Random differential equations and applications

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Abstract

We introduce some basic methods and results in the field of random differential equations. These methods and results center on the concepts of exponential dichotomy, Lyapounov type number, and rotation number. They are applied to two problems in random bifurcation theory.

1 Introduction

The purpose of this paper is to discuss some basic methods and results in the field of "random differential equations" and to delineate in some detail an application of those methods to bifurcation theory. Despite the use of the word "random", and despite a certain formal similarity to the theory of stochastic differential equations, the techniques we introduce are only partly probabilistic in nature, and in fact we will not consider stochastic differential equations. This being the case, it seems natural to begin the paper by delimiting in broad outline the problems one *does* discuss in the field of random differential equations and the basic techniques used to deal with them. We will indicate several areas in which the methods of this field have found application. Then we will turn to the particular case of bifurcation theory and consider a bifurcation scenario intermediate between those considered in the well-developed fields of smooth quasi-periodic and stochastic bifurcation theory.

This paper is a revised version of a talk given at the Colloquium entitled "Dynamical Systems and their Applications in Science", sponsored by the Royal Netherlands Academy of Arts and Sciences and held in Amsterdam from 26–28 January 1995. The author wishes to thank the Academy and the organizers, Prof. S. Verduyn-Lunel and Prof. S. van Strien for their invitation to speak and for their hospitality during the Colloquium.

As promised, we begin by discussing what we mean by the term "random differential equation". These are non-autonomous linear or non-linear differential equations, viewed from a direction which permits the use of ideas of topological dynamics and ergodic theory in their study. As we will see, our point of view encompasses a very wide variety of time-dependent equations. The time dependence may be periodic, or "deterministically" chaotic, or "indeterministically" chaotic so long as it is bounded. Though we do not consider stochastic differential equations as these are usually defined, the methods we discuss apply to all non-autonomous equations satisfying a boundedness condition with respect to the time variable.

Our starting point is, then, the non-autonomous differential equation

$$x' = \widehat{f}(t,x) \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{1.1}$$

where the t-dependence is defined by a *flow* $\{T_t \mid t \in \mathbb{R}\}$ on a compact metric space Y. That is, $\{T_t\}$ is a one-parameter group of homeomorphisms of Y [52]. This amounts to viewing (1.1) as just one of a family of equations

$$x' = f(T_t(y), x) \qquad y \in Y,\ x \in \mathbb{R}^n \tag{1.1$_y$}$$

where now $f : Y \times \mathbb{R}^n \to \mathbb{R}^n$ is a jointly continuous function which is at least Lipschitz continuous in x, so that standard theorems about existence, uniqueness, and continuity of solutions with respect to y are valid. Frequently (but not always), Y will support a probability measure μ which is *ergodic* with respect to the flow $\{T_t \mid t \in \mathbb{R}\}$. That is to say

(i) μ is *invariant*: $\mu(T_t(B)) = \mu(B)$ for each $t \in \mathbb{R}$ and each Borel set $B \subset Y$;

(ii) μ is *indecomposable*: if $B \subset Y$ is a Borel set such that $T_t(B) = B$ for all $t \in \mathbb{R}$, then $\mu(B) = 0$ or 1.

The presence of a topological structure on Y tends to distinguish our approach to "random" differential equations from that of the Bremen school (see, e.g. [1] for a review). Here the emphasis is on the measurable structure corresponding to an ergodic measure μ. Stochastic differential equations can be studied in this framework and indeed these have been considered in detail by L. Arnold, F. Colonius, H. Crauel, W. Kliemann, and co-workers. In this article we will make considerable use of the compact metric structure on Y. It is this structure that will allow us to apply tools of topological dynamics.

It is easiest to give examples of random ordinary differential equations when f is linear in x, and this is what we now do

Example 1 Let

$$x' = a(t)x \qquad x \in \mathbb{R}^n \tag{1.2}$$

be a linear differential equation with bounded measurable coefficient matrix $a(\cdot)$. We "randomize" equation (1.2) in the following way. Let $L^\infty(\mathbb{R}, M_n)$ be the set of bounded measurable functions in the algebra M_n of $n \times n$ real matrices. We introduce the weak-* topology in this space: $\widehat{a}_n \to \widehat{a}$ if and only if

$$\int_{-\infty}^{\infty} \widehat{a}_n(t)\varphi(t)\,dt \longrightarrow \int_{-\infty}^{\infty} \widehat{a}(t)\varphi(t)\,dt$$

for every function $\varphi \in L^1(\mathbb{R})$. Then closed norm-bounded subsets of $L^\infty(\mathbb{R}, M_n)$ are compact. We define a flow in $L^\infty(\mathbb{R}, M_n)$ by translation:

$$(T_t\widehat{a})(s) = \widehat{a}(t+s) \qquad \widehat{a} \in L^\infty(\mathbb{R}, M_n).$$

This is the so-called *Bebutov* flow [38]; see also [36]. Returning to equation (1.2), define

$$Y = \operatorname{cls}\{T_t(a) \mid t \in \mathbb{R}\} \subset L^\infty(\mathbb{R}, M_n).$$

Then Y is compact, the set $\{T_t \mid t \in \mathbb{R}\}$ defines a flow on Y, and equation (1.2) is one of the family of equations

$$x' = y(t)x \qquad y \in Y. \tag{1.2$_y$}$$

We have *randomized* equation (1.2). When convenient, we can fix an ergodic measure μ on Y and discuss properties of equations $(1.2)_y$ which are valid for μ-a.a. y (but not necessarily for the original equation (1.2)). This not infrequently leads to important insights which are not at all obvious if attention is restricted to equation (1.2). A basic example is the Oseledec theory [39] which will be discussed later. On the other hand the existence of an ergodic measure on Y (there is always at least one; see [38]) is quite irrelevant for the discussion of some questions which are posed naturally in terms of random differential equations. For example, stability and smoothness problems arising in the theory of exponential dichotomy are often solved without use of ergodic theory.

