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On Long-lived Vortices in 2D Viscous Flows and Most Probable States of Inviscid 2D Flows

Abstract

We discuss, in contraposition to the most probable states of quasi-inviscid theories, the status of the vorticity–stream function relations satisfied by the long-lived vortices observed in some numerical simulations of decaying two-dimensional turbulence and in experiments in stratified fluids.

The emergence of large-scale, long-lived vortices in 2D, viscous, incompressible flows has been well established through numerical simulations [1, 2, 3] and laboratory observations [4, 5]. The mechanisms that lead to their emergence have been studied, see, e.g., [6], as well as the important role they play in determining the dynamics and the statistical characteristics of the flow, e.g., [7]. While viscosity is essential for the coalescence of smaller vortices into larger ones, the segregation of opposite-signed vorticity regions is a striking characteristic of the frictionless dynamics, as had been pointed out by Onsager [8], already decades before the above-mentioned investigations and was verified by Joyce and Montgomery [9]. State-of-the-art numerical calculations [10] seem to indicate that some quasi-stationary states of two-dimensional, viscous flows are characterized by a hyperbolic-sine relation between the vorticity field $\omega(x, y, t)$ and the stream function $\psi(x, y, t)$, i.e., suppressing the weak time-dependence,

$$\omega(x, y) = \omega_0 \sinh(\psi(x, y)/\psi_0). \tag{1}$$

Moreover, some quasi-stationary states observed in recent laboratory experiments in stratified fluids [11] are consistent with such an ω–ψ relationship. Taking into account the general relation between vorticity and the two-dimensional stream function of incompressible flows, i.e., $\omega \equiv -\Delta\psi$, this implies that, in these cases, the quasi-stationary stream function satisfies the following differential equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\omega_0 \sinh(\psi/\psi_0), \tag{2}$$

with $\psi = 0$ on the boundary of the domain. It should be noticed that such an ω–ψ relation is not preserved by viscous dissipation, therefore, these states can be quasi-stationary only if the Reynolds number is still large enough so that the evolution is not dominated by viscous effects.

The possibility raised by eq. (2) is interesting for a number of reasons. Firstly, as pointed out in [10], because statistical mechanical studies of inviscid two-dimensional hydrodynamics predict that the most probable equilibrium state is a stationary solution of Euler's equations, i.e., its stream function $\Psi(x, y)$ satisfies $\Delta\Psi = F(\Psi)$, with F either the hyperbolic-sine function as in eq. (2) [9] or some other similar function [12, 13, 14], see below. Secondly, because such a relation could make possible the application of powerful mathematical methods related to soliton equations. Before analysing this second aspect, some comments on the above-mentioned statistical mechanical studies are in place.

Joyce and Montgomery [9] computed the most probable state of a Hamiltonian system consisting of identical positive and negative point vortices and obtained that, in the case of vanishing total vorticity, the stream function of the equilibrium state must satisfy eq. (2), see also [15]. Their calculations can be extended, e.g., by discretizing the continuous vorticity distribution with non-identical point vortices; by doing so, one obtains most probable states that are characterized by ω–ψ relations reminiscent of but different from (2). The dependence of results upon the arbitrary discretization of the field, had already been noticed by Onsager [8].

This observation is in agreement with the more elaborate mean-field theories recently developed by two groups [12, 13, 14], see also [16], which take into account, besides the conservation of kinetic energy, momentum, etc, an infinite number of quantities conserved by the dynamics, i.e.,

$$\frac{d}{dt}\int f(\omega)\, dx\, dy = 0, \tag{3}$$

where $f(\omega)$ is an arbitrary function of the vorticity field. Due to these conservation laws, initial conditions that have, e.g., equal energy, momentum and enstrophy, but different vorticity distributions, will lead to different stationary, most probable distributions; i.e., (2) is only one possibility among many. These theories lead to partition functions $Z(\beta, \psi(x, y))$ of the following form,

$$Z(\beta, \psi(x, y)) = \int \exp(\beta\mu(a) - \beta a \psi(x, y))\, da, \tag{4}$$

where the parameters β (analogous to an inverse temperature) and the function $\mu(a)$ (analogous to a chemical potential) are Lagrange multipliers chosen in such a way that the constraints on the conserved quantities are satisfied. More precisely, the value of the energy determines β and the area (density) on which the vorticity takes a value, call it a, determines the function $\mu(a)$. The integrand in (4) is proportional to the probability of observing the vorticity value a at a position (x, y) with stream-function value $\psi(x, y)$. Consequently, the average vorticity is given by

$$\omega(x, y) = Z^{-1} \int a \exp(\beta\mu(a) - \beta a\psi(x, y))\, da = -\frac{1}{\beta} \frac{\partial}{\partial\psi} \log Z(\beta, \psi). \qquad (5)$$

Identifying the $\psi(x, y)$ in this equation with the average stream function, i.e., using a mean-field approximation, this equation leads to a functional relation between the average vorticity $\omega(x, y)$ and the average stream function $\psi(x, y)$; therefore, these functions correspond to stationary solutions of Euler's equations. Moreover, Robert [13] has shown that the probability distribution is sharply peaked around the most probable state. It should be noticed that these theories do not conserve all the invariants of the inviscid dynamics, for example, the connectivity of isovortical lines is not conserved and, e.g., the enstrophy of the most probable state may be less than that of the initial state [12, 14]. Therefore, I shall use the term 'quasi-inviscid' in order to characterize them.

