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Determination of the Linear Stability of Modons on a Sphere by High-Truncation Time Integrations [1]

Abstract

The linear stability of modons on a sphere is determined from the growth of perturbations on an initial modon basic state. Numerical spectral time-integrations are performed for truncations T10-T43 and a few truncations up to T85. Three basic states are studied: a modon with an oscillatory exterior stationary in an eastward zonal flow, a modon with a decaying exterior stationary in a westward flow, and a Rossby wave. From the behaviour of the globally averaged energy and the relative vorticity at a particular point, the real and imaginary part of the eigenvalue of the fastest growing normal mode are obtained. Then from two stream function plots at different times, the real and imaginary part of this normal mode can be found. The modons and the Rossby wave are unstable: the decay time for the Rossby wave is 9 days, for the modons typically 5 days. The truncation runs confirm the results by Verkley (1987) who obtained the linear normal modes by solving the eigenvector problem up to T30. High-truncation runs show that the decay time slowly converges with truncation.

Introduction

The inviscid, unforced quasi-geostrophic equations possess a class of solutions known as modons. Modons are characterized by a multivalued relationship between potential vorticity and stream function in a comoving reference frame. Modons on the beta plane were introduced by Stern (1975) to describe Gulf Stream eddies and modons were put forward as models of atmospheric blocking by McWilliams (1980). The solutions have been extended to spherical geometry for several systems: dipole modon solutions for the barotropic vorticity equation by Verkley (1984, 1987, 1990), for the equivalent barotropic vorticity equation by Tribbia (1984), and for the two-layer quasi-geostrophic equations by Neven (1993). Quadrupole modon solutions are discussed in Neven (1992).

The decay of modons is an issue under discussion. On the beta plane, numerical stability analyses using time integrations were performed by Makino *et al.*

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(1981) and McWilliams *et al.* (1981). On a sphere, the normal modes of the linearized barotropic vorticity equation were determined by solving the eigenvalue problem for truncations up to T30 (Verkley 1987, 1990). The analytical proofs of stability and instability known in literature are still inconclusive (Nycander, 1992).

The aim of this paper is to provide a reliable and accurate determination of the stability properties of modons. The linear stability of modons on a sphere is determined from the growth of perturbations on an initial modon basic state. For three basic state fields, numerical spectral time integrations are performed for truncations T10-T43 and a few truncations up to T85. The results from the time-integration method of this paper confirm the results of the eigenvector method. The advantage of the time-integration method is that for a spectral model on a sphere with truncation TN the memory space required is proportional to N^3, whereas the eigenvector requires N^4. This enables us to investigate higher truncations and to determine more accurately the stability properties of modons. Investigation of the convergence of eigenvalues in dependence of truncation provides an estimate of the accuracy of the results.

Dynamical equations

For an inviscid, incompressible, unforced, homogeneneous fluid on a rotating sphere the conservation of potential vorticity q is expressed by the barotropic vorticity equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0 \tag{1}$$

where ψ is the stream function. The velocity is $\mathbf{v} = \mathbf{k} \times \nabla\psi$ with $\mathbf{k}$ the unit normal vector. With the radius of the earth $a = 6.371 \times 10^6$ m as the length scale, and the inverse of the angular velocity of the rotating earth $\Omega = 7.292 \times 10^{-5}\,\mathrm{s}^{-1}$ as the time scale, in spherical coordinates with longitude λ and latitude ϕ, the planetary vorticity $f = 2 \sin \phi$ and the potential vorticity $q = \nabla^2\psi + f$.

A decomposition of the field into a basic state field and a perturbation field

$$\begin{aligned} \psi &= \bar{\psi} + \psi' \\ q &= \bar{q} + q' \end{aligned} \tag{2}$$

with

$$\begin{aligned} \bar{q} &= \nabla^2\bar{\psi} + f \\ q' &= \zeta' = \nabla^2\psi' \end{aligned} \tag{3}$$

leads to the linearized barotropic vorticity equation

$$\frac{\partial q'}{\partial t} + J(\bar{\psi}, q') + J(\psi', \bar{q}) = 0 \tag{4}$$