When convenient, one can write equation $(1.2)_y$ in the form

$$x' = A(T_t(y))x \qquad x \in \mathbb{R}^n \tag*{$(1.2)'_y$}$$

where $A : Y \to M_n$ is a bounded Borel function. For example, define

$$A(y) = \lim_{n\to\infty} n \int_0^{1/n} y(s)\, ds \quad (y \in Y);$$

then for each $y \in Y$, $A(T_t(y))$ is defined and equals $y(t)$ Lebesgue-a.e. Sometimes it is useful to have a continuous function $A : Y \to M_n$ in $(1.2)'_y$; this is possible if (and only if) the original function $a(\cdot)$ is uniformly continuous on $\mathbb{R}$.

As a special case, suppose a is periodic with period 1: $a(t+1) = a(t)$. Then the above construction produces a circle $Y \subset L^\infty(\mathbb{R}, M_n)$. The flow $\{T_t \mid t \in \mathbb{R}\}$ is equivalent to translation on the standard unit circle: $T_t(e^{2\pi i\theta}) = e^{2\pi i(\theta+t)}$ $(0 \le \theta < 1)$. There is exactly one ergodic measure on Y, which corresponds to normalized Lebesgue measure on the circle $\{0 \le \theta < 1\}$.

Example 2 Suppose $a : \mathbb{R} \to M_n$ is quasi-periodic with k frequencies $\gamma_1, \ldots, \gamma_k$. That is, $a(t)$ is a uniform limit of trigonometric polynomials

$$\sum c_{n_1\ldots n_k} e^{2\pi i(n_1\gamma_1 t+\cdots+n_k\gamma_k t)}.$$

In this case the construction of Example 1 produces a k- torus Y (assuming the frequencies $\gamma_1, \ldots, \gamma_k$ are rationally independent), and the translation flow on Y is equivalent to a Kronecker twist flow:

$$T_t(\theta_1, \ldots, \theta_k) = (\theta_1 + \gamma_1 t, \ldots, \theta_k + \gamma_k t) \bmod \mathbb{Z}^k.$$

There is a unique invariant measure μ on Y, which corresponds to the normalized Lebesgue measure $d\theta_1 \wedge \ldots \wedge d\theta_k$ on T^k.

Example 3 We refer to the book of Doob [15] for the definitions of the terms from probability theory used below. Let (Ω, ν) be a probability space, and let $\{Z_t \mid t \in \mathbb{R}\}$ be a stationary ergodic, stochastically continuous family of differential equations

$$x' = Z_t(\omega)x \qquad (\omega \in \Omega). \tag*{$(1.3)_\omega$}$$

Let X be the uncountable product $X = \prod_{t\in\mathbb{R}} \Omega$, and let ν_∞ be the probability measure defined on the σ-algebra $\mathcal{B}$ generated by finite products $A = \{(\omega_t)_{t\in\mathbb{R}} \mid \omega_{t_1} \in B_{t_1}, \dots, \omega_{t_k} \in B_{t_k}\} \subset X$ (where $B_{t_1}, \dots, B_{t_k} \subset \Omega$ are ν-measurable) by the formula $\nu_\infty(A) = \nu(B_{t_1}) \cdots \nu(B_{t_k})$. Define

$$i : X \longrightarrow L^\infty(\mathbb{R}, M_n); \qquad i((\omega_t)) = Z_t(\omega_0)$$

where ω_0 means the zero-th "coordinate" of $(\omega_t) \in X$. Let $Y = \operatorname{cls} i(X)$ in the weak-* topology. Then it can be shown that i is a Borel map and that the image measure $i(\nu_\infty) = \mu$ is an ergodic measure on Y.

We take the point of view that the family of differential equation $(1.3)_\omega$ is equivalent to the random differential equation

$$x' = y(t)x \qquad (y \in Y)$$

with $Y = \operatorname{cls} i(X)$. Thus all methods we develop for random differential equations can be applied to equations $(1.3)_\omega$.

At this point it is instructive to observe that there is natural progression in the collection of all random ODEs as regards the "degree of randomness" of the triple $(Y, \{T_t\}, \mu)$. A periodic differential equation exhibits no randomness. It is rather surprising that, even though a quasi-periodic flow (Example 2) exhibits very strong recurrence properties, solutions of a random ODE with quasi-periodic flow $(Y, \{T_t\})$ can exhibit quite irregular behavior. This is evidenced by results concerning the Lyapounov exponents of such equations; see especially the examples of Millionščikov [37] and Vinograd [51]. More recently, it has been shown that the quasi-periodic Schrödinger operator can exhibit a "substantial amount" of point spectrum; see [18].

In any case, one can imagine that the ergodic flow on Y may satisfy mixing conditions, have positive entropy, etc. In particular the entire range of possibilities of "deterministic chaos" may be present in the flow on Y. It is to be expected that the randomness of the flow on Y will make itself felt in the behaviour of the solutions of equations $(1.1)_y$.

At this point one may object that the concept of random differential equation is too general. One of the lessons of the last twenty-five years is that potent tools are available for the study of all such equations, the application of which leads not infrequently to useful insights. These tools are (1) the concept of exponential dichotomy; (2) Lyapounov type numbers adapted to the random frameworks; (3) rotation number. We will illustrate all three of these concepts in our treatment of random bifurcation.

Some fields in which one or more of these concepts have been fruitfully applied in recent years are the following.

(1) The random Schrödinger operator. Textbooks are now available on this subject [9, 16]. It is interesting to compare their contents with the discussion in the early "reviews" ([26, 50]). It is clear that many interesting problems in this field need further study, for example the Cantor spectrum problem for quasi-periodic operators and the Schrödinger inverse problem together with its relation to the Korteweg-de Vries equation.

(2) The study of transversal homoclinic orbits and the numerical study of chaotic systems. These fields have benefited from the use of exponential dichotomy as a tool. Palmer [40, 41] first related exponential dichotomy to the existence of transversal homoclinic orbits. For further developments see, e.g., [2, 3, 42]. The Contemporary Mathematics volume [33] is devoted to chaotic numerics and contains several papers which use Palmer's exponential dichotomy approach to orbit shadowing.

