Hamiltonian Dynamics, Conservation Laws and the Vortex Stability-Instability Problem

Abstract

The concept of a Hamiltonian system is described and illustrated with three different types of models: shallow water equations with or without horizontal inhomogeneities and frontal geostrophic dynamics. Noether's Theorem relates symmetries and integrals of motion, and the latter are then used to obtain sufficient stability conditions -or necessary instability ones- suitable for the study of vortex dynamics.

Hamiltonian dynamics

This subject provides a **geometric** description of certain dynamical systems, thereby rendering some results (like the link between symmetries and conservation laws or the existence of extremal integrals of motion) less mysterious (see for instance Salmon, 1988, and Shepherd, 1990).

We start by imagining the state of the system at time t as a point φ in the state space E. In practice one has a particular set of fields $\varphi^a(\mathbf{x}, t)$ $\mathbf{x} \in \mathscr{D} \subset \mathbb{R}^n$ (*e.g.* depth and velocity components, for the shallow water equations), but an advantage of the Hamiltonian formalism is its manifest covariance under change of variables. An important object are the *functionals* $\mathscr{F}: E \to \mathbb{R}$, *e.g.* $\mathscr{F}[\varphi, t] = \int F(\varphi^a, \nabla\varphi^a, ..; \mathbf{x}, t)$, where "$\int$" represents an integration over the whole domain $\mathscr{D}$ and ∇ is the nabla operator in $\mathbb{R}^n$. Given any functional $\mathscr{F}$, the *covector* $D\mathscr{F}[\varphi]$ represents its gradient at φ (in practice, an array of functional derivatives), defined such that the first variation of $\mathscr{F}$ from φ to $\varphi + \delta\varphi$ is given by the linear expression $\delta\mathscr{F} = D\mathscr{F} \cdot \delta\varphi$, where the dot implies an integration in $\mathscr{D}$ and, probably, its boundary $\partial\mathscr{D}$. More precisely

$$\mathscr{F}[\varphi + \delta\varphi, t] = \mathscr{F}[\varphi, t] + D\mathscr{F}[\varphi, t] \cdot \delta\varphi + D^2\mathscr{F}[\varphi, t] \cdot (\delta\varphi, \delta\varphi) + O(\delta\varphi)^3. \quad (1)$$

A key ingredient of a Hamiltonian system is the Poisson tensor $\mathbb{J}[\varphi]$, used to construct the *vector* $\mathbb{J} \cdot D\mathscr{F}$ from the the covector $D\mathscr{F}$. The *Poisson bracket* of two functionals $\mathscr{F}$ & $\mathscr{G}$ is then defined by

$$\{\mathscr{F}, \mathscr{G}\} := D\mathscr{F} \cdot \mathbb{J} \cdot D\mathscr{G}. \quad (2)$$

(I will discuss the properties of $\mathbb{J}$ below.) The other key ingredient is a particular functional: the Hamiltonian $\mathscr{H}[\varphi, t]$, such that the evolution equations take the form

$$\partial_t \varphi = \mathbb{J}[\varphi] \cdot D\mathscr{H}[\varphi, t]. \tag{3}$$

The total time derivative of an arbitrary functional of state $\mathscr{F}[\varphi, \mathrm{t}]$ is then given by $\partial_t \mathscr{F} + \{\mathscr{F}, \mathscr{H}\}$, i.e. the sum of the local one (keeping φ fixed) plus the contribution due to the flow through state space.

Time is a parameter external to state space. Therefore, the functionals of state $\mathscr{F}[\varphi, t]$ may also be functions of time but the Poisson tensor $\mathbb{J}$ cannot have an explicit dependence in t (although $\mathbf{x}$ and ∇ may appear in its definition); $\{\ ,\ \}$ gives the *structure* of state space.

