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Entropies for 2D Viscous Flows

Abstract

Previously, two-dimensional Navier-Stokes (2D NS) flow was represented as the evolution of two non-negative vorticity fields whose difference is the physical 2D vorticity, and whose overlap corresponds to the viscous dissipation of vorticity. Any physical state in 2D periodic geometry can be achieved as some rearrangement of the (conserved) fluxes of these two fields. Defining an entropy for the system is equivalent to assigning statistical weights to the possible rearrangements. Maximizing this "two fluid" entropy leads to satisfactory predictions for the turbulent decay of the embedded 2D NS flow, as indicated by numerical solutions of the two-fluid equations. The entropy defined is independent of the absolute values of the fluxes.

1. Introduction

Recent spectral-method computations (Matthaeus *et al.*, 1991a; Matthaeus *et al.*, 1991b; Montgomery *et al.*, 1992) of freely-decaying, two-dimensional, Navier-Stokes (2D NS) turbulence at high Reynolds numbers have been carried out for periodic boundary conditions. In a few hundred large-scale eddy turnover times, a two vortex final state has been achieved, and appears to decay stably thereafter. The decay is on the much slower energy decay time scale, typically more than 10,000 eddy turnover times. The dominant dynamical mechanism that achieves the two vortex state is like-sign vortex merger, repeated over and over again at increasingly larger spatial scales.

A respectable fit (Montgomery *et al.*, 1992) of the computed data to a much earlier (Joyce and Montgomery, 1973; Montgomery and Joyce, 1974) statistical mechanical theory of many ideal parallel line vortices has been noted. In particular, there is an apparent hyperbolic-sinusoidal dependence of the vorticity ω upon the stream function ψ (where $\nabla^2\psi = -\omega$) that characterizes the two vortex final state and which requires a justification going beyond the mean-field theory for the discrete-particle Hamiltonian mechanics (Onsager, 1949; Kraichnan and Montgomery, 1980) that underlie the ideal line vortex model.

Seeking a most-probable state for any system to evolve toward requires some kind of an entropy to maximize. Entropy for continua is still a controversial topic. Here, we propose a two-fluid description, involving positive and negative vorticity fields, which contains embedded in it the 2D NS dynamics. The non-negative fluxes associated with the two vorticities are conserved, and their interpenetration is equivalent to a decay of the physical vorticity. As conserved quantities, the two non-negative fluxes adapt themselves to an information-theoretic (Jaynes, 1957) definition of entropy.

The two-fluid model is summarized in Section 2, and some supporting numerical evidence is described in Section 3, along with some closing remarks.

2. Two-fluid model

The two vorticity fields, $\omega^{\pm}$, both non-negative, are taken to obey

$$\frac{\partial \omega^{\pm}}{\partial t} + \mathbf{v} \cdot \nabla \omega^{\pm} = \nu \nabla^2 \omega^{\pm} \tag{1a,b}$$

where the physical vorticity field is the difference, $\omega \equiv \omega^+ - \omega^-$. The fluid velocity is $\mathbf{v} = \nabla \psi \times \hat{e}_z$, where the stream function $\psi = \psi\ (x, y, t)$ obeys the Poisson equation, $\nabla^2 \psi = -\omega$. The kinematic viscosity is $\nu \ll 1$. For all fields, $\partial/\partial z \equiv 0$, and for simplicity we assume periodic boundary conditions in the (x, y) plane over a square box of edge 2π. Subtracting Eq. (1b) from Eq. (1a) gives the 2D NS equation in the well known vorticity representation.

For reasons which are familiar (Kraichnan and Montgomery, 1980), the energy $E = (1/2) \int (\omega^+ - \omega^-)\, \psi d^2x$ decays slowly, according to Eqs. (1a,b), but the enstrophy $\Omega = (1/2) \int (\omega^+ - \omega^-)^2\, d^2x$ decays much more rapidly. The essence of the maximum entropy argument is to define an entropy, or measure of the likelihood of a particular state, then maximize that measure subject to the constancy of the conserved or nearly-conserved quantities that may exist. If the system exhibits ordinary statistical-mechanical behaviour, or something close to it, its time evolution should typically lead it toward that maximum-entropy state. Previously (Montgomery *et al.*, 1992), the entropy proposed was

$$S = -\int \omega^+ \ln \omega^+ d^2x - \int \omega^- \ln \omega^- d^2x,$$

the maximization of which leads to the "sinh-Poisson" equation, if only energy and fluxes are conserved, and positive-negative symmetry is assumed:

$$\nabla^2 \psi = \lambda^2 \sinh(\beta \psi)$$

where λ^2 (>0) and β (<0) are real constants.