(3) Control theory of non-autonomous systems. The present author and M. Nerurkar have discussed the relation between local and global controllability for linear systems using Lyapounov exponents [31]. Exponential dichotomy is very useful in studying the random linear stabilization problem [29]. We also wish to mention the papers of Bougerol [4, 5] who discusses the random Kalman filter using Lyapounov exponents.

(4) Random orthogonal polynomials have been studied systematically by J. Geronimo and his co-authors [20, 21]. In particular, an inverse problem for such polynomials has been formulated and solved by extending a basic result of Kotani from the theory of the random Schrödinger operator [34] and using ideas of algebraic curve theory [22].

(5) Random bifurcation theory. In Section 3 below we will discuss the random saddle node bifurcation (see also [1, 12]). In Section 4 a bifurcation scenario worked out by the author and Y.F. Yi [32] will be discussed.

2 Basic concepts

In this section we consider some basic definitions and facts having to do with exponential dichotomies, Lyapounov exponents, and rotation numbers.

Let

$$x' = A(T_t(y))x \qquad x \in \mathbb{R}^n \tag*{$(2.1)_y$}$$

be a random family of linear equations where $\{T_t \mid t \in \mathbb{R}\}$ is a flow on a compact metric space Y. We will assume for convenience that $A : Y \to M_n$ is continuous though, as discussed in § 1, it would suffice to assume that Y is a weak-* compact, translation invariant subset of $L^\infty(\mathbb{R}, M_n)$ (or more generally of $L^p_{\text{loc}}(\mathbb{R}, M_n)$ for $p \geq 1$).

Definition 2.2 Equations $(2.1)_y$ are said to have an *exponential dichotomy* (ED) if there are constants $C > 0$, $\gamma > 0$ and a continuous family $\{P_y \mid y \in Y\}$ of projections $P_y : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} \|\Phi_y(t)P_y\Phi_y(s)^{-1}\| &\leq Ke^{-\gamma(t-s)} && (t \geq s) \\ \|\Phi_y(t)(I - P_y)\Phi_y(s)^{-1}\| &\leq Ke^{+\gamma(t-s)} && (t \leq s). \end{aligned}$$

Here $\Phi_y(t)$ is the fundamental matrix solution of $(2.1)_y$ (i.e. it is an $n \times n$ matrix function which satisfies the differential equation and the initial condition $\Phi_y(0) = I$).

A basic fact concerning the existence of an exponential dichotomy is due to Sacker-Sell [45, 46] and Selgrade [48]. First recall that the flow on Y is called *chain recurrent* if given $y \in Y$, $T > 0$, and $\epsilon > 0$, there is a finite sequence $y = y_1, y_2, \dots, y_N = y$ of points in Y and a corresponding sequence $t_1 > T, \dots, t_{N-1} > T$ such that

$$\text{distance}\,(y_{i+1}, T_{t_i}(y_i)) < \epsilon \qquad 1 \le i < N.$$

Theorem 2.3 *Suppose that the flow on Y is chain recurrent. Then equations $(2.1)_y$ have an ED if and only if, for each $y \in Y$ the only solution $x(t)$ of $(2.1)_y$ which is bounded on all of $\mathbb{R}$ is the zero solution.*

The ED property is extremely useful because of the remarkable stability properties and smoothness properties of the projections P_y. Basic stability results are due to Coppel [14] and Sacker-Sell [46], while Palmer [40], Yi [53] and others have proved smoothness results.

We next give a brief discussion of Lyapounov exponents. Fix $y \in Y$; the *Lyapounov exponent* of a non-zero solution $x(t)$ of equation $(2.1)_y$ is

$$\beta(x) = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|. \tag{2.4}$$

(If the limit does not exists, one replaces lim by lim sup in (2.4)). Also the *maximal* Lyapounov exponent of equation $(2.1)_y$ is

$$\beta_y = \lim_{t\to\infty} \frac{1}{t} \ln \|\Phi_y(t)\|. \tag{2.5}$$

The limits in (2.4) and (2.5) need not exist; however in the random context one has the fundamental theorem of Oseledec [39] which has been reproved several times (e.g. [1]) and which has a non-linear version developed by Ruelle [43]. To state the linear version, we introduce the *skew-product* flow defined by equations $(2.1)_y$. This is the flow $\{\widehat{T}_t \mid t \in \mathbb{R}\}$ defined on the product space $Y \times \mathbb{R}^n$ in the following way:

$$\widehat{T}_t(y, x) = (T_t(y), \Phi_y(t)x).$$

The reason for the term "skew-product" is that the y-part of the flow does not depend on x.

Theorem 2.6 *Let μ be an ergodic measure on Y. There is a set $Y_0 \subset Y$ of full μ-measure such that, if $y \in Y_0$, then there are $k \le n$ Lyapounov exponents $\beta_1, \dots, \beta_k$ of equation $(2.1)_y$. That is, for each non-zero solution $x(\cdot)$ of equation $(2.1)_y$, the limit in (2.4) exists and is among $\{\beta_1, \dots, \beta_k\}$. The set $\{\beta_1, \dots, \beta_k\}$ is independent of*

$y \in Y_0$. Furthermore, there are measurable subbundles $W_1, \ldots, W_k \subset Y_0 \times \mathbb{R}^n$ which are invariant with respect to the flow $\{\widehat{T}_t \mid t \in \mathbb{R}\}$ and which satisfy

$$Y_0 \times \mathbb{R}^n = W_1 \oplus \cdots \oplus W_k;$$

the bundle W_i has the following description:

$$W_i = \{(y, x) \in Y_0 \times \mathbb{R}^n \mid \lim_{t \to \pm\infty} \frac{1}{t} \ln \|\Phi_y(t)x\| = \beta_i\}.$$

Finally, $\beta_y = \max\{\beta_1, \ldots, \beta_k\}$ for μ-a.a. $y \in Y$.