Let us see a few examples, using in all of them polar coordinates $\mathbf{x} = (r, s)$, more suitable for vortex dynamics, and a uniform Coriolis parameter f. Consider first the classical **shallow water equations**, but allowing for horizontal inhomogeneities (Ripa, 1993b). The dynamical fields are the buoyancy ϑ, the layer depth h, and the radial u and azimuthal v velocity components. The evolution equations are

$$\begin{aligned} &\partial_t \vartheta = -\mathbf{u} \cdot \nabla\vartheta,\ \partial_t h = -\nabla \cdot (h\mathbf{u}), \\ &\vartheta_t \mathbf{u} = -(f+\chi)\,\hat{\mathbf{z}} \times \mathbf{u} - \nabla b + \tfrac{1}{2} h \nabla\vartheta, \end{aligned} \tag{a-d}$$

where $b := \vartheta h + u^2/2 + v^2/2$ is the Bernoulli head, $\chi := (\partial_r(rv) - \partial_s u)/r$ the relative vorticity and $\hat{\mathbf{z}}$ the vertical unit vector. These equations can be derived from the following Hamiltonian and Poisson bracket:

$$\mathscr{H}[\vartheta, h, u, v] := \tfrac{1}{2} \int h(u^2 + v^2 + \vartheta h) \tag{4e}$$

and

$$\{\mathscr{F}, \mathscr{G}\} := \int \left[\frac{1}{h} \nabla\vartheta \cdot \frac{\delta\mathscr{F}}{\delta\mathbf{u}} \frac{\delta\mathscr{G}}{\delta\vartheta} - \frac{\delta\mathscr{F}}{\delta h} \nabla \cdot \frac{\delta\mathscr{G}}{\delta\mathbf{u}} + q \frac{\delta\mathscr{F}}{\delta u} \frac{\delta\mathscr{G}}{\delta v} \right] - (\mathscr{F} \leftrightarrow \mathscr{G}). \tag{4f}$$

Indeed, the first variation of (4e) gives $D\mathscr{H}[\vartheta, h, u, v] = (h^2/2, b, hu, hv)$, which in (4f) and (3) yields the system $(4a-d)$. □

This Hamiltonian corresponds with the total energy and it is conserved because $\partial_t \mathscr{H} = 0$. The potential vorticity $q := (f+\chi)/h$, satisfies

$$\partial_t q + \mathbf{u} \cdot \nabla q = J(h, \vartheta)/2h, \tag{5}$$

where $J(A, B)$ is the Jacobian $r^{-1}(\partial_r A \partial_s B - \partial_r B \partial_s A)$.

The standard shallow water model belongs to the submanifold of E represented by $\vartheta = \text{constant}$ ($=: g$, say), which is consistent with (4a). This equation is then no longer needed and the last term on the right hand side of ($4c$, d) disappears. The evolution equations in the reduced state space can be obtained from the same Hamiltonian (4e) and the Poisson bracket (4f) but without the first term. Notice that in this case, the right hand side of (5) vanishes, *i.e.*, potential vorticity is conserved. The two systems considered so far are the simplest examples of *inhomogeneous* and *homogeneous layers primitive equations* models; *ILPEM* & *HLPEM* for short (Ripa, 1993b).

Finally consider the **frontal geostrophic dynamics** (*FGD*) proposed by Cushman-Roisin (1986). For this model it is assumed that the local accelerations may be neglected in the slow manifold, *i.e.* (4*c*, *d*) may be approximated by $(f+\chi)\,\mathbf{u}=\hat{\mathbf{z}}\times\nabla b$. This balance means that u & v are no longer independent variables but, rather, functionals of h. To second order in ∇h it is indeed found that

$$\mathbf{u}=\hat{\mathbf{z}}\times\nabla\left(\mu h+\frac{\mu^2}{2f}\nabla h\cdot\nabla h\right)-\frac{\mu^2}{f}\nabla^2 h\hat{\mathbf{z}}\times\nabla h, \tag{6}$$

where $\mu:=g/f$. Finally, upon substitution in (4b), and using $\nabla\cdot(B\hat{\mathbf{z}}\times\nabla A)=J(A,B)$, it is obtained

$$\partial_t h=\frac{\mu^2}{f}J(h\nabla^2 h+\tfrac{1}{2}\nabla h\cdot\nabla h,h). \tag{7a}$$

This evolution equation may be obtained from the following Hamiltonian and Poisson bracket:

$$\mathscr{H}[h]:=\frac{1}{2}\mu^2\int h\nabla h\cdot\nabla h \tag{7b}$$

and

$$\{\mathscr{F},\mathscr{G}\}:=-\int(h/f)\,J(\delta\mathscr{F}/\delta h,\delta\mathscr{G}/\delta h). \tag{7c}$$

In fact, (7b) gives $D\mathscr{H}[h]=-\mu^2(\nabla h\cdot\nabla h/2+h\nabla^2 h)$; using this in (7c) and (3), the system (7a) is easily obtained. □

Notice that $\mathscr{H}$ equals the kinetic energy due to the geostrophic velocity field, first term in the right hand side of (6), but that h is changed by the advection and divergence of the ageostrophic part.

