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Characterization and Numerical Calculation of Plane Vortex Structures

Abstract

A variational characterization is given for coherent vortices in plane fluid flows. We characterize isolated vortices, such as mono-, di- and tripolar vortices, as (relative) equilibrium states of the dynamical equation. To calculate the vortices numerically, an iteration scheme is constructed to solve the variational problems. The Lagrange multipliers and the free boundary are solved implicitly in the iteration process, which makes it possible to calculate the complex vorticity configurations with distributed vorticity.

Confined vortices in two-dimensional fluid flow

The Euler equations which give the evolution equations for incompressible, homogeneous and inviscid fluid flow, are a dynamical system with a special structure, called a Poisson structure. In vorticity formulation the Euler equations for purely two-dimensional flow are given by:

$$\begin{cases} \partial_t \omega + \nabla\omega \cdot J\nabla\psi = 0 \\ \omega = -\Delta\psi \end{cases} . \tag{1}$$

which are the vorticity equation and the Poisson equation; here ω denotes the scalar vorticity, ψ the stream function and J the skew-symmetric matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For this dynamical system there exists a natural invariant functional, the Hamiltonian H, which is the kinetic energy of the system:

$$H(\omega) = \frac{1}{2} \int \psi\omega \, dx \, dy \tag{2}$$

Together with other invariant functionals the Hamiltonian determines special solutions of the evolution equation. Critical points of the Hamiltonian on level sets of the invariant integrals are called relative equilibria. They are equilibrium solutions of the equation (1).

For an arbitrary function f of ω, the invariant functional:

$$C(\omega) = \int f(\omega)\, dx\, dy \tag{3}$$

is a Casimir integral, called the generalized enstrophy. A special case is given by the circulation or total vorticity:

$$\Gamma(\omega) = \int \omega\, dx\, dy \tag{4}$$

which is indeed a constant of the motion, as was shown by Kelvin. Besides the invariants given by the Casimirs C, the system admits constants of the motion given by the momentum integrals (linear and angular momentum), if the flow domain is translationally or rotationally symmetric.

We can find time-independent solutions by looking for relative equilibria, solving variational problems of the following type:

$$\operatorname*{extr}_{\omega \in L_2} \{H(\omega) \mid C(\omega) = \gamma\} \tag{5}$$

The Lagrange multiplier rule supplies a constant λ such that an extremizing solution ω (depending on γ) satisfies:

$$\delta H(\omega) = \lambda \delta C(\omega) \tag{6}$$

or similarly:

$$\psi = \lambda f'(\omega) \tag{7}$$

This yields an explicit functional relationship between stream function and vorticity, which is directly related to the choice of integral constraints $C(\omega)$ in the variational principle.

Denoting by M a momentum integral (either linear or angular momentum), steadily translating or rotating solutions can be found from the variational principles:

$$\operatorname*{extr}_{\omega \in L_2} \{H(\omega) \mid C(\omega) = \gamma;\ M(\omega) = m\} \tag{8}$$

We are interested in finding confined vortices, i.e. vorticity distributions with compact support, surrounded by irrotational flow. To find confined solutions, the function $f'(\omega)$ in (7) must be multivalued in $\omega = 0$. To achieve this multivaluedness, we use non-differentiable constraints given by the positive and negative circulations:

$$\Gamma_+(\omega) = \int (\omega)_+ \quad \text{and} \quad \Gamma_-(\omega) = \int (\omega)_- \tag{9}$$

where $(\omega)_+ = \max(\omega, 0)$, $(\omega)_- = \min(\omega, 0)$.

Related to the confinement of the vorticity distributions is the free boundary of the vortex support. The vorticity distribution and consequently the domain of non-zero vorticity have to be found from the variational inequalities for the vorticity.

We look at an algorithm that solves the variational inequality for the vorticity. The algorithm is able to find the multipliers and to adapt the free boundary within the process. First it is observed that the Lagrange multipliers satisfy a variational characterization (see also Eydeland and Van Groesen, 1989). Secondly the method by Eydeland *et al.* (1988, 1990) for non-convex optimization is applied, by linearizing the non-convex part of the functionals around the previous iteration step.