The normal mode solutions of (4) can be written as

$$\psi'(\lambda, \phi, t) = \mathrm{Re}[\xi(\lambda, \phi)\, e^{\nu t}]$$
$$\zeta'(\lambda, \phi, t) = \mathrm{Re}[p(\lambda, \phi)\, e^{\nu t}]$$
$$p(\lambda, \phi) = \nabla^2 \xi(\lambda, \phi) \tag{5}$$

with complex fields

$$\xi(\lambda, \phi) = \xi_r(\lambda, \phi) + i\xi_i(\lambda, \phi)$$
$$p(\lambda, \phi) = p_r(\lambda, \phi) + ip_i(\lambda, \phi) \tag{6}$$

and complex eigenvalue

$$\nu = \nu_r + i\nu_i \tag{7}$$

Substitution of the fields (5) in the linearized equation (4) gives

$$\nu p + J(\bar{\psi}, p) + J(\xi, \bar{q}) = 0 \tag{8}$$

which is an eigenvalue problem with eigenvalue ν and the harmonic coefficients of the field $p(\lambda, \phi)$ as the components of the eigenvector. The definition of the real and imaginary part of the complex fields contains an arbitrary global phase factor due to the choice of origin of time. The normal mode of which the eigenvalues have the largest real part determines the stability properties of the modon. For the eigenvalue corresponding to this fastest growing normal mode, the real part is related to the e-folding time of the mode

$$\nu_r = \frac{1}{\tau} \tag{9}$$

and the imaginary part is related to the oscillation time of the mode

$$\nu_i = \frac{2\pi}{T} \tag{10}$$

There are three cases. If ν_r is negative all modes decrease exponentially and the stationary modon basic state solution is asymptotically stable. If $\nu_r = 0$ the modon is neutrally stable. If ν_r is positive there is at least one exponentially growing mode and the modon is unstable.

Basic states

We have investigated three stationary basic states centred at midlatitude $(\lambda, \phi) = (270°, 45°)$. The Rossby wave is characterized by a single linear $q(\psi)$-relationship with negative slope in the $q(\psi)$-diagram. The wavenumber $n = 8$ and

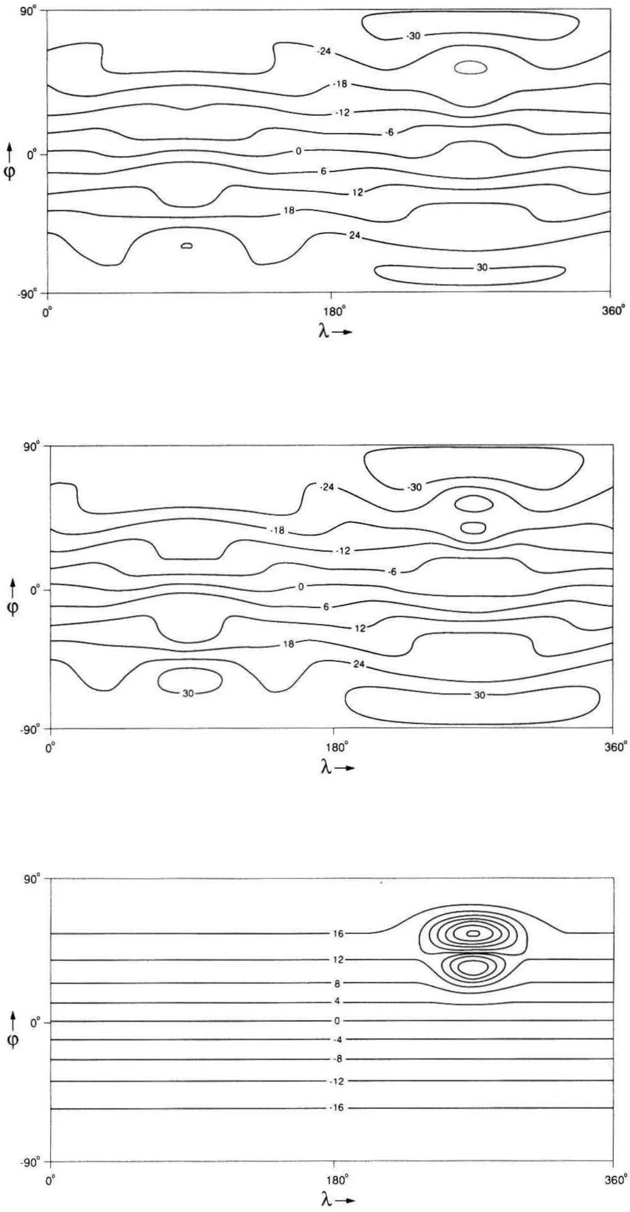


Fig. 1. The stream function for the stationary basic states (a) A Rossby wave with wavenumber $n=8$ in an eastward zonal flow. The contour interval is 0.006 (b) A wavelike modon with wavenumbers $\kappa_o = 7.78$ and $\kappa_i = 12.28$ in an eastward zonal flow. The contour interval is 0.006 (c) A localized modon with wavenumbers $\kappa_o = 10$ and $\kappa_i = 10$ in a westward zonal flow. The contour interval is 0.004.

