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A Contour Dynamics Approach to Large-Scale Atmospheric Flow [1]

Abstract

A simple model of the large-scale atmospheric circulation is studied. The model is based on the observation that the potential vorticity changes rapidly at the tropopause. We idealize the potential vorticity distribution on an isentropic surface (cutting through the tropopause) by taking it to be piecewise uniform with a discontinuity at the tropopause. If it is assumed that the time evolution of the atmosphere can be described by the equivalent barotropic vorticity equation, a dynamical system is obtained which can be formulated entirely in terms of the discontinuity. We first discuss a zonally symmetric flow to demonstrate that the model leads to quite realistic zonal velocity profiles. We then consider infinitesimal-amplitude (linear) waves superimposed on the basic zonal flow. It is found that these waves are neutral and move westward with respect to the basic zonal flow like Rossby-Haurwitz waves on a solid-body background flow. Using a numerical iteration procedure we also construct families of finite-amplitude (nonlinear) waves. The possible use of stationary waves of this kind as models for atmospheric blocking is discussed.

Introduction

The tropopause, the boundary between troposphere and stratosphere, is traditionally defined as the surface where the vertical gradient of potential temperature undergoes a large, discontinuous, change. It has been stressed by Shapiro (1980) that also the potential vorticity undergoes a large change at the tropopause. In fact he proposes to replace the traditional thermodynamic definition of the tropopause by Reed's (1955) dynamic definition in which the tropopause is identified with a surface of constant potential vorticity with a value in the range of rapid change. The fact that stratospheric values of potential vorticity are indeed much larger than tropospheric ones is illustrated by Fig. 1a. This figure shows a meridional cross-section through the atmosphere at the 45° east meridian. The dashed lines are isolines of potential temperature, the solid

[1] This contribution is an extended summary of the author's paper 'Tropopause dynamics and planetary waves', which will be published in the Journal of the Atmospheric Sciences.