Instead of the variational inequality that has to be satisfied outside the vortex support, we use the Laplace equation for the stream function, with $\Delta\psi = 0$ and ψ sufficiently smooth over the boundary of the vortex domain.

The iteration scheme and some calculations are shown for mono-, di-, tri- and quadrupolar vortices.

Variational principle for Lagrange multipliers

Suppose a family of equilibrium solutions is parametrized by the value of γ in the following variational problem for the energy:

$$\operatorname*{extr}_{\omega} \{H(\omega) \mid C(\omega) = \gamma\} \tag{10}$$

If H and C are sufficiently smooth, there exists a constant $\lambda \in \mathbb{R}$ such that:

$$\delta H(\omega) - \lambda \delta C(\omega) = 0 \tag{11}$$

The value of λ depends on the specified value of the constraint, $\lambda = \lambda(\gamma)$, and the variational problem can be written as: $\text{extr}_{\omega}\, H(\omega) - \lambda C(\omega)$. The multiplier λ can be found from the following variational principle:

$$\operatorname*{extr}_{\lambda} \operatorname*{extr}_{\omega} H(\omega) - \lambda(C(\omega) - \gamma) \tag{12}$$

The extremum for ω, denoted by $\omega(\lambda)$, satisfies (11) and the remaining critical value problem for λ yields the value for λ for which $C(\omega(\lambda)) = \gamma$. In this way the constrained variational problem is transformed to an unconstrained problem. Without any restriction this can be generalized to variational problems with a family of constraints. After substitution of the extremizer $\omega(\lambda)$ into (12), the finite dimensional optimization problem for the multiplier(s) can be solved by a steepest descent method. The optimization for ω in (12) is infinite dimensional and the functional $H(\omega) - \lambda(C(\omega) - \gamma)$ might be non-convex in ω. We implement a numerical optimization method for non-convex functionals by Eydeland *et al.* (1988, 1990).

The iteration process

An iteration scheme is constructed, combining the non-convex optimization and the optimization for the Lagrange multipliers. Consider the variational principle (10). Choose a starting value for the vorticity ω_0 with corresponding stream function ψ_0. At iteration step k, with stream function ψ_k, the linearization of the energy functional is given by: $\int \delta H(\omega_k)\,\omega = \int \psi_k \omega$. If we define for each iteration step the functional:

$$\mathscr{L}_k(\omega, \lambda) = \int \psi_k \omega - \lambda(C(\omega) - \gamma) \tag{13}$$

the iteration process is given by:

$$\begin{cases} \omega_{k+1}(\lambda) = \arg \operatorname*{extr}_{\omega} \mathscr{L}_k(\omega, \lambda) \\ \lambda_{k+1} = \arg \operatorname*{extr}_{\lambda} \mathscr{L}_k(\omega_{k+1}(\lambda), \lambda) \end{cases} \tag{14}$$

From ω_{k+1} one finds ψ_{k+1} by application of a Laplace solver for the Poisson equation:

$$-\Delta \psi_{k+1} = \omega_{k+1} \tag{15}$$

with suitable boundary conditions for ψ_{k+1}. At each iteration step the multipliers are found such that the constraints are satisfied. This means that no distraction takes place from the level sets of the integrals.

The method is applied to steadily rotating vortices in the plane and to dipolar vortices in a bounded domain with periodic boundary conditions.

Calculation of confined vortices

Rotating vortices in the plane, with continuous and distributed vorticity, can be found from the following variational problem:

$$\operatorname*{extr}_{\omega \in S} \{H(\omega) - \alpha P(\omega)\} \tag{16}$$

with the constrained set S given by:

$$\{W(\omega) = w;\ \Gamma_+(\omega) = \gamma_+;\ \Gamma_-(\omega) = \gamma_-\} \tag{17}$$

The fixed value of α corresponds to the rotation rate of the vortices. Here $H = \frac{1}{2}\int \psi\omega$ denotes the kinetic energy, $P = \frac{1}{2}\int r^2\omega$ the angular momentum, $W = \frac{1}{2}\int \omega^2$ the enstrophy or squared vorticity and Γ_+ and Γ_- the positive and negative circulations (9).