So far I have shown that the evolution equations for the three different systems can be written in the Hamiltonian form (3), with suitable chosen $\mathscr{H}$ and $\mathbb{J}$, providing a common geometric description of them. (The form (3) is not enough to guarantee that a dynamical system is Hamiltonian: the Poisson tensor must also have certain general properties which I will point out in a short while.)

Of course there is more to Hamiltonian dynamics than equation (3). For instance, any functional $\mathscr{M}[\varphi,t]$ may be used to define an infinitesimal transformation with parameter s, by

$$\partial_s\varphi=-\mathbb{J}[\varphi]\cdot D\mathscr{M}[\varphi,t]. \tag{8}$$

It is more illustrative to work with the effect of the transformation on an arbitrary functional $\mathscr{F}[\varphi, t]$, rather than φ. Let $R(\Delta s)$ denote the operator that performs a *finite* transformation; from (8) it follows $\delta_s \mathscr{F} = -\{\mathscr{F}, \mathscr{M}\}\, \Delta s$ and therefore $R\mathscr{F} = \mathscr{F} - \{\mathscr{F}, \mathscr{M}\}\, \Delta s + \{\{\mathscr{F}, \mathscr{M}\}, \mathscr{M}\}\, \Delta s^2/2 + O(\Delta s)^3$. (If s is the azimuth, like in the models above, then $\mathscr{M}$ is the angular momentum and R a finite rotation.)

The operator $T(\Delta t)$ for a finite time evolution is similarly found to give $T\mathscr{F} = \mathscr{F} + (\partial_t \mathscr{F} + \{\mathscr{F}, \mathscr{H}\})\, \Delta t + ..\Delta t^2 + O(\Delta t)^3$.

Now, let us combine both operations and ask what is the difference between making the s-transformation and then letting the time run and vice versa. The answer is

$$
\begin{aligned}
(RT - TR)\,\mathscr{F} = & -(\{\mathscr{F}, \partial_t \mathscr{M}\} + \{\{\mathscr{F}, \mathscr{M}\}, \mathscr{H}\} - \{\{\mathscr{F}, \mathscr{H}\}, \mathscr{M}\}) \\
& \times \Delta s\, \Delta t + O(\Delta s, \Delta t)^3. \qquad (9)
\end{aligned}
$$

This is an appropriate place to point out the properties that the Poisson tensor must have in order for (3) to be a Hamiltonian system: antisymmetry and Jacobi identity, *i.e.*

$$
\mathbb{J} \left\{
\begin{aligned}
&\{\mathscr{F}, \mathscr{G}\} = -\{\mathscr{G}, \mathscr{F}\} \\
&\{\mathscr{F}, \{\mathscr{G}, \mathscr{I}\}\} + \{\mathscr{G}, \{\mathscr{I}, \mathscr{F}\}\} + \{\mathscr{I}, \{\mathscr{F}, \mathscr{G}\}\} = 0
\end{aligned}
\right. \qquad (10)
$$

for any pair or triad of *admissible* functionals of state (sufficiently smooth and satisfying appropriate boundary conditions). All the Poisson tensors mentioned above satisfy these conditions (see Ripa, 1993b).

By using the antisymmetry in the right hand side of (9) we get $\{\{\mathscr{F}, \mathscr{M}\}, \mathscr{H}\} - \{\{\mathscr{F}, \mathscr{H}\}, \mathscr{M}\} = \{\mathscr{M}, \{\mathscr{F}, \mathscr{H}\}\} + \{\mathscr{H}, \{\mathscr{M}, \mathscr{F}\}\}$, which by the Jacobi identity is then equal to $-\{\mathscr{F}, \{\mathscr{H}, \mathscr{M}\}\}$. Using this in (9) it is then found

$$
(RT - TR)\,\mathscr{F} = -\{\mathscr{F}, \partial_t \mathscr{M} + \{\mathscr{M}, \mathscr{H}\}\}\, \Delta s\, \Delta t + O(\Delta s, \Delta t)^3. \qquad (11)
$$

A **symmetry** is a statement in the sense that it is equivalent to let the time run and then make the transformation $R(\Delta s)$ than vice versa, $(RT - TR)\,\mathscr{F} \equiv 0$ $\forall \Delta s, \Delta t, \mathscr{F}$, *i.e.* the dynamics is invariant under the transformation R. In such a case, does (11) imply $\partial_t \mathscr{M} + \{\mathscr{M}, \mathscr{H}\} = 0$ (*i.e.* that $\mathscr{M}$ is an integral of motion)?