An unfortunate feature of this definition of entropy is that it is dependent upon the absolute value of the (conserved) fluxes, equal for periodic boundary conditions:

$$\int \omega^{\pm} d^2x \equiv V\langle \omega^{\pm} \rangle,$$

where V is $(2\pi)^2$, the area of the periodic box. However, both Eqs. (1) and all the other dynamics are invariant to the addition of the same positive constant to ω^+ and ω^-.

There is a more subtle way of counting states and assigning entropies that is invariant to the addition of constants to the two vorticity fields. For any pair of initial vorticities, ω^+ and ω^-, we may define four auxiliary fields by the following relations ($< >$ means a spatial average):

$$\begin{aligned}
\omega^{++} &= \omega^+ - \langle \omega^+ \rangle, && \text{if } \omega^+ > \langle \omega^+ \rangle, && \text{zero otherwise;} \\
\omega^{+-} &= -\omega^+ + \langle \omega^+ \rangle, && \text{if } \omega^+ < \langle \omega^+ \rangle, && \text{zero otherwise;} \\
\omega^{-+} &= -\omega^- + \langle \omega^- \rangle, && \text{if } \omega^- < \langle \omega^- \rangle, && \text{zero otherwise;} \\
\omega^{--} &= \omega^- - \langle \omega^- \rangle, && \text{if } \omega^- > \langle \omega^- \rangle, && \text{zero otherwise.}
\end{aligned}$$

Then any other spatial redistribution of these four non-negative fluxes associated with ω^{++}, ω^{+-}, ω^{-+}, and ω^{--} can account for all possible states into which the system might evolve, and do it in a way that is invariant to the addition of constants to ω^+ and ω^-. An entropy which measures the likelihood of any such redistribution may be taken to be

$$S = -\int (\omega^{++} \ln \omega^{++} + \omega^{+-} \ln \omega^{+-} + \omega^{-+} \ln \omega^{-+} + \omega^{--} \ln \omega^{--})\, d^2x. \quad (2)$$

Given the four auxiliary fields, ω^+ may for example be written as $\omega^+ = \omega^{++} - \omega^{+-} + \langle \omega^+ \rangle$ and similarly, $\omega^- = -\omega^{-+} + \omega^{--} + \langle \omega^- \rangle$.

Maximization of this S, subject to constant values of the fluxes of the four auxiliary fields and the nearly constant value of E yields, as most probable values, $\omega^{++} = \exp[-\alpha^{++} - \beta\psi]$, $\omega^{+-} = \exp[-\alpha^{+-} + \beta\psi]$, $\omega^{-+} = \exp[-\alpha^{-+} - \beta\psi]$, and $\omega^{--} = \exp[-\alpha^{--} + \beta\psi]$. Here, the α's and β are five Lagrange multipliers, to be determined from the conservation laws. If an assumption of complete symmetry is made among the four auxiliary fields, the four α's may be taken to be equal, and a hyperbolic-sinusoidal connection between $\omega = \omega^+ - \omega^- = \omega^{++} - \omega^{+-} + \omega^{-+} - \omega^{--}$ and ψ results. As will be seen in Section 3, this symmetry is not quite fulfilled by the computations, for reasons that are apparently rather specific to the system.

