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Stability of 2-D Circular Vortices

Abstract

The second variation of a linear combination of energy and angular momentum is used to investigate the stability of circular vortices. For the simplest model of an isolated vortex the linear stability regime is found to coincide with the formal stability regime. The method uses Lagrangian displacements of vorticity contours and can be applied to vortices consisting of several nested rings of piecewise constant vorticity.

Introduction

With regard to the stability of planar circular vortices in an ideal, incompressible and unbounded fluid, Rayleigh's inflection-point theorem (Drazin & Reid, 1981) states that a sufficient condition for linear stability is that the vorticity gradient does not change sign anywhere, i.e. no inflection-point in the azimuthal velocity profile. Rayleigh's theorem is of little use when one considers isolated vortices, i.e. vortices with zero circulation and finite energy. Such vortices always have at least one inflection-point, but this does not guarantee instability, and the question arises whether in some cases they are actually stable. Linear stability, i.e. stability with respect to vanishingly small perturbations, can be established with normal-modes analysis, but here, however, a method is discussed to establish a stronger form of stability, i.e. formal stability. A stationary flow is called formally stable if there is a conserved quantity such that the first variation of this quantity (i.e. the lowest order change due to arbitrary infinitesimal perturbations) is zero while the second variation is sign-definite. In finite dimensional systems formal stability implies nonlinear stability whereas in infinite dimensions it is a necessary prerequisite for nonlinear stability.

The present method was developed after it was noted that Arnol'd's (1966) method cannot be applied to isolated vortices. It is applied here to the simplest possible model of an isolated vortex and the single, circular vortex patch (Rankine vortex). Details can be found in Kloosterziel & Carnevale (1992).

Variations of conserved quantities

For an ideal incompressible fluid, the area enclosed by a material curve Γ is a constant of the motion, as well as the kinetic energy E and angular momentum L. The dependence of E and L on the vorticity distribution q(x,y) is

$$E = \frac{1}{2} \int\int_{\mathscr{D}} q\psi \, dx \, dy, \tag{1}$$

$$L = \int\int_{\mathscr{D}} (x^2 + y^2) \, q \, dx \, dy, \tag{2}$$

where the stream function ψ is

$$\psi(\mathbf{r}) = -\frac{1}{2\pi} \int\int_{\mathscr{D}} q(\mathbf{r}') \log|\mathbf{r} - \mathbf{r}'| \, dx' \, dy'. \tag{3}$$

Consider the case of a circular vortex, which is a steady state. In particular, the models to which our analysis are most easily applied are circular vortices with piecewise constant vorticity, i.e. vortices with (in polar coordinates (r, θ))

$$\begin{aligned} q &= q_1 \qquad (0 < r < d_1), \\ &= q_2 \qquad (d_2 < r < d_2), \\ &= q_j \qquad (d_j < r < d_{j+1}), \qquad etc. \end{aligned}$$

At each radial position $r = d_i$ the vorticity jumps by $\Delta q_i = q_{i+1} - q_i$. The vortex is perturbed by slightly deforming the circular vorticity contours without breaking or folding them. Energy, angular momentum and area enclosed by each contour are conserved during the subsequent evolution. It appears plausible therefore that the flow cannot develop towards a radically different state if it can be shown that a further growth of the perturbation amplitude would violate a conservation law. Such is the case if at $O(\varepsilon)$, with ε some measure for the perturbation amplitude, energy and angular momentum are unchanged while the second order variation of some combination of them is sign-definite, in other words, in the case that the particular functional has a maximum/minimum at the stationary state with respect to area preserving perturbations.

In order to apply classical calculus of variations, one expands all functionals involved in a power series in ε. First, however, one has to prescribe the perturbation. For instance, in polar coordinates a perturbed circular contour could be written as $d_i + \varepsilon\delta r_i(\theta)$, with d_i the radius of the circle and $\varepsilon\delta r_i$ the perturbation (with $|\delta r_i|$ at most $O(1)$). The index i labels the particular contour under consideration here. The change in area is $\varepsilon \int_0^{2\pi} d_i \delta r_i(\theta) \, d\theta + \frac{1}{2}\varepsilon^2 \int_0^{2\pi} \delta r_i(\theta)^2 \, d\theta$, and although at $\mathcal{O}(\varepsilon)$ area conservation can be satisfied, it cannot at $\mathcal{O}(\varepsilon^2)$. A more general perturbation is introduced