there is a closed circular streamline at radius 15° from the centre. The modons have two regions separated by a boundary circle, and a piecewise linear $q(\psi)$-relationship. For the wavelike modon (oscillatory behaviour in both regions) there is a negative slope in the $q(\psi)$-diagram for both regions. We have investigated a wavelike modon with wavenumber in the outer region $\kappa_o = 7.78$ and wavenumber in the inner region $\kappa_i = 12.28$. For the localized modon (exponential decaying behaviour in the outer region) there is a positive slope in the $q(\psi)$-diagram in the outer region, and a negative slope in the inner region. We have investigated a localized modon with wavenumber in the outer region $\kappa_o = 10$ and wavenumber in the inner region $\kappa_i = 10$. For both modons the radius is approximately 15°. The three basic states are shown in Fig. 1.

Evolution

The initial basic states were provided on a Gaussian grid. Extended time integrations of the linearized barotropic vorticity equation on a sphere using the spectral method were performed on the newly installed CRAY-C90 at the ECMWF. During the numerical integration, the global perturbation energy $\mathscr{E}$ and the local perturbation relative vorticity ζ_p at a point in the inner region were observed. Two series of spectral runs were made: first, for each triangular truncation from T10 to T43 the time integration was performed over 256 days with timesteps of 15 minutes. Second, for the Rossby wave and wavelike modon a T85 run was made, whereas for the localized modon a truncation run for every other run from T75 to T85 was made. The time integration was performed over 128 days with timesteps of 5 minutes. The initial perturbation consisted of noise with energy 10^{-3} of the basic state. The perturbation field was rescaled to unity when its energy exceeded 100 times its initial energy.

The solution of the eigenvalue problem for a given spectral truncation yields a discrete set of eigenvalues. If the existence of a single eigenvalue with largest real part is assumed, then there is a single fastest growing normal mode. The behaviour of the system after some time in a time integration is dominated by this fastest growing normal mode. From substitution of (6) and (7) in (5) the stream function and relative vorticity are obtained in complex notation

$$\psi'(\lambda, \phi, t) = e^{\nu_r t} \cdot (\xi_r(\lambda, \phi) \cos \nu_i t - \xi_i(\lambda, \phi) \sin \nu_i t) \tag{11}$$

$$\zeta'(\lambda, \phi, t) = e^{\nu_r t} \cdot (p_r(\lambda, \phi) \cos \nu_i t - p_i(\lambda, \phi) \sin \nu_i t) \tag{12}$$

The real part of the eigenvalue of the fastest growing mode ν_r is determined from the evolution of the globally averaged perturbation energy

$$\mathscr{E}(t) = \frac{1}{2} \int (\nabla \psi'(\lambda, \phi, t))^2 \, dS = -\frac{1}{2} \int \psi'(\lambda, \phi, t) \, \zeta'(\lambda, \phi, t) \, dS \tag{13}$$

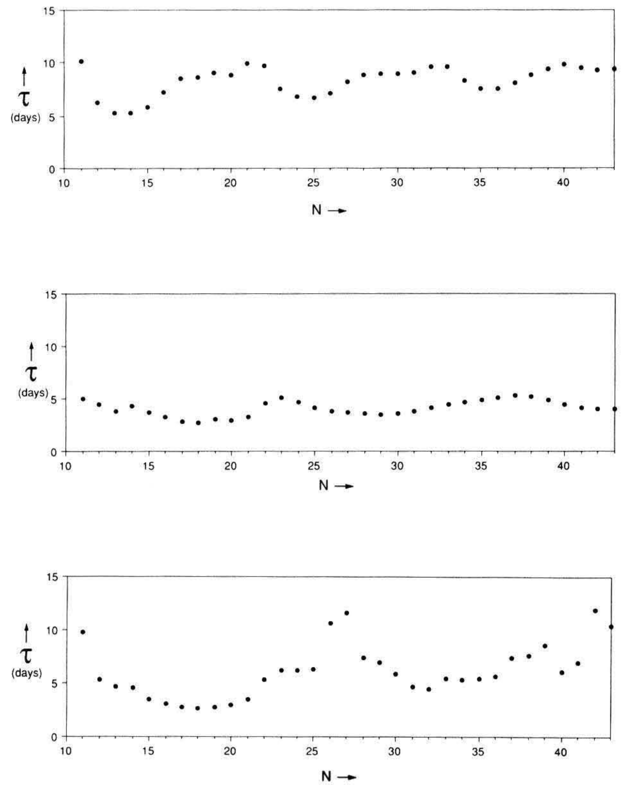


Fig. 2. The lifetime vs. truncation for (a) the Rossby wave (b) the wavelike modon (c) the localized modon.