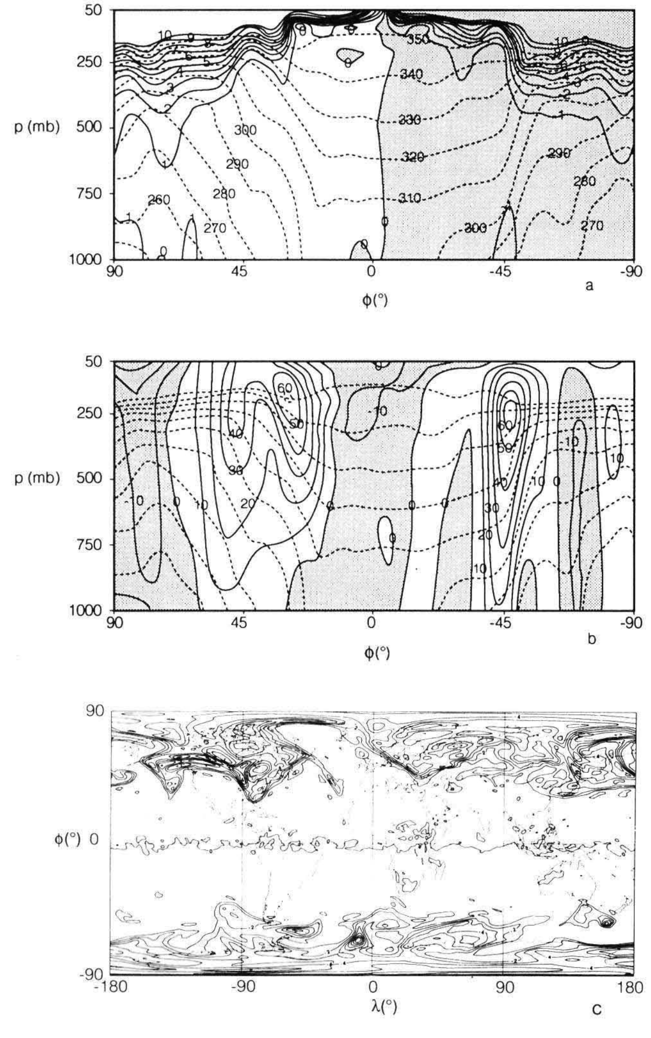

Fig. 1. (a) Meridional cross-section of the atmosphere at the 45° east meridian at January 27 1987, 12.00 GMT. Solid curves are isolines of potential vorticity in units of $10^{-6}\,m^2\,s^{-1}\,Kkg^{-1}$ (PVU), dashed curves are isolines of potential temperature in K. The contour interval for potential vorticity is 1 PVU, the contour interval for potential temperature is 10 K. The region where potential vorticity is negative is shaded. (b) The same meridional cross-section but now with zonal velocity (in ms^{-1}) and potential temperature (in K). The contour interval for the zonal velocity is $10\,ms^{-1}$, the contour interval for potential temperature is 10 K. Regions with negative zonal velocity are shaded. (c) Isolines of potential vorticity on the 310 K isentropic surface for the same date and time as the cross-sections of (a) and (b). The contour interval for the potential vorticity is 1 PVU. Figs. 1a and 1b were prepared by dr. P. Berrisford, Fig. 1c by dr. T. Davies, both using archived meteorological fields from the European Centre for Medium Range Weather Forecasts (ECMWF).

lines are isolines of potential vorticity. The tropopause could, in Reed's definition, be placed at the +2 PVU level in the Northern Hemisphere and at the −2 PVU level in the Southern Hemisphere. (For the definition of one unit of potential vorticity (PVU) see the caption of Fig. 1a.) The potential vorticity involves both the vertical temperature gradient and the absolute vorticity, and usually both are involved in any change of its value. As a consequence one also finds the maximum velocities (the center of the jet stream) in the neighbourhood of the tropopause. This is illustrated by Fig. 1b, which shows the same meridional cross-section with isolines of potential temperature and zonal velocity.

Due to the large change in potential vorticity at the tropopause, its position determines, to a large extent, the structure of the potential vorticity distribution in the atmosphere. Its position therefore largely determines the state of the atmosphere as a consequence of the invertibility principle (assuming balanced flow and given the distribution of potential temperature at the ground, see Hoskins *et al.*, 1985). In this paper we wish to explore this fact to devise and study a simple model of the atmospheric large-scale circulation. Our first assumption is that we can limit ourselves to a single surface of constant potential temperature (isentropic surface) at a representative height in the atmosphere and such that it intersects the tropopause. Viewed from this perspective the tropopause emerges as a small band of closely packed isolines of potential vorticity, as illustrated by Fig. 1c. Our next assumption is that the large change at the tropopause is indeed the dominant feature of the potential vorticity distribution and that, on the isentropic surface, it can be assumed to be piecewise uniform. We then assume that large-scale atmospheric flow is adiabatic and frictionless, which means that both potential vorticity and potential temperature are conserved following the motion of fluid particles. This implies that the dynamics of potential vorticity is given by advection within each isentropic surface. To close the system we finally assume that the dynamics of potential vorticity is governed by the equivalent barotropic vorticity equation on a rotating sphere. Because potential vorticity only changes by advection, the dynamics reduces to the dynamics of a single line or contour. For this type of system a new theoretical and numerical technique has been developed recently which is called contour dynamics (see the review by Dritschel, 1989).

The model

As announced in the Introduction, we will consider the evolution of potential vorticity on a single isentropic surface, say the 310 K surface in Fig. 1. It is assumed that this surface does not intersect the earth and that the surface can be approximated by a sphere with radius $a = 6.371 \times 10^6$ m, rotating with the earth's angular velocity $\Omega = 7.292 \times 10^{-5}\ \mathrm{s}^{-1}$. Distances will be expressed in units a, time in units Ω^{-1}, and points $\mathbf{r}$ on the sphere will be denoted by their geographical coordinates (λ, ϕ).