This theorem states that there is a "measurable decomposition" of $Y \times \mathbb{R}^n$ into invariant measurable subbundles, where the subbundles are defined by the μ-Lyapounov exponents of the random differential equation $(2.1)_y$. Of course the measurable decomposition can be viewed as a random analogue of the decomposition of $\mathbb{R}^n$ into the generalized eigenspaces of a constant matrix A, because the generalized eigenspaces of A are invariant under the action of the fundamental matrix solution $\Phi(t) = e^{At}$ of the constant-coefficient system $x' = Ax$.

Next, we discuss the relation between the Sacker-Sell theory of exponential dichotomy and the Oseledec theory. Define the *dynamical spectrum* Σ of the random differential equation $(2.1)_y$ as follows: $\Sigma = \{\lambda \in \mathbb{R} \mid$ the translated equations $x' = [-\lambda I + A(T_t(y))]x$ do *not* admit an exponential dichotomy$\}$. Then it is proved in [46] that Σ is a union of finitely many closed intervals $[a_1, b_1] \cup \ldots \cup [a_r, b_r]$ where $a_1 \leq b_1 < a_2 \leq b_2 \leq \ldots < a_r \leq b_r$ and $r \leq k$. There are *continuous* subbundles $\widehat{W}_1, \ldots, \widehat{W}_r \subset Y \times \mathbb{R}^n$ such that

(i) $\widehat{W}_1 \oplus \cdots \oplus \widehat{W}_r = Y \times \mathbb{R}^n$;

(ii) if $(y, x) \in \widehat{W}_i$, then $\overline{\lim_{t \to \pm\infty}}, \underline{\lim_{t \to \pm\infty}} \frac{1}{t} \ln \|\Phi_y(t)x\| \in [a_i, b_i]$.

Furthermore, each continuous bundle $\widehat{W}_i$ is a direct (measurable) sum of Oseledec bundles: $\widehat{W}_i = W_{j_1} \oplus \cdots \oplus W_{j_i}$. The endpoints $e \in \{a_1, b_1, \ldots, a_r, b_r\}$ of the spectral intervals are distinguished in the sense that, for each such number e, there is an ergodic measure $\mu = \mu_e$ on Y with respect to which e is an almost everywhere Lyapounov exponent in the sense of the Oseledec theory.

Suppose now that the linear random differential equation $(2.1)_y$ is 2-dimensional, i.e., $x \in \mathbb{R}^2$. Let $x(t)$ be a non-zero solution of $(2.1)_y$, and let $\theta(t)$ be the polar angle of $x(t)$ in the $x = (x_1, x_2)$-plane. Of course we suppose that $\theta(t)$ is determined in a continuous way. We define the *rotation number*

$$\alpha = \lim_{t \to \infty} \frac{\theta(t)}{t}. \tag{2.6}$$

The right hand side of (2.6) is clearly independent of the initial value $\theta(0)$. The limit need not exist for every $y \in Y$ (though it does if $(Y, \{T_t\})$ admits exactly one ergodic measure). As in the case of the Lyapounov exponent, if μ is an ergodic measure on Y, then the limit in (2.6) exists for μ-a.a. y and is constant, independent of y and of the initial value $\theta(0)$. Proofs of these statements are mostly in [28], though the results stated there are given for the almost periodic Schrödinger operator. The proofs in [28] are generalized to the case we are considering in [23].

The rotation number is of fundamental importance in the theory of the one-dimensional random Schrödinger operator, and the paper [28] marks the foundation of the systematic development of that theory in the 1980s. The utility of the rotation number derives from its strong continuity properties with respect to parameters. This feature will be illustrated in our discussion of bifurcation theory. There is a higher-dimensional version of the rotation number, defined for example if the function $A(\cdot)$ takes values in the Lie algebra $sp(n, \mathbb{R})$ of infinitesimally symplectic matrices, or more generally in $u(p, q)$. This quantity is discussed in [27, 30] and an application to the random feedback stabilization problem of control theory is given in [29]. We will not discuss the higher-dimensional rotation number here because its definition would take us too far afield and because we will not use it in the sequel.

3 Random bifurcation theory: the random saddle node

We consider one of the simplest random bifurcation problems which, however, still has instructive features. See [6, 12]. The random saddle node is modelled by the random differential equation

$$x' + x^2 = -\lambda + q(T_t(y)) \tag{3.1$_y$}$$

where $(Y, \{T_t\})$ is a compact metric flow and $q : Y \to \mathbb{R}$ is a continuous function. If $q = 0$, then one checks directly that $x^{\pm} = \pm\sqrt{-\lambda}$ determines, for each $\lambda < 0$, a pair of fixed points, one of which is attracting and one of which is repelling. On the other hand, if $\lambda > 0$, then all solutions $x(t)$ of $x' + x^2 = -\lambda$ tend to $-\infty$ in finite time.

If q is non-zero, the situation is similar but there are some interesting possibilities that merit mention. Let us begin by changing variables, writing $\varphi = \frac{x'}{x}$. Then the equation for φ is simply the random one-dimensional Schrödinger equation where λ plays the role of an eigenvalue parameter:

$$-\varphi'' + q(T_t(y))\varphi = \lambda\varphi. \tag{3.2$_y$}$$

Since q is bounded, the operator $L_y = -\frac{d^2}{dt^2} + q(T_t(y))$ is self-adjoint and bounded from below in $L^2(\mathbb{R})$ for each $y \in Y$.

We now make use of some of the most basic facts from the theory of the random Schrödinger operator (see [28, 50, 9, 16]). Fix an ergodic measure μ on Y and suppose for convenience that the "topological support" of μ is all of Y (that is, $\mu(V) > 0$ for every open subset $V \subset Y$). Then the spectrum $\Sigma \subset \mathbb{R}$ of the operator L_y is constant (as a closed subset of $\mathbb{R}$) for μ-a.a $y \in Y$. Let λ_0 be the left endpoint of Σ. Then, for

$\lambda < \lambda_0$, the associated differential equation

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(T_t(y)) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \tag{$3.3)_y$}$$

has an exponential dichotomy. Using the fact that the coefficient matrix in $(3.3)_y$ has trace zero, this implies that, in the $\varphi - \varphi'$ space, there is an expanding direction and a contracting direction, and these directions vary continuously with y. More precisely, one has

$$Y \times \mathbb{R}^2 = W^+ \oplus W^-$$

where $W^\pm$ are the continuous invariant subbundles discussed in § 2: each is one-dimensional.