After substitution of (11) and (12) in (13) we obtain

$$\mathscr{E}(t) = Ae^{2\nu_r t}\,(1 + \varepsilon\cos(2\nu_i t + \eta)) \tag{14}$$

with A, ε and η constants depending on the initial condition. The contribution to the time evolution of $\mathscr{E}$ due to ε usually turns out to be small, and the global energy follows a nearly exponential curve. Then from an exponential fit over a long time series that includes several oscillations the real part of the eigenvalue ν_r and therefore the e-folding time τ can be obtained. The lifetime vs. truncation of the three basic states is shown in Fig. 2.

The imaginary part of the eigenvalue of the fastest growing mode ν_i is determined from the evolution of the perturbation relative vorticity at a particular point p on the sphere from (12)

$$\zeta'_p \equiv \zeta'(\lambda_p, \phi_p, t) = A_p e^{\nu_r t}\cos(\nu_i t + \eta_a(\lambda_p, \phi_p)) \tag{15}$$

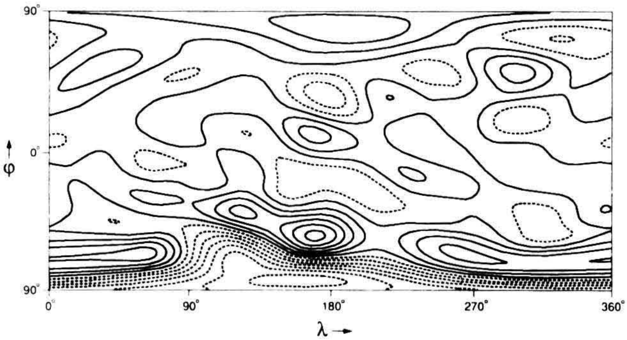


Fig. 3. For the Rossby wave the T38 perturbation stream function.

If the real part of the eigenvalue ν_r obtained above is used, the exponential growth can be divided out from (15) and the result almost follows a sinusoidal curve. From a nonlinear fit over a long time series that includes several oscillations, the imaginary part of the eigenvalue ν_i and the oscillation time of the mode T can be obtained.

For the Rossby wave the T38 results are $\tau = 8.89$ days and $T = 36.8$ days, and the T85 results are $\tau = 8.62$ days and $T = 24.6$ days. For the wavelike modon the T30 results are $\tau = 3.59$ days and $T = 34.3$ days, and the T85 results are $\tau = 4.48$ days and $T = 37.3$ days. For the localized modon the T30 results are $\tau = 5.86$ days and $T = 2.12$ days, and the T85 results are $\tau = 4.01$ days and $T = 4.19$ days.

Normal mode fields

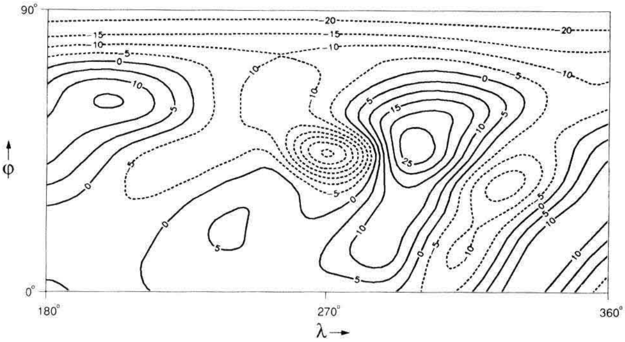
The normal mode fields $\xi_r(\lambda, \phi)$ and $\xi_i(\lambda, \phi)$ are obtained from the stream function fields at two different times t_1 and t_2. For each point from the fields $\psi'_1 = \psi'(\lambda, \phi, t_1)$ and $\psi'_2 = \psi'(\lambda, \phi, t_2)$ the real part and the imaginary part can be obtained from (11)

$$\xi_r(\lambda, \phi) = \frac{1}{\sin \nu_i(t_2 - t_1)} \cdot (e^{-\nu_r t_1} \sin \nu_i t_2 \cdot \psi'_1 - e^{-\nu_r t_2} \sin \nu_i t_1 \cdot \psi'_2)$$

$$\xi_i(\lambda, \phi) = \frac{1}{\sin \nu_i(t_2 - t_1)} \cdot (e^{-\nu_r t_1} \cos \nu_i t_2 \cdot \psi'_1 - e^{-\nu_r t_2} \cos \nu_i t_1 \cdot \psi'_2) \qquad (16)$$

where the real part of the eigenvalue ν_r and the imaginary part of the eigenvalue ν_i are obtained as indicated above. This definition is not unique, since there is an arbitrary global phase factor related to the choice of origin of time. The stream function of the perturbation field at a particular instant in time for the Rossby wave is shown in Fig. 3.