We assume that the potential vorticity on an isentropic surface can be approximated by

$$q = \zeta + f - F\psi. \tag{1}$$

Here ζ is the relative vorticity, given by the vertical component of the curl of $\mathbf{v}$, where $\mathbf{v}$ is the velocity field. The velocity field has components u and v along the unit vectors $\mathbf{i}$ and $\mathbf{j}$ and is assumed to be nondivergent. Therefore, $\mathbf{v}$ can be written in terms of the stream function ψ,

$$\mathbf{v} = \mathbf{k} \times \nabla\psi, \tag{2}$$

where $\boldsymbol{k}$ is a unit vector pointing vertically upwards. This implies that the relative vorticity can be written as

$$\zeta = \nabla^2\psi. \tag{3}$$

The second contribution to the potential vorticity is the planetary vorticity f which, in the unit Ω, can be expressed as

$$f = 2 \sin \phi. \tag{4}$$

The last contribution to q is the 'stretching term' $-F\psi$. This term is an approximate way of taking into account the effects of vertical stratification. For the factor F in the stretching term we write

$$F = L_R^{-2}, \tag{5}$$

where L_R is the Rossby radius in units of a. We will take $L_R = 1/10$ which amounts to a Rossby radius of 637.1 km. (For a discussion of this particular choice we refer to the next section.) The fact that potential vorticity is conserved following the fluid motion thus leads to a single closed system in terms of the potential vorticity q

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0, \tag{6}$$

This equation is the equivalent barotropic vorticity equation.

Matters are simplified further by assuming that the q-field is piecewise uniform. We will thus assume that q has the constant value q_1 in a region R_1 (around the north pole) and another constant value q_0 in the rest of the sphere, denoted by R_0. The boundary between R_1 and R_0 is assumed to be a single closed curve B, see Fig. 2.

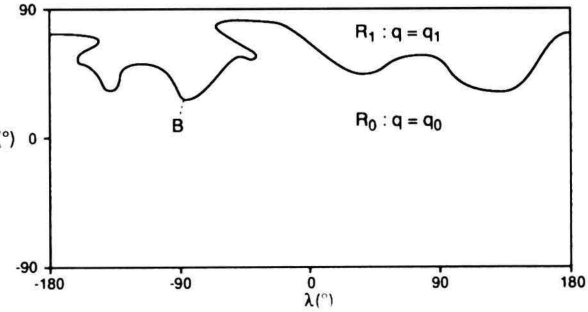

Fig. 2. Schematic picture (inspired by Fig. 1c) of the idealized potential vorticity distribution on an isentropic surface. The potential vorticity q has the constant value q_1 in the region R_1 and constant value q_0 in the region R_0. The region R_1 is to be associated with the stratosphere, the region R_0 with the troposphere and the boundary B with the tropopause.

So the potential vorticity q is assumed to be of the form

$$q(\mathbf{r}) = q_0 + \begin{cases} q_1 - q_0 & \mathbf{r} \in R_1 \\ 0 & \mathbf{r} \in R_0 \end{cases} \tag{7}$$

It can then be demonstrated that the gradient of the stream function (from which the velocity follows according to (2)) is given by

$$\nabla\psi(\mathbf{r}) = \frac{\nabla f(\mathbf{r})}{F+2} + (q_1 - q_0) \oint_B dl' [\mathbf{n}' \cdot \mathsf{T}(\mathbf{r}; \mathbf{r}')] \, G(\mathbf{r}; \mathbf{r}'). \tag{8}$$

where dl' is a line element along the boundary B and $\mathbf{n}'$ is a unit vector locally perpendicular to the boundary and to $\mathbf{k}$ and pointing away from R_1. In this expression T is a tensor defined by

$$\mathsf{T}(\mathbf{r}; \mathbf{r}') \equiv -\frac{\cos\phi'}{\cos\phi} \mathbf{i}'\mathbf{i} + \sin\phi' \sin(\lambda - \lambda') \, \mathbf{i}'\mathbf{j} - \cos(\lambda - \lambda') \, \mathbf{j}'\mathbf{j}. \tag{9}$$