$$r_i(\theta) = d_i + \varepsilon\delta r_{i,1}(\theta) + \tfrac{1}{2}\varepsilon^2 \delta r_{i,2}(\theta) + \cdots,$$

and for given $\delta r_{i,1}$ area can now be conserved at orders higher than one also by an appropriate choice of the $\delta r_{i,m}$ ($m=2, 3, \cdots$). The area enclosed by the perturbed circle is

$$A(\Gamma_i+\delta\Gamma_i)=A_{i,0}+\varepsilon A_{i,1}+\tfrac{1}{2}\varepsilon^2 A_{i,2}+\cdots,$$

where $A_{i,0}=\pi d_i^2$ and

$$A_{i,1}=\int_0^{2\pi} d_i\delta r_{i,1}\,d\theta=\int_0^{2\pi} dA_{i,1}(\theta),$$

$$A_{i,2}=\int_0^{2\pi} (\delta r_{i,1}^2+d_i\delta r_{i,2})\,d\theta=\int_0^{2\pi} dA_{i,2}(\theta).$$

Area conservation means $A_{i,1}=A_{i,2}=\cdots=0$. In a similar vein changes in stream function, energy and angular momentum are expanded

$$\begin{aligned}\psi&=\psi_0+\varepsilon\psi_1+\tfrac{1}{2}\varepsilon^2\psi_2+\cdots,\\ E&=E_0+\varepsilon E_1+\tfrac{1}{2}\varepsilon^2 E_2+\cdots,\\ L&=L_0+\varepsilon L_1+\tfrac{1}{2}\varepsilon^2 L_2+\cdots.\end{aligned}\tag{4}$$

first variations

After some calculations (see Kloosterziel & Carnevale, 1992) the following first variations are found:

$$L_1=-\sum_i \triangle q_i d_i^2\int_0^{2\pi} dA_{i,1}(\theta),$$

$$E_1=-\sum_i 2\triangle q_i\psi_0(\Gamma_i)\int_0^{2\pi} dA_{i,1}(\theta).$$

The unperturbed vortex is a steady flow and therefore the stream function ψ_0 is constant on each contour Γ_i. It is thus seen that under area preserving perturbations, i.e. $\int dA_{i,1}=0$, the first variations of energy and angular momentum are zero. It can be shown more generally that the first variation of energy is zero for any stationary flow, while for angular momentum it is zero only when the flow is circularly symmetric.

second variations

For the second variations it is convenient to introduce the variable ϕ_i defined as $\phi_i(\theta)=d_i\delta r_{i,1}(\theta)$. First order area conservation is then equal to $\int\phi_i d\theta=0$. Also the real inner product on $L^2[0,2\pi]$ is introduced $\langle f,g\rangle=\int_0^{2\pi} fg\,d\theta$, and the norm $\|\cdots\|$ in L^2, i.e. $\|f\|=\langle f,f\rangle^{1/2}$. If one imposes second order area conservation, one finds, using the notation introduced above, the following expressions for the second variations

$$L_2 = -2 \sum_i \Delta q_i \|\phi_i\|^2, \tag{5}$$

$$E_2 = \sum_i \frac{2v_\theta(d_i)}{d_i} \Delta q_i \|\phi_i\|^2 - \sum_i \sum_j \Delta q_i \Delta q_j \langle \mathscr{L}_{i,j}\phi_i, \phi_j \rangle. \tag{6}$$

Here v_θ is the azimuthal velocity of the unperturbed vortex at the indicated radial position and $\mathscr{L}_{i,j}$ is an integral operator

$$\mathscr{L}_{i,j}\phi(\theta) = \frac{1}{\pi} \int_0^{2\pi} \log|d_j e^{i\theta} - d_i e^{i\theta'}| \, \phi(\theta') \, d\theta'. \tag{7}$$

It can be shown that the eigenvalues of $\mathscr{L}_{i,j}$ are

$$\lambda_n(i,j) = -\Lambda_{i,j}^n \qquad (n = 1, 2, \cdots), \tag{8}$$

with

$$\Lambda_{i,j} = \min \left\{ \frac{d_i}{d_j}, \frac{d_j}{d_i} \right\}, \tag{9}$$

and eigenfunctions $\cos n\theta$ and $\sin n\theta$.