We simulate the vortices in the plane by considering a numerical spatial domain D, with boundary condition imposed upon the stream function given by either $\psi(\mathbf{x})=0$ or:

$$\psi(\mathbf{x}) = -\frac{\Gamma(\omega)}{2\pi}\log|\mathbf{x}|, \qquad \mathbf{x}\in\partial D \tag{18}$$

The variational inequality for a maximizer of (16) reads:

$$\begin{cases} \psi - \frac{1}{2}\alpha r^2 = \mu\omega + \sigma_+, & \text{if} \quad \omega > 0 \\ \psi - \frac{1}{2}\alpha r^2 = \mu\omega + \sigma_-, & \text{if} \quad \omega < 0 \\ \sigma_- \leqslant \psi - \frac{1}{2}\alpha r^2 \leqslant \sigma_+, & \text{if} \quad \omega = 0 \end{cases} \tag{19}$$

Here μ, σ_+, σ_- are the multipliers associated to W, Γ_+, Γ_- respectively. We linearize the energy (with the correction for the rotation), and define the functional $\mathscr{L}_k$ at iteration step k by:

$$\mathscr{L}_k(\omega,\mu,\sigma_+,\sigma_-) = \int\left(\psi_k - \frac{1}{2}\alpha r^2\right)\omega - \mu\left(\frac{1}{2}\int\omega^2 - w\right)$$
$$-\sigma_+\left(\int(\omega)_+ - \gamma_+\right) - \sigma_-\left(\int(\omega)_- - \gamma_-\right) \tag{20}$$

Given ψ_k at step k, the next iteration step reads with $\mu > 0$:

$$\omega_{k+1} = \frac{1}{\mu}\left(\psi_k - \frac{1}{2}\alpha r^2 - \sigma_+\right)_+ + \frac{1}{\mu}\left(\psi_k - \frac{1}{2}\alpha r^2 - \sigma_-\right)_- \tag{21}$$

Substitution of ω_{k+1} into $\mathscr{L}_k$ yields $\mu_{k+1} = \arg\operatorname{extr}_\mu \mathscr{L}_k(\omega_{k+1},\mu,\sigma_+,\sigma_-)$ and after substitution of μ_{k+1} into $\tilde{\mathscr{L}}_k$ the problem reduces to a convex minimization, of a function $\tilde{\mathscr{L}}_k(\sigma_+,\sigma_-)$.

Numerical results

We chose the numerical domain $D = [-1, 1]\times[-1, 1]$. The calculations are performed on a 64x64 equidistant grid.

Monopolar vortices:

By taking $\gamma_- = 0$ in the variational principle (16), the extrema are positive monopolar vortices surrounded by irrotational flow. Figures 1 and 2 show typical vorticity distributions for maximization and minimization, respectively. The parameter values are: $w = 20$, $\gamma_+ = 4$, $\gamma_- = 0$ for the calculations on the monopolar vortices.

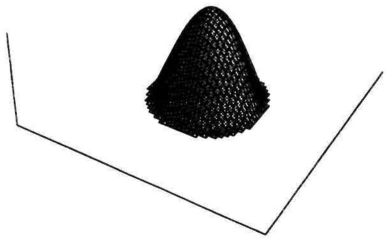

Fig. 1. Spatial plot of the vorticity solution of (16) for $w=20$, $\gamma_+=4$, $\gamma_-=0$, $\alpha=7$.

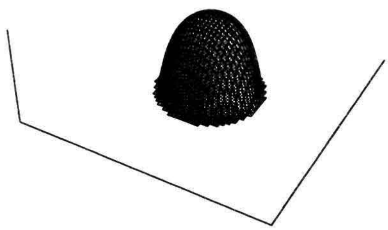

Fig. 2. Spatial plot of the vorticity solution of (16) for $w=20$, $\gamma_+=4$, $\gamma_-=0$, $\alpha=-7$.