and G is the Green's function of the Helmholtz operator for a sphere,

$$G(\mathbf{r}; \mathbf{r}') = -[4\cosh(\pi\kappa)]^{-1} \times P^0_{-1/2+i\kappa}(-\cos\theta''), \tag{10}$$

where $P^m_\nu(x)$ is a Legendre function with integer order m, real or complex degree ν and real argument x. The order m of the Legendre function is 0 whereas the degree ν is given by $-1/2 + i\kappa$. The parameter κ is related to F by

$$F = 1/4 + \kappa^2. \tag{11}$$

The argument x is $-\cos\theta''$ where θ'' is the angular distance between the points $\mathbf{r}$ and $\mathbf{r}'$ for which we have

$$\cos\theta'' = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda - \lambda'). \tag{12}$$

It is also possible to derive an analogous expression for the stream function. We have

$$\psi(\mathbf{r}) = -\frac{q_0}{F} + \frac{f(\mathbf{r})}{F+2} - \frac{q_1 - q_0}{F}\frac{A_1}{4\pi} + \frac{q_1 - q_0}{F}\oint_B dl'\mathbf{n}' \cdot \nabla' V(\mathbf{r}; \mathbf{r}'), \tag{13}$$

where V is a scalar defined by

$$V(\mathbf{r}; \mathbf{r}') \equiv G(\mathbf{r}; \mathbf{r}') - H(\mathbf{r}; \mathbf{r}'), \tag{14}$$

with

$$H(\mathbf{r}; \mathbf{r}') \equiv \frac{1}{4\pi} \ln \sin^2(\theta''/2). \tag{15}$$

By A_1 we mean the area of R_1. These expressions give the stream function and its gradient (and therefore the velocity) in terms of the boundary B. The expression for the velocity allows us to determine the evolution of B in terms of B itself. Each point of B is advected by the velocity field at the corresponding point and this velocity is determined by the gradient of the planetary vorticity (the first term on the right-hand side of (8)) and a line integral over B (the second term on the right-hand side of (8)). This is the essence of what is called *contour dynamics*.

The material presented in this section is a straightforward application of known techniques in contour dynamics to the equivalent barotropic vorticity equation on a rotating sphere. Expression (8) was derived by Kimura and Okamoto (1987) for the barotropic vorticity equation on a sphere. A new element, according to the author's knowledge, is expression (13) for the stream function in terms of a line integral. Details of the derivations can be found in Verkley (1993).

Zonal flow

In order to check whether our model is capable of reproducing flows that resemble the global atmospheric circulation, we first discuss the flow resulting from a discontinuity in the q-field that coincides with a latitude circle. In other words, we first consider the case in which the region R_1 is separated from the rest of the sphere R_0 by a boundary B which is given by

$$\phi(\lambda) = \phi_B, \tag{16}$$

We note that although this is a highly idealized case, the corresponding velocity profile will serve as a first approximation of the velocity field associated with a contour of arbitrary shape.

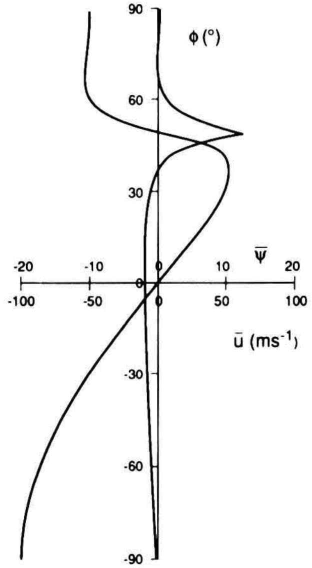

Fig. 3. An example of zonal flow profile, resulting from a piecewise uniform potential vorticity distribution. The value of ϕ_B is 50° and the value of $q_1 - q_0$ is 2.99. The value of q_0 is 0. The cusped line is the zonal velocity $\bar{u}$ (in ms^{-1}, values below the horizontal axis), the smooth line is the nondimensional stream function ψ (multiplied by 1000, values above the horizontal axis). Note the sharp maximum of u at the tropopause $\phi(\lambda) = \phi_B$.