A detail of the stream function for the fastest growing normal mode for the wavelike modon is given in Fig. 4, and for the localized modon in Fig. 5.



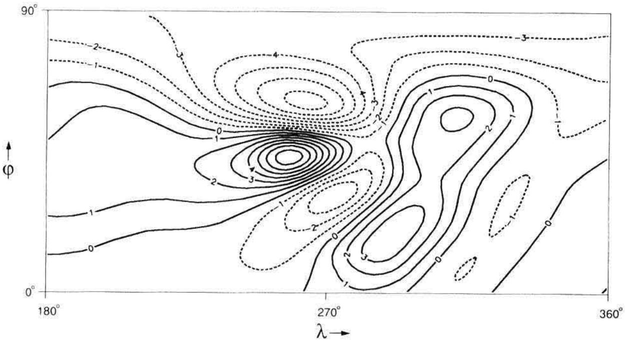
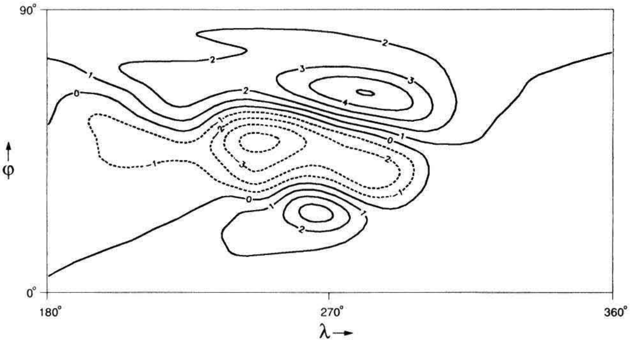


Fig. 4. For the wavelike modon the T30 perturbation stream function (a) the real part with contour interval 0.005 (b) the imaginary part with contour interval 0.01. The modulus of the perturbation relative vorticity field integrated over the sphere is normalized to unity.

The contour plots show that the fastest growing normal mode for the modons are localized in the inner region of the modon basic state, whereas for the Rossby wave the perturbation field is not localized in a particular region. These results agree with the comparison of modons with Rossby waves (Verkley, 1987). They also agree with barotropic instability being generated on gradients of potential vorticity. For a Rossby wave there is no concentrated potential vorticity in a particular region of the sphere, and normal modes in different regions of the sphere compete. A modon is characterized by concentrated potential vorticity in the inner region, so normal modes are likely to develop over the modon: a modon attracts a normal mode.

Since for a localized modon the total field is the sum of the dipole basic state field and the perturbation field, this suggests that in a nonlinear evolution the dipole shifts and tilts. This oscillation is observed if a normal mode field, with its energy normalized to a given fraction of the energy of the basic state field, is added to the basic state field. The behaviour of modons in an equivalent barotropic model on a beta plane can be modelled by a pair of point vortices



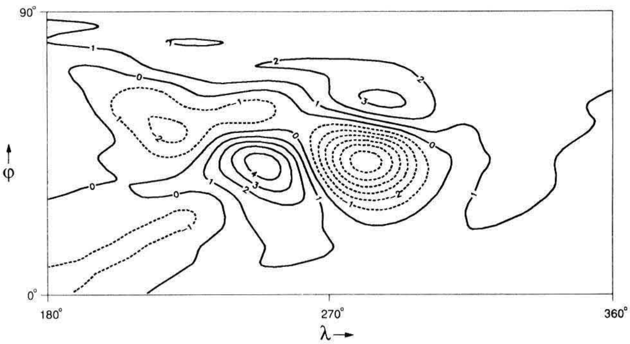


Fig. 5. For the localized modon the T38 perturbation stream function (a) the real part with contour interval 0.01 (b) the imaginary part with contour interval 0.01. The modulus of the perturbation relative vorticity field integrated over the sphere is normalized to unity.

(Aref, 1980; Hobson, 1991; Matsuoka and Nozaki, 1992). From a feedback argument the wobbling of a westward moving localized modon and the tumbling of the eastward wavelike modon in the absence of zonal flow was shown. We have shown that the localized modon is a much more rigid oscillator than the wavelike modon. During the lifetime of the localized modon the dipole wobbles, whereas the wavelike modon develops a tilt and disappears.

Conclusions

By performing numerical time integrations in a spectral model with high truncation, we have shown that both stationary wavelike and localized modon solutions of the barotropic vorticity equation on a sphere are linearly unstable.

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