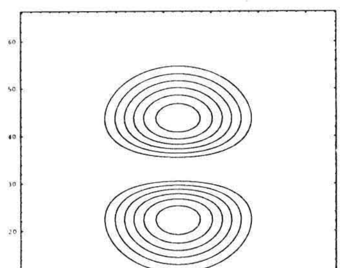

Fig. 3. Contour plot of the vorticity solution of (16) with $w=3.2$, $\gamma_+=1.6$, $\gamma_-=-1.6$, $\alpha=0.0$.

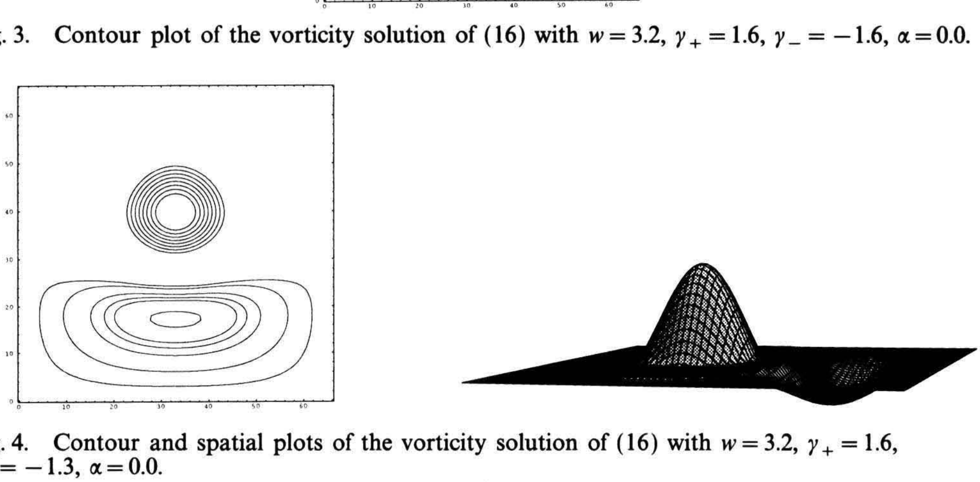

Fig. 4. Contour and spatial plots of the vorticity solution of (16) with $w=3.2$, $\gamma_+=1.6$, $\gamma_-=-1.3$, $\alpha=0.0$.

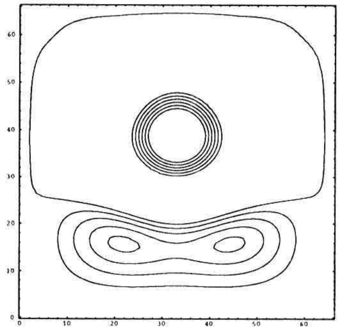

Fig. 5. Contour plot of the vorticity solution of (16) with $w = 3.2$, $\gamma_+ = 1.6$, $\gamma_- = -1.15$, $\alpha = 0.0$.

Non-symmetric dipolar vortices:

By taking $\alpha = 0$ in the variational principle (16) and $\psi = 0$ on ∂D we find a characterization of dipolar vortices on a periodic domain. We chose parameter values $w = 3.2$ and $\gamma_+ = 1.6$ and different values for γ_- in the neighbourhood of $\gamma_- = -1.6$. Figure 3 shows the vorticity in a contour plot for $\gamma_- = -1.6$. For this choice of parameter values there exists a solution which is a confined and symmetric dipolar vortex. The multipliers σ_+ and σ_- satisfy $\sigma_+ = -\sigma_-$.

There exists a branch of confined dipolar solutions, parametrized by the value of γ_-. In a neighbourhood of $\tilde{\gamma}_- = -\gamma_+$ non-symmetric, confined solutions of (16) are found. In figure 4 a non-symmetric dipole is shown for $\gamma_- = -1.3$. In figure 5 we show the result of the calculations with $\gamma_- = -1.15$. For this value of γ_- the value of the multiplier σ_- has become positive, such that the vorticity solution is not confined.

Tripolar and quadrupolar vortices:

For special choices of the parameter values we find tripolar and even quadropolar vortical structures. The structures rotate as a whole around the center of vorticity (the origin). We chose parameter values $w = 10$, $\gamma_+ = 1$, γ_- in the neighbourhood of $\gamma_- = -1$ and α in $\mathbb{R}$.