The velocity and stream function corresponding with a zonal contour can be calculated using (8) and (13). We used an alternative (simpler) way and the result of an example is given in Fig. 3. The value of ϕ_B is 50° and the value of q_0 is 0. For the difference in potential vorticity, $q_1 - q_0$, we have taken the value 2.99. This value is chosen such that a linear wave with zonal wavenumber 3 is stationary for this zonal flow, as will be seen in the next section. The figure shows the zonal velocity u (the cusped lines) and the zonal stream function ψ (the smooth lines) as a function of the latitude ϕ. The numbers below the horizontal axes are velocity in ms^{-1}, the numbers above the horizontal axis are nondimensional stream function values multiplied by 1000. The figure shows that the zonal velocity u has a sharp maximum at the tropopause. Around its maximum the wind is westerly and further away it is weakly easterly. The value of the velocity at the tropopause is in this case 63.09 ms^{-1}. It can be shown that the velocity at the tropopause depends mainly on the difference in potential vorticity between troposphere and stratosphere. The velocity at the tropopause is nearly independent of the latitude of the tropopause except when the tropopause lies close to the north pole. We remark that, in choosing the value of F, we were guided by the corresponding zonal velocity profiles. As can be seen from expression (8), the contribution of the Coriolis parameter to the velocity field is inversely proportional to $F + 2$. This means that if F becomes small this contribution becomes large. For $F = 0$, this contribution is so large that it is impossible to obtain a zonal flow profile that behaves realistically at all points on the sphere.

The barotropic vorticity equation, i.e., equation (6) without the 'stretching term' $-F\psi$ in the expression for q, would therefore be quite unacceptable as a basis for our contour dynamics model of the atmosphere.

Linear waves

In this section we consider linear waves, i.e., waves with infinitesimal amplitude on the basic zonal flows discussed in the previous boundary B. In the case of linear waves on a zonal background the contour B is written as

$$\phi(\lambda)=\phi_B+\delta\phi(\lambda) \tag{17}$$

with

$$\delta\phi(\lambda)=\mathrm{Re}[\varepsilon \exp im(\lambda-\omega t)], \tag{18}$$

where Re denotes the real part, ε is a number with infinitesimally small absolute value, m is a nonnegative integer and ω is the angular velocity of the wave propagating along the basic zonal flow. Note that ε as well as ω can be complex.

It can be shown that wave solutions of the form above can indeed be found. It follows that for positive q_1-q_0 the waves move westward with respect to the basic zonal flow. This can be understood qualitatively as follows. Fig. 4 shows, in addition to the perturbed contour, the corresponding stream function for q_1-q_0 and ε equal to 1. Cells with positive values are marked with H, cells with negative values are marked with L. The arrows give the direction of maximum meridional velocity associated with the perturbation stream function. These arrows show that the perturbation stream function is such as to induce a

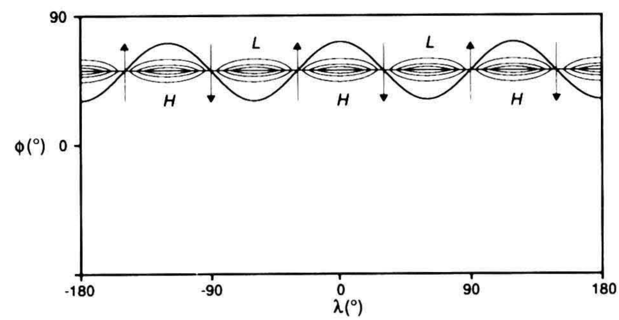

Fig. 4. The stream function associated with a linear wave. Both q_1-q_0 and ε are given the value 1. Cells with positive values are marked by H, cells with negative values by L. The outer isoline of each cell has value ± 0.01, the next isoline denotes a value of ± 0.02 and the third isoline has value ± 0.03. The arrows denote the direction of the largest meridional velocity. The figure illustrates that the perturbed contour induces a westward phase velocity of the perturbation pattern.

westward motion of the wave pattern. In fact, the wave propagation mechanism encountered here is a specific example of the more general case discussed by Hoskins *et al.* (1985, p. 919).