In order to answer this question we need to solve $(\{\mathscr{F}, \mathscr{X}\} = 0\ \forall \mathscr{F})$ for $\mathscr{X}$, *i.e.* to investigate the general solution of $\mathbb{J} \cdot D\mathscr{X} = 0$. For a canonical system $\mathbb{J}$ is represented by a constant non-singular matrix and therefore $\mathbb{J} \cdot D\mathscr{X} = 0$ implies that $\mathscr{X}$ does not depend on φ, *i.e.*, it is but a number (or a function of time). However, for the three examples discussed above, the operator $\mathbb{J}$ is singular, in the sense that there are non-trivial solutions of $\mathbb{J} \cdot D\mathscr{X} = 0$.

Recall that $\mathbb{J}$ does not depend explicitly on time, so we define a **Casimir** as a non-trivial solution of

$$
\mathbb{J}[\varphi] \cdot D\mathscr{C}[\varphi] = 0, \qquad (12)
$$

in which t does not appear explicitly. (Notice that the Casimirs are a property of the Poisson bracket, *i.e.* of the geometrical structure of E, independent of the Hamiltonian.) The general solution of $\mathbb{J} \cdot D\mathscr{L} = 0$, called a *distinguished functional*, is but a function of the Casimirs and time.

If R represents a symmetry, $RT = TR$, then (11) implies that $\partial_t \mathscr{M} + \{\mathscr{M}, \mathscr{H}\}$ is, at most, equal to a distinguished functional. However redefining $\mathscr{M}$ by subtracting from it the time integral of this functional, the infinitesimal transformation (8) is not altered and the redefined functional *is* conserved. In sum

$$RT - TR = 0 \Rightarrow \partial_t \mathscr{M}' + \{\mathscr{M}', \mathscr{H}\} = 0, \tag{13}$$

where $\mathscr{M}' = \mathscr{M}$ up to the addition of an appropriate distinguished functional. This is Noether's Theorem for singular Hamiltonian systems. Notice that for its derivation one needs to assume neither $\partial_t \mathscr{M} = 0$ nor $\partial_t \mathscr{H} = 0$ (*i.e.* the Hamiltonian may not be conserved).

For the shallow water equations represented by (4) the angular momentum and the Casimirs are given by

$$\mathscr{M} = \int hrv + hfr^2/2, \mathscr{C} = \int h(A(\vartheta) + qB(\vartheta)), \tag{14}$$

where $A(\ )$ & $B(\ )$ are arbitrary. At the hypersurface $\vartheta = g$ the latter is essentially $\int h$ or $\int hq$; however, if the state is restricted to this submanifold ($\vartheta = g$ & $\delta\vartheta = 0$) then there are more Casimirs, namely $\mathscr{C} = \int hF(q)$ with arbitrary $F(\ )$.

For the FGD model, on the other hand, the angular momentum and the Casimirs are given by

$$\mathscr{M} = \int hfr^2/2, \quad \mathscr{C} = \int C(h), \tag{15}$$

where $C(\ )$ is arbitrary. The leading term of $\int hrv$ in (14) vanishes here (because of geostrophy) and $\int hF(q)$ becomes $\int C(h)$ because in the region of validity of *FGD* the potential vorticity is $\tilde{q}f/h$.

Notice that the potential energy $\int gh^2/2$ is but a Casimir, which could be added to the Hamiltonian (7b), without any change in the evolution equations. Indeed, any Casimir -with the appropriate units- can be added to $\mathscr{H}$ or $\mathscr{M}$ without any change in equations (3) and (8).

Lyapunov stability

Let the *basic state* Φ be some exact solution of (3) and let us study the free evolution of a perturbation from it, $\delta\varphi = \varphi - \Phi$. Stability of Φ represents some statement on the inability of $\delta\varphi$ to grow. The weakest definition is that of **normal modes stability**, in which it is assumed that the linearized perturbation has

a coherent evolution for all $\mathbf{x} \in \mathscr{D}$. The strongest concept corresponds to **nonlinear stability**, for which some measure $\|\delta\varphi\|$ is always bounded by a multiple of its initial value (this stability depends on the definition of the metric $\| \; \|$).