Rankine vortex

The Rankine vortex is a single circular 'patch' of constant vorticity q_1. For this case the second variation of angular momentum is according to (5) simply (there is only one contour $r = d_1$)

$$L_2 = 2q_1 \int_0^{2\pi} \phi_1^2 \, d\theta. \tag{10}$$

This expression is sign-definite for any perturbation $\phi_1(\theta) = d_1 \delta r_{1,1}(\theta)$, and it is seen that the Rankine vortex minimizes angular momentum when q_1 is positive, and is therefore formally stable.

Moreover, equation (5) shows that a vortex with, say, maximum positive vorticity at the centre, which decays monotonically with increasing radius (all $\Delta q_i < 0$), is also formally stable. Nonlinear stability for this case has been proven by Dritschel (1988), also by essentially using the angular momentum invariant and the area constraint. Similar vortices with smooth, monotonically decreasing vorticity can be shown to be nonlinearly stable with Arnol'd's method (Carnevale & Shepherd, 1990).

For vorticity distributions that have both positive and negative $\triangle q_i$ (this corresponds to vortices with inflection points), it appears necessary to use energy in addition to angular momentum. Before turning to this more complicated case, it is first shown here that formal stability of the Rankine vortex can also be inferred from a consideration of the energy. For this the perturbation is expanded in the eigenfunctions of the operator $\mathscr{L}$

$$\delta r_1(\theta) = \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta,$$

where no $k = 0$ component has been allowed so that area conservation at $\mathcal{O}(\varepsilon)$ is not violated. After substitution in (6) one gets

$$E_2 = -\pi q_1^2 d_1^2 \sum_{k=1}^{\infty} \left(\frac{k-1}{k}\right) \{a_k^2 + b_k^2\}, \tag{11}$$

where we have used $v_\theta(d_1) = \frac{1}{2} q_1 d_1$. It is seen that E_2 is sign-definite except for the case of a pure wavenumber 1 perturbation (in polar coordinates the perturbations are proportional to $\exp(ik\theta)$, where k is the wavenumber). Such a perturbation corresponds to a displacement of the vortex, and as expected this does not change the energy. This proves that the Rankine vortex with, say, positive vorticity, is a local maximum in energy with respect to all area preserving perturbations (modulo translations).

An isolated vortex

A simple example of an isolated vortex is one for which the azimuthal velocity increases monotonically from the centre until it reaches a maximum at some finite radius, and then falls off to zero monotonically. A velocity profile is then called 'steeper' than another, similar one, when the velocity falls off to zero faster in this outer region. Laboratory observations (Kloosterziel & van Heijst, 1991), numerical analysis (Gent & McWilliams, 1986; Carton & McWilliams, 1989) and analytical studies (Flierl, 1988) indicate that an isolated vortex of this type is unstable if steep enough. The simplest model for such vortices consists of a core of constant vorticity $q_1 = 1$ within the non-dimensional radius $r = 1$ plus an annulus of oppositely-signed vorticity $q_2 = -q < 0$ between $r = 1$ and $r = d$. These vortices all have vanishing circulation at $r = d$ if one takes $q = 1/(d^2 - 1)$. A steeper vortex corresponds here to larger q and correspondingly smaller d.

Normal-modes analysis shows (Flierl, 1988) that for large enough d (small q) they are linearly stable to all wavenumber perturbations. In the notation introduced above one has $d_1 = 1$, $d_2 = d$, $\triangle q_1 = -(1+q)$ and $\triangle q_2 = q$. Substitution in (5) and (6) yields

$$L_2 = (1+q)\, \|\phi_1\|^2 - q\|\phi_2\|^2, \tag{12}$$

$$E_2 = -(1+q)\|\phi_1\|^2 - (1+q)^2\langle \mathscr{L}_{1,1}\phi_1, \phi_1\rangle$$
$$-q^2\langle \mathscr{L}_{2,2}\phi_2, \phi_2\rangle - 2q(1+q)\langle \mathscr{L}_{1,2}\phi_1, \phi_2\rangle. \qquad (13)$$

It is clear from (12) that angular momentum is not sign-definite, and stability can no longer be inferred from it alone.