Define now

$$x_y^+ = \frac{\varphi'_+(0)}{\varphi_+(0)}, \qquad x_y^- = \frac{\varphi'_-(0)}{\varphi_-(0)}$$

where $\varphi_+(t)$ ($\varphi_-(t)$) is a solution of $(3.3)_y$ in the expanding (contracting) direction. It can be shown that $x_y^\pm$ are finite for all $y \in Y$, or equivalently that $\varphi_\pm(t) \neq 0$ for all $t \in \mathbb{R}$, for each $y \in Y$.

A moment's reflection shows that the sections $\{(y, x_y^\pm) \mid y \in Y\} \subset Y \times \mathbb{R}$ are analogues of the attracting-repelling fixed points arising for $\lambda < 0$ when $q = 0$. These sections support ergodic measures $\mu^\pm$ (the natural lifts of μ under the projection $\pi : Y \times \mathbb{R} \to Y$). Thus one can also speak of attracting and repelling invariant measures in $Y \times \mathbb{R}$.

On the other hand, if $\lambda > \lambda_0$, then the rotation number $\alpha = \alpha(\lambda)$ of equations $(3.3)_y$ is strictly positive ([28, 23]). This implies (see § 2) that, for μ-a.a y, all non-zero solutions $\begin{pmatrix} \varphi(t) \\ \varphi'(t) \end{pmatrix}$ of $(3.3)_y$ rotate around the origin in $\varphi - \varphi'$ space infinitely often as $t \to \infty$. This means that all solutions of the x-equation $(3.1)_y$ blow up in finite time for μ-a.a. y.

These features have of course direct analogues when $q = 0$. When $\lambda = \lambda_0$, however, an interesting possibility arises which has no analogue in the case $q = 0$ (nor in the case when q is periodic). Namely, at $\lambda = \lambda_0$, the two ergodic measures μ^+ and μ^- need not collapse together to form one measure (this is what does happen if $q = 0$ or if q is periodic), but rather they may remain distinct. It so happens that they remain distinct if and only if the maximal Lyapounov exponent $\beta(\lambda_0)$ with respect to μ of equations $(3.3)_y$ when $\lambda = \lambda_0$ is strictly positive. This phenomenon in turn is common when the flow $(Y, \{T_t\})$ admits non-trivial recurrence properties. Its discovery for almost periodic flows is due to Millionščikov ([37]; see also Vinograd [51]). For "highly random" flows it is due to Furstenberg and Kesten [19].

In any case, consider now the linearization of $(3.1)_y$ around a given solution $x(t)$: one obtains

$$(\delta x)' + 2x(t)\delta x = 0. \tag{$3.4)_y$}$$

We may linearize equations $(3.1)_y$ "around μ^+", where μ^+ is the limit as λ increases to λ_0 of the measures $\mu^+(\lambda)$. It is easy to make sense of this idea; intuitively speaking one substitutes in $(3.4)_y$ solutions $x(t)$ of $(3.1)_y$ which are in the support of μ^+. Since $x = \frac{\varphi'}{\varphi}$, we see that the Lyapounov exponent of equations $(3.4)_y$ which corresponds to the measure μ^+ is $-2\beta(\lambda_0)$.

The moral of these remarks is that the *stability* of an invariant measure (in the sense that the corresponding Lyapounov exponent of equations $(3.4)_y$ is strictly negative) does not guarantee the *continuability* of the invariant measure. Indeed $Y \times \mathbb{R}$ carries no measures invariant for equations $(3.1)_y$ (which project to μ) if $\lambda > \lambda_0$. If the continuability is *given*, however, then negativity of the Lyapounov exponent does indeed guarantee the stability of the continued invariant measure for nearby parameter values [6].

4 Random bifurcation theory II: a two-dimensional problem

We begin by formulating a quite general one-parameter bifurcation problem with two degrees of freedom. Let Y be a compact metric space with flow $\{T_t \mid t \in \mathbb{R}\}$, and let $I \subset \mathbb{R}$ be an open interval containing the origin $\lambda = 0$. Consider the random differential equations

$$x' = l_\lambda(T_t(y))x + n_\lambda(T_t(y), x) \qquad x \in \mathbb{R}^2 \tag{4.1$_y$}$$

where $n_\lambda(y,x)$ is jointly continuous in (λ, y, x), is C^2-smooth in x, and satisfies $n_\lambda(y,x) = O(\|x\|^2)$ as $x \to 0$. The flow $\{T_t\}$ is allowed to vary with λ; we assume that $T_t = T_t^\lambda$ is jointly continuous.

Suppose now that $x = 0$ is an asymptotically stable solution of $(4.1)_y$ for each $\lambda < 0$, but that asymptotic stability is lost as λ passes through zero. A natural and important question arises: is there a new asymptotically stable invariant set (attractor) if $\lambda > 0$? If so, what does it look like?

In a moment we will consider two situations in which variants of this general problem arise. First let us rephrase the problem slightly. Note that, for fixed λ, the solutions of $(4.1)_y$ define a skew-product flow $\{\widehat{T}_t\}$ on $Y \times \mathbb{R}^2$ in the following way:

$$\widehat{T}_t(y, x_0) = (T_t(y), x(t))$$

where $x(t)$ is the solution of $(4.1)_y$ satisfying $x(0) = x_0$. It is easy to see that $\{\widehat{T}_t\}$ defines a flow on $Y \times \mathbb{R}^2$, at least if solutions of $(4.1)_y$ exist on $-\infty < t < \infty$. But this latter condition can be assured by multiplying $n_\lambda(y, \cdot)$ by a suitable bump function of x centered at $x = 0$.