Somewhere in between is **Lyapunov stability**, based upon the existence of a functional $\mathscr{I}[\varphi, t]$ which (**i**) is an integral of motion, $\partial_t \mathscr{I} + \{\mathscr{I}, \mathscr{H}\} = 0$, and (**ii**) has a vanishing first variation from the basic state, $D\mathscr{I}[\Phi, t] = 0$. This condition implies, with (1), a quadratic restriction on the evolution of the linearized perturbation, namely

$$D^2\mathscr{I}[\Phi, t] \cdot (\delta\varphi, \delta\varphi) = \text{constant } \forall \delta\varphi. \tag{16}$$

Furthermore, (**iii**) if this second variation is sign definite, then the basic state is Lyapunov (or formally) stable. This concept is stronger than normal modes stability because the perturbation may have arbitrary shape and a time dependence not necessarily exponential. However, unlike the finite dimensional cases, formal stability does not necessarily imply normed stability.

It may seem a formidable task to find integrals of motion that satisfy (**ii**) at some basic state. However, here is where the Hamiltonian formalism comes to our rescue: If the basic state is steady, $\partial_t \Phi = 0$, and/or s-symmetric, $\partial_s \Phi = 0$, then equations (3) and (8) imply that $\mathbb{J}[\Phi] \cdot D\mathscr{H}[\Phi, t] = 0$ and/or $\mathbb{J}[\Phi] \cdot D\mathscr{M}[\Phi, t] = 0$. Except for some pathological cases, (12) then shows that the desired functional $\mathscr{I}$ will be $\mathscr{H}$ and/or $\mathscr{M}$ plus some distinguished functional. In the examples above $\mathscr{H}$ & $\mathscr{M}$ are not explicit functions of t, and therefore

$$\mathscr{I} = \begin{cases} \mathscr{H} + \mathscr{C}_E\text{: pseudoenergy,} & \text{if } \partial_t \Phi = 0, \\ \mathscr{M} + \mathscr{C}_M\text{: pseudomomentum,} & \text{if } \partial_s \Phi = 0. \end{cases} \tag{17}$$

This is Arnol'd's method to find the extremal functional $\mathscr{I}[\varphi]$.

Let me now illustrate the construction of these extremal integrals of motion, in the case of the *FGD*, *i.e.* for $h = H + \delta h$. If the basic state H is a steady solution of (7a), then it must satisfy

$$H\nabla^2 H + \tfrac{1}{2}\nabla H \cdot \nabla H = \Psi(H). \tag{18}$$

Let $\mathscr{I} = \mathscr{H} + \mathscr{C}_E$ (pseudoenergy): $D\mathscr{I}[H] = 0 \; \forall \mathbf{x}$ requires $dC_E(h)/dh = \mu^2 \Psi(h)$. The second variation is then found to be

$$D^2\mathscr{I} \cdot (\delta\varphi, \delta\varphi) = \frac{1}{2}\mu^2 \int H(\nabla \delta h)^2 + (\Psi(H)' - \nabla^2 H)\, \delta h^2. \tag{19}$$

Clearly if $\Psi(H)' > \nabla^2 H \; \forall \mathbf{x}$ then the basic state is formally stable.

Now assume that the basic state is also axial-symmetric, $H = H(r)$; we may use $\mathscr{I} = \mathscr{H} + \mathscr{C}_E - \alpha(\mathscr{M} + \mathscr{C}_M)$, where α is arbitrary. A positive definite second variation is now guaranteed if $\exists\alpha|\mu^2\Psi(H)' > \mu^2\nabla^2 H - \alpha fr/H(r)'\ \forall\mathbf{x}$. Using (18) the stability condition can be written in terms of $H(r)$ and its derivatives:

$$\text{If } \exists\alpha \left| \frac{V_0(r)-\alpha r}{fH(r)'} < 0 \quad \forall r \right. \tag{20}$$

then the basic state is formally stable, where
$V_0(r) := (\mu^2/f)(H\nabla^2 H)' - (\mu^2/fr)(H')^2$. A necessary instability condition is that inequality (20) must be violated $\forall\alpha$. [For a parallel basic state $H = H(y)$, the stability condition is $(U_0(y)-\alpha)/fH(y)' > 0\ \ \forall y$ and some α, where $U_0(y) := (\mu^2/f)(HH'')'$.]