As an aside it may be noted here that the structure of unstable normal-modes can be uncovered using these expressions. The reason for this is the following. It turns out that the linearized equations of motion, from which the normal-modes equations follow, conserve both L_2 and E_2. This implies that unstable modes can only correspond to those cases for which $L_2 = E_2 = 0$, because a growing normal-mode would otherwise violate the conservation law (E_2 and L_2 are proportional to the square of the amplitude of the mode). So, by putting (12) and (13) equal to zero, one has two equations in two unknowns: the ratio of the perturbation amplitudes on the inner and outer boundary, and the phase difference between the two (see Kloosterziel & Carnevale, 1992). For a normal-mode one has

$$\phi_1(\theta) = d_1\delta r_{1,1}(\theta) = r_1 \cos m\theta$$
$$\phi_2(\theta) = d_2\delta r_{2,1}(\theta) = r_2 \cos(m\theta + m\theta_0),$$

where θ_0 is the phase difference between the perturbation on the inner and outer circle.

For instance, for m = 2 one has, after using (12) to eliminate the occurrence of $\|\phi_1\|$ in E_2,

$$E_{2,m=2} = q\|\phi_2\|^2\left\{-\frac{1}{2} + q - q\sqrt{\frac{q}{1+q}}\cos 2\theta_0\right\}.$$

Depending on the choice of θ_0, $E_{2,m=2}$ varies between

$$-\frac{1}{2} + q - q\sqrt{\frac{q}{1+q}} \leqslant E_{2,m=2}/(q\|\phi_2\|^2) \leqslant -\frac{1}{2} + q + q\sqrt{\frac{q}{1+q}}.$$

It follows that if the upper and lower bound are of the same sign, the vortex is linearly stable to wavenumber 2 perturbations. This happens only for q $q < q_{\text{crit}} = 1/3$, and then E_2 is negative definite. The situation can be interpreted as that all small $m = 2$ perturbations lead to an increase in energy. This critical value was previously found by Flierl (1988) by means of a normal-modes analysis of the linearized equations of motion. With q above the critical value the phase difference for a possible unstable, wavenumber-2 mode is determined by the relation

$$\cos 2\theta_0 = \frac{(q-\frac{1}{2})\sqrt{1+q}}{q\sqrt{q}},$$

while (12) determines the ratio of the amplitudes of the normal mode on the inner and outer boundary, i.e.

$$\frac{\|\phi_1\|^2}{\|\phi_2\|^2}=\frac{q}{1+q}.$$

Formal stability is proven if one can show that along the manifold defined by $L_2=0$, E_2 is sign definite, or vice versa. The same is accomplished in an elegant fashion if one can find a Lagrange multiplier μ such that the quadratic form $E_2+\frac{1}{2}\mu L_2$ is sign definite. The calculations are rather involved, and the reader is referred to Kloosterziel & Carnevale (1992) where it is shown that such a μ exists whenever $0\leqslant q\leqslant 1/3$. This is exactly the linear stability range as found with normal-modes analysis, but with wavenumber 1 perturbations excluded. Such perturbations correspond to imparting linear impulse to the system and lead to a translating vortex (see Stern, 1987). Modulo translations, it is thus concluded that the isolated vortex with $0\leqslant q\leqslant 1/3$ is formally stable.

Final remarks

In this paper only two cases have been discussed, i.e. the Rankine vortex and the simplest possible model of an isolated vortex, but clearly with (5) and (6) formal stability of other cases can also be investigated. Furthermore, by taking the limit of ever smaller vorticity jumps and closer and closer jump positions, one can derive the equivalent expressions for continuous vorticity distributions too.

A major question to be answered in the future is whether if formal stability is found for an isolated vortex, one can also find a proof of normed, nonlinear stability. For this the remainder of the Taylor series expansion has to be estimated and many technical complications surface. But, in view of many observations of long-lived, stable isolated vortices such a proof appears not impossible. It has been found that the above analysis proceeds along exactly the same lines when, instead of the variable $r(\theta)$, the variable $y(\theta)=r(\theta)^2/2$ is introduced. Angular momentum is then exactly quadratic in this variable, and the remainder of the functional $E+\frac{1}{2}\mu L$ beyond second order stems from just the energy functional. Unlike in Arnol'd stability, this remainder can be of both signs, depending on the amplitude of the perturbation, for instance, and the type of stability to be expected can only be conditional, i.e. stability with respect to perturbations that are 'small enough' initially. Arnol'd stability is unconditional, i.e. applies to perturbations of any size. It is similar to the finite-dimensional case of a marble in an infinitely-deep well. The case of conditional stability, however, is similar to that of a marble in a well of finite depth, surrounded by more 'holes' or planes. In this case a large-enough perturbation may take the marble far from its original location.

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