Nonlinear waves

In this section we will investigate whether a system consisting of a single discontinuity in potential vorticity supports finite amplitude waves around an average zonal flow. We will restrict ourselves to stationary waves, i.e., waves of which the contours coincide with an isoline of the corresponding stream function. The approach of this issue is numerical and we therefore introduce a discrete label i, ranging from 1 to N, which labels the different points $\mathbf{r}_i$ by which we represent any contour B. We then define the following functional K,

$$K \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2} [\psi(\mathbf{r}_i) - \psi_{av}]^2, \tag{19}$$

where ψ_{av} denotes the average value of ψ over B, which is defined as

$$\psi_{av} \equiv \frac{1}{N} \sum_{i=1}^{N} \psi(\mathbf{r}_i). \tag{20}$$

The functional K is nonnegative, and zero if and only if ψ is constant along the contour, i.e., if and only if the contour is stationary. If contours exist for which K is indeed zero we might expect to find them from appropriate first guesses by an adjustment process in which K is minimized. This is the basic idea of the method. For the minimization we used the routine E04KAF from the NAG Fortran Library, Mark 13, in double precision implementation, which is based on a quasi-Newton algorithm. We applied the numerical procedure to obtain families of stationary nonlinear waves on the basic zonal contour of which the velocity and stream function profiles are given in Fig. 3. We recall that the latitude of this basic zonal contour is $\phi_B = 50°$ and that the value of $q_1 - q_0$ is chosen such that this contour supports a stationary linear wave with zonal wavenumber $m = 3$.

We will denote the positions of the k-th member of a family of stationary contours by (λ_i^k, ϕ_i^k), where i runs from 1 to N and the first guess from which each member is obtained by $(\hat{\lambda}_i^k, \hat{\phi}_i^k)$. Then the 0-th member of the family is the basic zonal contour represented by

$$\lambda_i^0 = -\pi + (i-1)\left(\frac{2\pi}{N}\right), \tag{21a}$$

$$\phi_i^0 = \phi_B. \tag{21b}$$

The first guess for contour (λ_i^1, ϕ_i^1) is of the form of a linear wave with a small but finite amplitude, i.e.,