Note now that the set $Y \times \{0\} \subset Y \times \mathbb{R}^2$ is compact and invariant with respect to the flow $\{\widehat{T}_t = \widehat{T}_t^\lambda\}$ for each $\lambda \in I$. By hypothesis this set is asymptotically stable for $\lambda < 0$ but ceases to be so at $\lambda = 0$. We will search for compact, invariant, asymptotically stable subsets Z of $Y \times \mathbb{R}^2$ which are near $Y \times \{0\}$ when $\lambda > 0$.

Let us now consider two problems which motivate the study of $(4.1)_y$. The first is that of the breakdown of stability of an invariant two-torus in a non-linear dynamical system. Consider a one-parameter family of equations

$$z' = f_\lambda(z) \qquad z \in \mathbb{R}^N \tag{4.2$_\lambda$}$$

where as before λ lies in an open interval $I \subset \mathbb{R}$ containing zero. Suppose that $\{Y_\lambda \mid \lambda \in I\}$ is a continuous family of invariant 2-tori, which is stable for $\lambda < 0$ but loses stability at $\lambda = 0$. One asks if $(4.7)_y$ admits an invariant attractor for $\lambda > 0$.

There is a well-known way, indicated by Ruelle and Takens [44], by which a family of invariant 2-tori can arise in parametrized family of nonlinear dynamical systems. Thus this problem is of significant interest. Two approaches to studying the problem were developed in the late 1970s by Sell and Flockerzi ([49, 17]) and by Chenciner-Iooss ([10, 11]). In both approaches, strong smoothness conditions together with a diophantine assumption on the flow on the 2-torus at the critical value $\lambda = 0$ are of crucial importance.

It should be emphasized that Chenciner and Iooss wrote down a (strong) condition guaranteeing the *existence* of a smooth family of invariant 2-tori for $\lambda > 0$, i.e., after the breakdown of asymptotic stability. Our theory presupposes the persistence of the family of 2-tori for $\lambda > 0$. In situations where this persistence does not hold, our theory is not applicable.

If persistence does hold, however, then $(4.2)_\lambda$ can be reduced to $(4.1)_y$ by the device of linearizing $(4.2)_\lambda$ around the compact invariant set Y_λ and making appropriate assumptions concerning the existence of a center manifold. See [32] for details.

A second type of problem, to which our methods apply directly, is illustrated by the noisy Duffing van der Pol oscillator

$$v'' = (\alpha + y(t))v + \beta v' - v^2 v' - v^3. \tag{4.3}$$

Here $y(\cdot) \in Y$, and Y is a weak-* compact, translation invariant subset of $L^\infty(\mathbb{R}, M_n)$. Thus one has a "parameter-disturbed" bifurcation problem. Note that $v = v' = 0$ is a solution of (4.3); one studies the stability of this solution as α and β vary. The problem (4.3) was studied by Holmes and Rand [24] when $y = 0$. They divided the α-β parameter space into eight regions, with various bifurcation scenarios as one crosses the boundary between one region and another.

When $y(t) = \xi(t) =$ white noise, this problem has been studied numerically by K.R. Schenk of Bremen [47]. Motivated by the work of Holmes-Rand, he also divides the α-β parameter space into eight regions. He describes the "attracting invariant measures" in $Y \times \mathbb{R}^2$ in each region (here Y is the path space of white noise). He observes that the attracting invariant measure has a "two-peak" structure in certain regions, and a "crater" structure in others. The two-peak structure is produced by what he calls a stochastic pitchfork bifurcation, and the crater structure by a stochastic Hopf bifurcation.

Schenk does not give an analytical discussion of these very interesting bifurcation patterns. The bifurcation scenario we now discuss resembles in a general way Schenk's stochastic pitchfork bifurcation (and not the stochastic Hopf bifurcation, despite the title of [32]). It must be quickly noted that the randomness $y(\cdot)$ which we study is defined by a flow on a 2-torus, very far indeed (one might think) from white noise. Yet we have the impression that our scenario has more in common with that described by Schenk than with those discussed in quasi-periodic bifurcation theories ([10, 11, 49, 17] and more recently [25, 35, 7, 8]).

Let us now return to equations $(4.1)_y$. Recall that the parameter λ takes values in an open interval I containing $\lambda = 0$. We suppose that $y \in Y$, a fixed 2-torus, and write

$$y = (y_1, y_2)$$

where $y_1, y_2 \in [0, 1) \simeq \mathbb{R}/\mathbb{Z}$ are angular coordinates on Y. The flow $\{T_t^\lambda \mid t \in \mathbb{R}\}$ is assumed to be generated by a vector field V_λ on Y. We take for granted that standard existence and uniqueness results are satisfied by the solutions of the equations $y' = V_\lambda(y)$. We shall assume throughout that the vector fields V_λ $(\lambda \in I)$ are jointly continuous in (λ, y) and that they are all transversal to a fixed simple closed curve $K \subset Y$ which is not homotopic to a point. Changing coordinates on Y, we can and will assume that K is given by $\{(y_1, y_2) \mid y_1 = 0\}$.

As an example of the flows on Y which we have in mind, consider

$$T_t^\lambda(y_1, y_2) = (y_1 + t, y_2 + \rho(\lambda)t)$$

(the quasi-periodic case). The frequencies are 1 and $\rho(\lambda)$. The quantity $\rho(\lambda)$ is obviously the classical rotation number [13] of the first return map $m_\lambda : K \to K : y_2 \to y_2 + \rho(\lambda)$, which in this case coincides with the time-one map.

In general we will write $\rho(\lambda)$ for the classical rotation number of the first return map $m_\lambda : K \to K$ $(\lambda \in I)$, which need not coincide with any time-t_0 map of $\{T_t^\lambda\}$. Since m_λ is a homeomorphism of the circle K to itself, $\rho(\lambda)$ has all the usual properties, which we will use below with limited further comment.