The following examples should illustrate the many possible most probable states allowed by these quasi-inviscid theories of 2-D flows. The chemical potential $\mu(a)$ is closely related to the initial probability distribution of vorticity [12, 14]. If the initial state corresponds to a random 'turbulent' field, it makes sense to consider Poissonian and Gaussian distributions. It turns out that if the chemical potential $\mu(a)$ is given by $\exp(\beta\mu(a)) = \exp(-|a|/q)$ where q is a positive constant, i.e., a Poisson distribution of the initial vorticity values a, then one obtains

$$\nabla^2\bar{\psi} = 2\beta q^2 \frac{\bar{\psi}}{1 - \bar{\psi}^2} \quad \text{with} \quad \bar{\psi}(x, y) = \beta q\psi(x, y) \quad \text{and} \quad |\beta q\psi(x, y)| < 1. \qquad (6)$$

This functional relation fits the experimental results as satisfactorily as the hyperbolic-sine relation (2) does [11, 17], see Fig. 1; it should be noticed that a cubic polynomial does the job as well [11]. Therefore, it would be interesting to check whether it is capable of fitting also the results of the numerical simulations.

Besides regular solutions, eq. (6) allows also non-regular ones. Circular, non-regular solutions of eq. (6) behave like $\bar{\psi} = 1 - r/\sqrt{2} + O(r^2)$, where r measures the distance from a cusplike singularity; such a singularity carries finite circulation and kinetic energy but has infinite enstrophy.

On the other hand, if the chemical potential is taken to be quadratic in the vorticity, $\beta\mu(a) = -(a/2q)^2$, i.e., a Gaussian distribution of initial vorticity values, then one obtains a linear relation between vorticity and stream function [12],

$$\nabla^2\psi = \beta q^2\psi(x, y). \qquad (7)$$

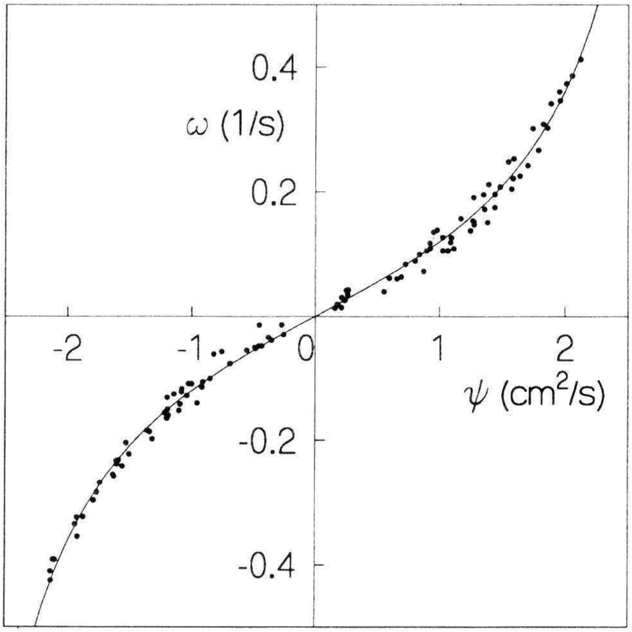

Fig. 1. The vorticity ω, in s^{-1}, as a function of the stream function ψ, in cm^2/s. The dots are experimental results from [11], the line is the best fit obtained using eq. (6) with $\beta = -1.88$ (s/cm)^2 and $q = 0.17$/s. Courtesy of Flór and van Heijst [17].

The ω-ψ relations obtained in the numerical simulations of Morel and Carton, fig. 5c of ref. [18], as well as the hyperbolic-sine relation in (2) can be obtained as follows: Consider the case

$$\exp(\beta\mu(a)) = A^2\delta(a-q_1) + B^2\delta(a) + C^2\delta(a-q_2), \tag{8}$$

with $A^2 + B^2 + C^2 = 1$ and $A^2 q_1 + C^2 q_2 = 0$, i.e., the initial vorticity takes only the values q_1, q_2 and 0 with probabilities A^2, C^2 and B^2 respectively, such that the total vorticity is 0; this agrees with the initial conditions used in these simulations [18]. This leads to

$$\nabla^2\bar{\psi} = -\beta A^2 q_1^2 \frac{\exp(A^2\bar{\psi}/C^2) - \exp(-\bar{\psi})}{1 + A^2(\exp(-\psi) - 1) + C^2(\exp(A^2\psi/C^2) - 1)} \tag{9}$$

with $\bar{\psi}(x, y) \equiv \beta q_1 \psi(x, y)$. Joyce and Montgomery's result, eq. (2) with $\beta q_1 \psi_0 = 1$ and $\omega_0 = q_1 C^2$, follows if one takes identical point vortices, i.e., the limits of $A^2 = C^2 \to 0$ and $q_1 = -q_2 \to \infty$ such that the circulation $q_1 A^2$ remains finite and $\beta \to 0$ such that $q_1\beta$ remains finite. One sees then that it is possible to obtain $\omega(\psi)$ functions that remain finite for all values of ψ, as in eq. (9), or that diverge for $\psi \to \pm\infty$, as in eqs. (2) and (7), or that diverge at a finite value of ψ, as in eq. (6). All the above examples illustrate a point which was already realized by Onsager [8]: in a bounded domain, localized solutions exist with $\beta < 0$, i.e., there are 'negative temperature' states.

