantity, while the angular momentum only determines the position of the structure: we choose the origin such that it vanishes. We then define the x axis parallel to **P**, and use the usual polar coordinates r, θ. At large distance from the vorticity region, the stream function is the dipolar field

$$\Psi = \frac{P \sin\theta}{2\pi r}$$

(obtained as an expansion of the Biot-Savart formula).

We first notice that a continuous maximum entropy solution in the whole domain is now impossible (except for the degenerate case $\beta = 0$, with zero energy). Indeed, each probability p_i would be a function of $(\Psi - Uy)$. Since $\Psi \to 0$ at large distance, the argument is then dominated by Uy ($U \neq 0$), which takes all the values between $-\infty$ to $+\infty$, so that the probability could not vanish everywhere at large distance. Therefore the equilibrium probabilities cannot decay smoothly in the whole domain, *and for each vorticity level, we must have a discontinuity, beyond which* $p = 0$ (The same result holds for $U = 0$, although the justification is a little less simple). We shall make the hypothesis that all the probabilities are continuous inside a domain D, and drop to zero on the same line: the boundary of D. Inside this domain, the flow must be steady in the frame of reference translating at a velocity U. Therefore the line of probability discontinuities must keep the same shape, and the whole domain D translates at constant speed U: the boundary of D is a streamline in the translating frame of reference.

Outside D, the flow is irrotational, and is therefore uniquely determined by the boundary of D and its velocity U. This velocity can be related to the momentum P and a boundary integral, using an integration by parts

$$P = U|D| + \oint y\mathbf{u} \cdot \mathbf{dl} \equiv C_D U|D|.$$

The velocity on the boundary depends only on the outside flow, so that the coefficient C_D depends only on the shape of the domain D. It is convenient to use the non-dimensional variables r', a' and the stream function Φ in the translating frame or reference

$$\mathbf{r} = R\mathbf{r}', \qquad \text{with } \pi R^2 = |D|, \qquad a = a_0 a', \qquad \Psi(\mathbf{r}) - Uy = a_0 R^2 \Phi(\mathbf{r}'),$$

where a_0 is a characteristic vorticity of the initial condition. We can write the energy as the sum of the internal energy and the translation energy

$$E = (a_0 R^2)^2 \frac{1}{2} \int_D \Phi(-\Delta\Phi)\, d^2\mathbf{r}' + \frac{P^2}{2\pi C_D R^2}.$$

In fact, because of the continuity of the velocity at the boundary of D, the two terms must be of the same order, which imposes a strong constraint on the domain area $\pi R^2 \approx P^2/E$. Then, the determination of the equilibrium states requires to solve a non-dimensional version of (8), with the new integral conditions (denoted by primes):

$$g'(a') = \frac{Fa_0}{R_2} g(a_0 a'), \qquad E' = (a_0 R^2)^{-2} \left[E - \frac{P^2}{2\pi C_D R^2} \right].$$

The probability distribution g(a) is relative to the initial area F where the vorticity is non-zero (the value $a = 0$, for which g(a) is not defined must be excluded of all the integrals involving a). We have again the impermeable boundary condition $\Phi = 0$, but also an imposed velocity on the boundary, by continuity with the outside flow. The necessity for these two conditions will determine the shape of the boundary, and the value of R.

Therefore the shape and structure of the modon is in principle predicted by the statistical theory in terms of the conserved quantities, although the general problem is difficult. However we can use the expansion presented in section 3. At the first order, we get the classical modon, translating at velocity U = U/P, with the vorticity confined to a circular domain with $R = P/(2\pi E)^{1/2}$. The internal stream function

$$\Psi - Uy = \sqrt{\frac{2E}{\pi}} \frac{J_1(x_1 r') \sin\theta}{-x_1 J_0(x_1)}$$

is expressed in terms of the Bessel functions J_o and J_1. x_1 must be a zero of J_1, and the condition of highest entropy selects the first zero ($x_1 \approx 3.83$). Indeed, the expansion of the entropy gives at this order

$$S = const - \frac{x_1^2 E'}{2\langle a^2 \rangle} + \dots$$

The next order correction can be obtained by a perturbation method, and the shape of the domain must be corrected together with the function Φ. Such an expansion is valid if

$$\frac{E}{a_0^2 F R^2} \ll 1 \qquad \left(\text{with } R^2 = \frac{P^2}{E}\right) \tag{13}$$

which is true if the scale R of the modon is sufficiently large. In the opposite limit, the area R^2 defined by P^2/E may become smaller than the initial vorticity area F. A possible organization in that case is the formation of two modons with partly opposing directions, such that the momentum of each individual modon is greater than the initial one, allowing a larger scale P^2/E. In any case, such a system of two modons occupies a wider area, and it has clearly a higher entropy than a single modon. Therefore, we can understand that the condition (13), and the corresponding linear relationship between vorticity and stream function is a quite common feature.

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