3. Numerical results

A 2D spectral-method periodic code has been written by Shan (Montgomery *et al.*, 1993) to solve Eqs. (1a,b). In the process, of course, a solution for the

2D NS equation is generated, but extra information on the two-fluid system is provided. Interpreted in terms of the 2D NS variables (ω, $\mathbf{v}$, ψ), the evolution is familiar (Matthaeus *et al.*, 1991a, Matthaeus, 1991b; Montgomery *et al.*, 1992). For Reynolds numbers greater than about 1000, like-sign vortex mergers occur until only one vortex of either sign remains. Here R is the large scale Reynolds number defined from the initial rms velocity $\langle \mathbf{v}^2 \rangle^{1/2}$ (typically $= 1$), unit length scale, and ν^{-1} which has ranged from 1000 to about 14,000 in recent runs. With these conventions, the Reynolds number is in effect ν^{-1}, and the characteristic energy decay time is also of the order of ν^{-1}. The vortex captures are typically completed in a time more than an order of magnitude less than the energy decay time.

As illustrated in Fig. 1, a scatter plot of the computed ω^+ and ω^- vs. ψ at late times is typically well fit by the maximum entropy predictions, which are the curves drawn through the scatter plots. The α's and β for the curves are determined by a least squares fitting procedure. The symmetry requirements necessary to convert the exponentials into hyperbolic sines for ω are typically not quite so well fulfilled, for what seems to be the following rather specific reason.

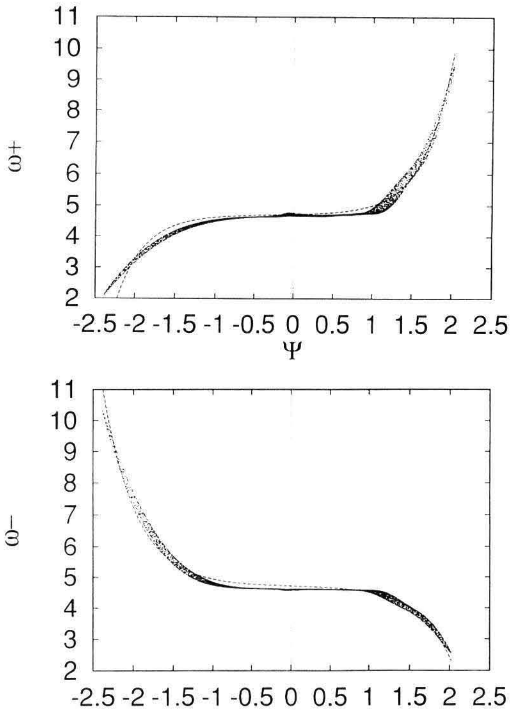

Fig. 1a,b. Scatter plots of the computed ω^+ vs. ψ and ω^- vs. ψ at $t = 390$ initial large-scale eddy turnover times at $R = 10,000$. The dashed lines are the least-squares fit of the maximum entropy predictions (from Montgomery *et al.* (1993)).

In the vortex merger process, there is a conservation law that guarantees that the two final-state vortices should have equal absolute values of integrated vorticity; but there is no corresponding reason why they should acquire equal energies. While a roughly equal distribution of energy might be expected, there is nothing in the dynamics of the sequence of like-sign vortex mergers leading to the final state that guarantees an equal sharing of energy. In the runs that have been done, it has been typical to have a ten percent difference in the kinetic energy associated with the half of the basic box containing the positive vortex, compared with the half containing the negative vortex, and this difference persists.

A consequence of this difference is that the structure of the positive and negative vorticity parts of the final state can be better fit separately by the two values of the reciprocal temperature β that differ by perhaps 15% than by a single common value of β. There are comparable differences in the α's, since they are not independent of the β. The mergers seem to lead to vortices that are individually maximum entropy structures more accurately than the whole system is fit by the overall maximum entropy prediction, somewhat in the manner suggested by Smith (1991). However, the overall fit with a single maximum-entropy sinh-Poisson prediction is not a bad fit to the computed data (Matthaeus *et al.*, 1991a; Matthaeus *et al.*, 1991b; Montgomery *et al.*, 1992).

Considerable work remains, in defining entropies precisely for the three-dimensional case (Chorin, 1991). There has also been a suggestion that a maximum-entropy analysis can be made to fit the solutions of the Euler equations ($\nu = 0$), which appears somewhat more problematical (Robert and Sommeria, 1992).

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