The results briefly reviewed in the previous paragraphs, raise a number of interesting questions: What is the relation, if any, between quasi-stationary states generated by viscous flows and the most probable ones predicted by quasi-inviscid theories? Why should viscous dissipation lead into a quasi-stationary state before reaching the final stages of dissipation? Do some quasi-stationary states, out of infinitely many possibilities, actually satisfy the hyperbolic-sine relation (2)? How universal is this relation, i.e., do other initial conditions or boundary conditions lead to different quasi-stationary states? If yes, then what are the other possibilities or 'universal classes'? Does eq. (6) provide a better or worse fit of the same observations or does it correspond to another universal class? These questions still remain open; only some partial answers and conjectures have been advanced: It has been pointed out that the above-mentioned mean-field theories introduce, through coarse graining, some irreversibility and dissipation [12, 14] so that the 'dressed' vorticity distribution which measures the long-time, coarse-grained vorticity field is, in general, different from the 'bare', initial one so that, for example, the enstrophy is effectively dissipated. In fact, Pomeau has argued that such an effective dissipative behaviour should be generic to nonlinear, non-integrable classical fields [19]. Even if this were true, it remains to explain why the predictions of these quasi-inviscid theories should agree with the quasi-stationary states reached through standard viscous dissipation since other types of dissipation, like hyperviscosity or Ekman damping, could and in fact seem to lead to different quasi-stationary states [14]. Robert and Sommeria [20] have proposed a diffusive algorithm that conserves energy and all other constants of the motion and converges into the most probable stationary state; however, there is no reason why this algorithm should reproduce the actual inviscid dynamics, let alone the viscous one. Miller has shown that the most probable states predicted by the quasi-inviscid theories coincide with the maximum-energy states under the constraint of the dressed vorticity distribution [12]. Therefore, this approach can also be viewed as a generalization of the 'selective-decay' approach [27] in which only the second moment of the distribution, i.e., the enstrophy, is taken into account. This property may turn out to be the most relevant one for the possible application of this quasi-inviscid approach to viscous flows.

With respect to universal classes and their relation to initial conditions, it is worthwhile noticing that while turbulent-jet injection leads to a nonlinear ω-ψ relation similar to (2) or to (6), laminar-jet injection leads to a linear relation [11] while the initial conditions used in the numerical simulations of Morel and Carton [18] lead to a relation close to eq. (9). Also, it has been proposed by Farge and Holschneider that the presence of quasi-singularities in the initial conditions of some numerical simulations leads to the appearance of cusplike axisymmetric vortices [21].

In spite of all these uncertainties, the hyperbolic-sine relation (2) may play a role in the characterization of large vortices in decaying 2D turbulence; there fore, it is worthwhile studying some mathematical aspects of this differential equation. This equation, known in the plasma-physics literature as the sinh-Poisson equation, is the elliptic version of the sinh-Gordon equation, i.e., in dimensionless form,

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\Psi=\sinh\Psi. \tag{10}$$

The sinh-Gordon equation, like the sine-Gordon equation, is known to be exactly solvable by the inverse scattering method, to have soliton solutions, etc [22]. Therefore, one may ask whether some of the known nice properties of this integrable equation also hold for its elliptic version, eq. (2). The desirable properties one has in mind are, e.g., 1) exact solutions expressible in terms of known functions and 2) relatively simple algorithms (like Bäcklund transformations [23] or nonlinear superposition formulas) for the generation of new solutions from known ones [1]. A positive answer to these questions was given by Ting, Chen and Lee [24] who showed that, in the case of periodic boundary conditions, the exact solutions can be expressed in terms of Riemann theta functions and presented superposition formulas. Since the numerical simulations of Matthaeus *et al.* [10] were done with periodic boundary conditions, it follows that the quasi-stationary structures found by them should be described by these exact solutions, if they satisfy eq. (2), as it is claimed. In a more extended version of this work [25], it is shown that in the case of axisymmetry, the sinh-Poisson eq. (2) reduces to a particular case of the IIIrd Painlevé transcendent [26] and examples of regular solutions that vanish at infinity are presented as well as a Bäcklund transformation [23] from axisymmetric solutions into dipolar ones.

Besides continuing with the theoretical analysis, there is a clear need for more and better results obtained through numerical simulations and experiments in order to determine, e.g., the relevance of the initial conditions in the selection of quasi-stationary states and the interplay between the inviscid part of the dynamics on one hand and dissipation (and forcing) on the other hand.

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