We next introduce a useful decomposition. Write

$$l_\lambda(y) = \begin{pmatrix} a_\lambda(T_t(y)) & 0 \\ 0 & a_\lambda(T_t(y)) \end{pmatrix} + b_\lambda(y) \quad \text{where } \operatorname{tr} b_\lambda(\cdot) \equiv 0,$$

and let $\beta_b(\lambda)$ be the maximal Lyapounov exponent of the "traceless" equation

$$x' = b_\lambda(T_t(y))\, x.$$

Then the maximal Lyapounov exponent $\beta(\lambda)$ of the linearization of $(4.1)_y$:

$$x' = l_\lambda(T_t(y))\, x \tag{4.4$_y$}$$

is the sum:

$$\beta(\lambda) = \beta_a(\lambda) + \beta_b(\lambda). \tag{4.5}$$

It is necessary to add a caveat to this discussion. If the rotation number $\rho(\lambda)$ is irrational, then there is exactly one measure on Y which is ergodic with respect to the flow $\{T_t^\lambda\}$. In this case β_a, β_b and β are all well-defined and (4.5) is true. On the other hand, if $\rho(\lambda)$ is rational, then there may be several ergodic measures on Y. One must make a choice of ergodic measure in order to define the maximal

Lyapounov exponents. This will not always be convenient, so in what follows we will tacitly assume that $\rho(\lambda)$ is irrational whenever we apply the formula (4.5).

We now suppose that $\beta(\lambda) < 0$ for $\lambda < 0$ but that $\beta(\lambda) > 0$ for at least some λ in every interval $(0, \lambda_1)$ with $\lambda_1 > 0$. We pose the question: do equations $(4.1)_y$ possess an attractor in $Y \times \mathbb{R}^2$ for at least some $\lambda > 0$? We shall see that, subject to some more or less reasonable assumptions, the answer is yes. In the scenario we study, there will be an attractor for *some* but not *all* λ in each interval $(0, \lambda_1)$ with $\lambda_1 > 0$. In fact there will be an attractor for a "large" set of positive λ, but these attractors definitely do *not* form a continuous family on $(0, \lambda_1)$.

It is convenient to divide the possible relations between β_b and β into three cases:

(R1) $\beta_b(\lambda) \gg \beta(\lambda)$ for $\lambda > 0$, λ near zero;

(R2) $\beta_b(\lambda) \simeq \beta(\lambda)$ for $\lambda > 0$, λ near zero;

(R3) $\beta_b(\lambda) \ll \beta(\lambda)$ for $\lambda > 0$, λ near zero.

In what follows we will assume that the first relation holds. It holds in particular if $\beta_b(0) > 0$ and if β_b is continuous at $\lambda = 0$. There are grounds for believing that these conditions are verified rather often for the random linear equations $(4.4)_y$. However satisfactory rigorous results are not yet available which would bolster this belief, so we omit further discussion of the matter. Moreover, for technical reasons we shall have to assume that $\beta_b(0) = 0$ in order to prove our main result. We feel, however, that our main theorem is true if $\beta_b(0) > 0$, and that proving our results under this hypothesis would contribute substantially to understanding the breakdown of stability of the zero solution of equations $(4.1)_y$.

An important hypothesis of our main theorem will be that $l_\lambda(\cdot)$ is not too smooth as a function of y. In fact we will require that l_λ be no more than $C^{1-\delta}$-smooth for some $\delta > 0$. This is because the conclusion of our Theorem 4.6 is almost certainly false if l_λ is C^r-smooth for $r > 1$. Thus our bifurcation scenario should be viewed as complementary to theories in which a high degree of smoothness is required.

Before turning to a discussion of our results, we remark that condition (R3) is satisfied in theories where a stable 3-torus bifurcates cleanly from the family of 2-tori at $\lambda = 0$ ([10, 11, 49, 17]). Roughly speaking, the "hyperbolic" part of the linearized system $(4.4)_y$ is dominated by the "elliptic" part. The relation (R2) seems rather unpleasant from a theoretical point of view. An example which displays transversal homoclinic behaviour in $(4.1)_y$ when (R2) holds is given in [32].

We now begin the analysis of equations $(4.1)_y$ when $x = 0$ loses asymptotic stability at $\lambda = 0$ and when condition (R1) above holds. We will impose further assumptions of a "generic" nature, meant to hold for as large a class of problems as possible. We begin with the family of vector fields $\{V_\lambda\}$. If the rotation number $\rho(\lambda)$ of the first-return map $m_\lambda : K \to K$ is rational, then (generically) the circle K supports q attractor-repeller pairs, i.e., there is frequency locking. While it is certainly reasonable that frequency locking should occur for an open dense set of $\lambda \in I$, we assume that it does *not* occur at $\lambda = 0$. If it did, then our loss-of-stability problem would reduce to a (non-smooth!) version of the bifurcation problem studied in [44]. Further investigation of this matter would be interesting. Here, however, we will assume that, at $\lambda = 0$, the rotation number $\rho(0)$ is irrational. In fact the important set of λ-values will be those for which $\rho(\lambda)$ is irrational. With this in mind, we define

$$\Lambda_* = \mathrm{cls}\,\{\lambda \in I \mid \rho(\lambda) \notin \mathbb{Q}\},$$

and note that $0 \in \Lambda_*$.

Next we consider the linear equations

$$x' = l_\lambda(T_t(y))x. \tag*{$(4.4)_y$}$$

The solutions of this equation exhibit a rich range of behaviour; its theory is far from completely developed. On the other hand one has sufficient knowledge to permit our analysis of equations $(4.1)_y$. The basic result which we will need states that, roughly speaking, a generic one-parameter family $x' = b_\lambda(T_t(y))x$ of trace-zero linear systems has an exponential dichotomy for an open dense set of parameter values in Λ_*.

To state this result more precisely, let $\mathrm{sl}(2,\mathbb{R})$ be the Lie algebra of 2×2 real matrices with trace zero. Let B be the set of all $C^{1-\delta}$-mappings from the 2-torus Y into $\mathrm{sl}(2,\mathbb{R})$ where $0 < \delta \leq 1$. Further let s be any non-negative number ($s = \infty$ is allowed). Define $C^s(I, B)$ to be the collection of all C^s-mappings from the interval I to the Banach space B. The number s is not important in our theory, but the number δ is. The result we now enunciate is very likely false if $1-\delta$ is replaced by r for $r > 1$.