If one uses the second variation of the pseudomomentum alone, then only the term proportional to α is obtained: a sufficient stability condition is that $H(r)$ be monotonous. [The same is true for $H(y)$ in the parallel case.]

The stability conditions for the shallow water equations with a homogeneous layer are two (Ripa, 1991): $(V(r)-\alpha r)/Q(r)' > 0$ & $(V(r)-\alpha r)^2 < gH(r)$, whereas the quasi-geostrophic model (Ripa, 1992*a*, 1993*a*) and the *FGD* one (for which $Q' \sim fH'/H^2$) have only the first one. This is not surprising, because violation of the second condition, by some unstable flow, results in growing perturbations which are Poincaré-like, and these modes are absent in models of the slow manifold. Notice, however, that V_0 in (20) is *not* equal to the azimuthal velocity V.

Conclusions

Hamiltonian dynamics provides a unified description of diverse models of interest in geo-hydrodynamics, represented by evolution equations in the form (3) and infinitesimal transformations of the form (8); equations (10) & (12) describe the geometry of state space. This representation is manifestly covariant under a change of state variables, in the same sense that equation (4), written in vectorial notation, is coordinate independent.

Given a steady basic state, it is possible to construct a conserved pseudoenergy, $\mathcal{I}_E = \mathcal{H} + \mathcal{C}_E$, quadratic to lowest order in the deviation $\delta\varphi$ from this state. A conserved pseudomomentum, $\mathcal{I}_M = \mathcal{M} + \mathcal{C}_M$, is similarly constructed given a symmetric basic state. This might seem rather "miraculous": $D\mathcal{I}_E = 0$ for the *ILPEM* represents the annihilation of four fields (at every point) by means of the choice of two functions of one variable. However, this is no miracle but a result from Hamiltonian dynamics.

Consider the problem linearized in $\delta\varphi$: The original Hamiltonian is not appropriate for this problem, in the sense that the first variation of (3) gives $\partial_t\delta\varphi = \mathbb{J}\cdot\delta D\mathcal{H} + \delta\mathbb{J}\cdot D\mathcal{H}$ and the last term is non-Hamiltonian. However, because of (12) $\mathcal{I}_E$ may be used instead of $\mathcal{H}$ in (3), and then it is $\partial_t\delta\varphi = \mathbb{J}[\Phi]\cdot\delta D\mathcal{I}_E$ because $D\mathcal{I}_E = 0$ by construction. Furthermore $\delta D\mathcal{I}_E \equiv D\mathcal{H}_C$, where $\mathcal{H}_C := D^2\mathcal{I}_E[\Phi]\cdot(\delta\varphi, \delta\varphi)$ is an appropriate Hamiltonian for the linearized problem. Similarly, $\mathcal{M}_C := D^2\mathcal{I}_M[\Phi]\cdot(\delta\varphi, \delta\varphi)$ is a momentum. Finally $\mathcal{H}_C - \alpha\mathcal{M}_C$ is the Hamiltonian in a frame rotating with speed α relative to the f-plane.

The link between symmetries and conservation laws given by Noether's Theorem was found useful in the search for extremal integrals of motion $\mathcal{I}$, but that relationship also represents a formidable limitation on the class of Arnol'd stable states: $D^2\mathcal{I}[\Phi]\cdot(\delta\varphi, \delta\varphi) > 0\ \forall\delta\varphi$ implies that the basic state Φ must have the same symmetries of whole system (Andrew's Theorem).

Moreover, this method for the derivation of sufficient stability conditions has not had much success in models with more physical breadth, *e.g.* the ILPEM have no stability conditions at all (Ripa, 1993b). However, the condition $D^2\mathcal{I}[\Phi]\cdot(\delta\varphi, \delta\varphi) = 0$ for the unstable manifold may be used to characterize the types of instability modes (Ripa, 1992b). On the other hand, the limitations of Arnol'd's method may be overcomed by restricting the class of $\delta\varphi$ in $D^2\mathcal{I}[\Phi]\cdot(\delta\varphi, \delta\varphi)$ (*e.g.* see Kloosterziel & Carnevale 1992).

The most important limitation in using the Hamiltonian formalism is the lack of dissipative processes, although some progress has been made in that sense (*e.g.* Kaufman 1984 or Morrison 1986).

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