$$\hat{\lambda}_i^1 = -\pi + (i-1)\left(\frac{2\pi}{N}\right), \tag{22a}$$

$$\hat{\phi}_i^1 = \phi_B + \varepsilon \cos(m\hat{\lambda}_i^1), \tag{22b}$$

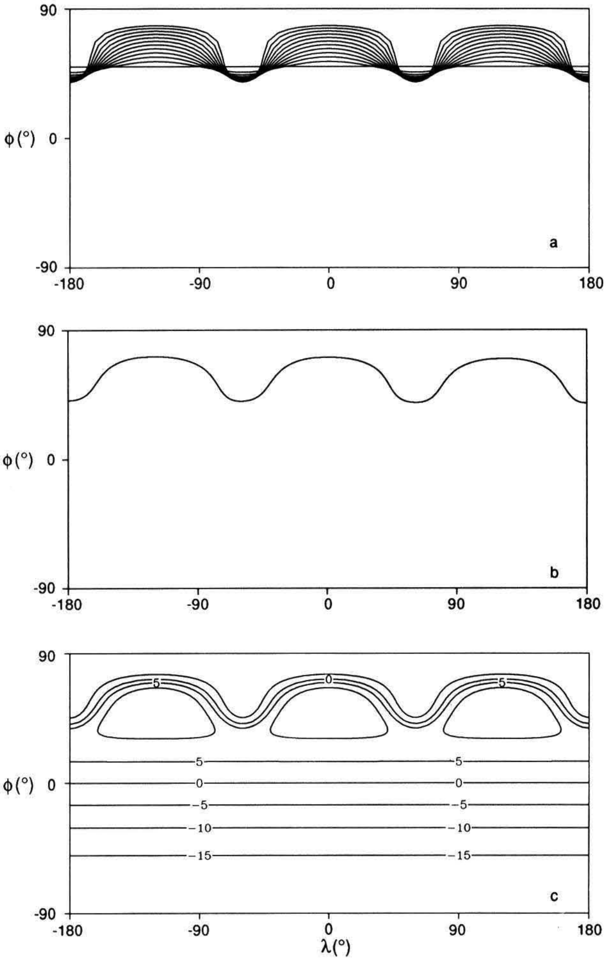

Fig. 5. (a) Families of stationary nonlinear waves on the basic zonal contour of which the stream function and velocity profiles are given in Fig. 3. In (a) we show the contours with $k = 0, 2, 4, \ldots, 22$ resulting from the numerical procedure described in the text. In (b) a representative member $(k = 14)$ is shown. In (c) we show the stream function corresponding with the contour of (b). The nondimensional values of the stream function are multiplied by 1000 and the contour interval is 5. The stream function is calculated on a regular grid of 144×72 points using a numerical discretization of expression (13).

where $\varepsilon = 2°$. We build up a family of contours by using as a first guess for any new contour a linear extrapolation of the two previous contours. For the stationary contour (λ_i^k, ϕ_i^k) the first guess is thus taken to be

$$\hat{\lambda}_i^k = \lambda_i^{k-1} + [\lambda_i^{k-1} - \lambda_i^{k-2}], \tag{23a}$$

$$\hat{\phi}_i^k = \phi_i^{k-1} + [\phi_i^{k-1} - \phi_i^{k-2}]. \tag{23b}$$

For the number of points N we take $N = m \times 30$, i.e., 30 points for each wavelength.

For the basic zonal flow of Fig. 3 and $m = 3$ we could continue the process of finding stationary contours until $k = 22$. After this the obtained contours did not change appreciably but fell back to their predecessors. The average of the intial value of K for these 22 cases was 2.91×10^{-10}, the average of the final value of K was 8.63×10^{-13}. In Fig. 5a we show the contours for $k = 0, 2, 4, ..., 22$. In Fig. 5b we show the contour for $k = 14$ from the family in Fig. 5a in isolation. The corresponding stream function, calculated by using a numerical discretization of (13), is shown in Fig. 5c. We observe that the isolines of the stream function are closely packed around the contour which signifies, of course, that the velocity field is sharply peaked at the tropopause. We also note that in the ridges of the waves closed cells of the stream function have formed.

We note that, although the procedure has brought us quite far into the realm of nonlinear stationary waves, it is quite likely that one can go much further. In the contours shown in Fig. 5a there is a tendency for the contours to touch upon themselves in the vicinity of the throughs, in much the same way as in the study of Pratt (1988). This suggests that, if the process were to be continued, a transition might occur in which the solution changes from a single contour into a combination of several isolated contours.

Summary and discussion

Our main finding is that the contour dynamics approach to large-scale atmospheric flow leads to a simple and concise picture of the atmosphere. First of all, the assumption of a single line of discontinuity representing the tropopause leads to a reasonable zonal flow profile. The velocity field is westerly and sharply peaked at the tropopause and falls off rapidly and becomes easterly away from the tropopause. Zonal flows of this form support linear neutral waves which, like Rossby-Haurwitz waves on a solid-body rotation, propagate westward with respect to the basic zonal flow. Also in analogy with Rossby-Haurwitz waves there exist families of nonlinear waves, although the form of the waves changes if the amplitudes increase. The nonlinear waves are obtained numerically using an iterative technique.

The results reported in the present paper only constitute a first step in the analysis of large-scale atmospheric phenomena from the viewpoint of the tropopause. In particular, the finite-amplitude waves we obtain can probably be extended much further into the nonlinear domain. It should also be possible, we believe, to construct finite amplitude waves with a localized character in much the same way as modon solutions can be found. These waves, if they exist, would be an appropriate model of atmospheric blocking. The advantage of such a model would be that it quite naturally incorporates the basic finding of Illari (1984) and Crum and Stevens (1988), namely that in the blocking region the potential vorticity is relatively low and uniform.

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