We show the vorticity at different stages in the iteration process in figures 6 and 7. It is observed that the symmetries of the initial vortex configuration are not necessarily conserved. The succeeding iterates form no approximation of any real dynamics.

In the experiments by Van Heijst and Kloosterziel (1989, 1990) only tripolar vortices have been observed with positive core and negative satellites rotating counter-clockwise (corresponding to α negative). The sign of rotation corresponds with the sign of vorticity of the vortex core. For most confined tripoles that are found by our numerical calculations, the rotation corresponds to the vorticity in the core. However, also tripoles are found that rotate in the

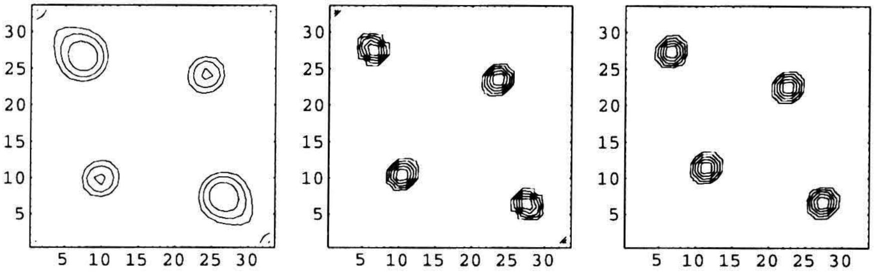

Fig. 6. Contour plots of the vorticity at different stages in the iteration. The calculation is performed with $w = 10$, $\gamma_+ = 1$, $\gamma_- = -1$, $\alpha = -0.04$.

other direction. In fact, tripolar and quadrupolar vortices are found with rotation rates positive, negative or even zero. Also in numerical simulations by Polvani and Carton (1990) tripolar vortices have been found with positive, negative or no rotational velocity.

With the chosen parameter values, the regions of vorticity are not far from the boundary of the numerical domain. This means that the imposed boundary condition on the square domain has a large influence on the solutions. It might be possible to find vorticity distributions that are confined to a small neighbourhood of the origin, by changing the parameter values for w, γ_+, γ_-, p and α. This however needs further investigation.

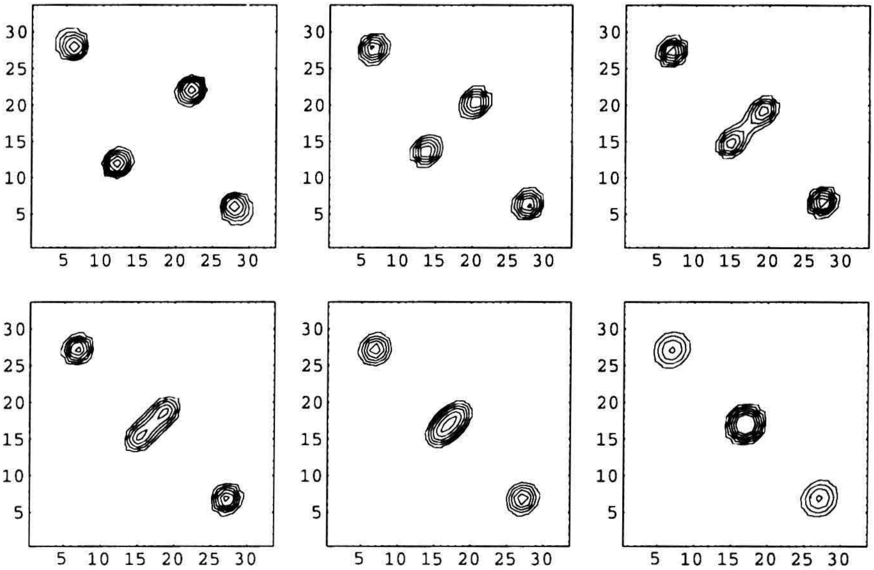

Fig. 7. Contour plots of the vorticity at different stages in the iteration. The calculation is performed with $w = 10$, $\gamma_+ = 1$, $\gamma_- = -1$, $\alpha = -0.08$.

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