Theorem 4.6 *Let $\{V_\lambda \mid \lambda \in I\}$ be a family of vector fields on Y satisfying the conditions enunciated earlier. There is a residual subset $E \subset C^s(I, B)$ with the following property: if $b = b_\lambda(\cdot) \in E$, then the equations*

$$x' = b_\lambda(T_t(y))x \tag*{$(4.7)_y$}$$

have an exponential dichotomy for all λ in an open dense subset of Λ_*.

The idea in what follows is quite simple. Since ED is an extremely robust property, it is natural to study the non-linear equations $(4.1)_y$ for parameter values $\lambda \in \Lambda_*$ for which an exponential dichotomy is present in equations $(4.7)_y$. One expects that the presence of ED in the linear equations will make itself felt in the behavior of solutions of the nonlinear equations.

We return to equations $(4.1)_y$. The fact is that these are too general for us to handle, even with the relation (R1) and Theorem 4.6 at our disposal. However, we *can* deal with the problem

$$x' = \lambda[l_\lambda(T_t(y))x + n_\lambda(T_t(y), x)] \tag*{$(4.8)_y$}$$

by using a generalization of the method of averaging, as described in [32].

To analyze equations $(4.8)_y$, we write

$$l_\lambda = a_\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \delta_\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \epsilon_\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let (r, θ) be polar coordinates in the x-plane. Then equations $(4.8)_y$ take the form

$$\begin{aligned} r' &= \lambda\{[a_\lambda + U_\lambda]r + d_\lambda r^2 + \widehat{f}_\lambda\} \\ \theta' &= \lambda\{L_\lambda + G_\lambda r + \widehat{g}_\lambda\} \end{aligned}$$

where a_λ is a function of $T_t(y)$; $U_\lambda, d_\lambda, L_\lambda$ and G_λ are functions of $(T_t(y), r(t), \theta(t))$. One has explicitly:

$$\begin{aligned} U_\lambda(y, \theta) &= \delta_\lambda(y)\cos 2\theta + \epsilon_\lambda(y)\sin 2\theta \\ L_\lambda(y, \theta) &= \gamma_\lambda(y) + \epsilon_\lambda(y)\cos 2\theta - \delta_\lambda(y)\sin 2\theta. \end{aligned}$$

The function $d_\lambda(y, \theta)$ depends on the nonlinearity n_λ. In addition $\widehat{f}_\lambda = O(r^3)$ and $\widehat{g}_\lambda = O(r^2)$ as $r \to 0$, uniformly in (y, θ).

Define another function

$$p_\lambda(y, \theta) = \frac{\partial U_\lambda}{\partial \theta} = 2\epsilon_\lambda(y)\cos 2\theta - 2\delta_\lambda(y)\sin 2\theta.$$

Introduce the following hypotheses.

(H1) $\beta_b(\lambda) \gg \beta(\lambda)$ for small positive λ, and $\beta_b(0) = 0$.

(H2) When $\lambda = 0$, there is a unique measure on the "projective bundle" $Y \times \mathbb{P}^1(\mathbb{R})$ which is invariant under the natural flow-of lines on $Y \times \mathbb{P}^1(\mathbb{R})$ defined by $x' = l_0(T_t(y))x$. See [32]. This hypothesis can be viewed as a very weak version of the diophantine condition in the smooth quasi-periodic theory. It allows our averaging procedure to work.

(H3) The average $\overline{d}_0 = \lim_{T\to\infty} \frac{1}{T}\int_o^T d_0(T_s(y), \theta(s))\, ds$ is less than zero. The average is well-defined by (H2). This is a generalized weak-attractor condition.

(H4) The mean values $\overline{p}_\lambda = \lim_{T\to\infty} \frac{1}{T}\int_0^T p_\lambda(T_s(y), \theta(s))\, ds$ satisfy $\overline{p}_0 = 0$ and $\overline{p}_\lambda \leq c\beta_b(\lambda)$ for a certain constant c. This may be viewed as a weak normal form condition; it is satisfied if, at $\lambda = 0$, $l_0 = \begin{pmatrix} 0 & -\gamma_0 \\ \gamma_0 & 0 \end{pmatrix}$.

(H5) Four quantities which we do not write explicitly do not deviate too much from their mean values.

We can now state our main theorem.

Theorem 4.9 *Let Λ_b be the set of points $0 < \lambda \in \Lambda_*$ such that $x' = b_\lambda(T_t(y))x$ has an ED. Assume that hypotheses (H1)-(H5) hold.*

(a) If $\lambda \in \Lambda \cap (0, \lambda_1)$ for sufficiently small λ_1, then equations $(4.1)_y$ admit an attractor-repeller pair $Z^-(\lambda), Z^+(\lambda)$ in $Y \times \mathbb{R}^2$. These sets tend to $Y \times \{0\} \subset Y \times \mathbb{R}^2$ as $\lambda \to 0$.

(b) The sets $Z^-(\lambda)$ vary discontinuously as $\lambda \to 0$ in a precise sense as $\lambda \to 0^+$; so do the sets $Z^+(\lambda)$.

We finish this paper with a brief discussion of point (b) of Theorem 4.9. It is here that the rotation number $\alpha(\lambda)$ of equation $(4.7)_y$ plays a role. Here we speak of the rotation number defined in § 2 and not of the classical quantity $\rho(\lambda)$.

The main point is that there is a "time-changed" version $\widehat{\alpha}(\lambda)$ of the rotation number such that, if equations $(4.7)_y$ have an ED at λ, then

$$\widehat{\alpha}(\lambda) = n_\lambda + m_\lambda \rho(\lambda)$$

where n_λ, m_λ are integers. Now, by removing a set of first category from the set E of Theorem 4.6, one can assume that $\widehat{\alpha}(0)$ is *not* of the form $n + m\rho(0)$ for integers n, m. Since $\widehat{\alpha}(\cdot)$ is continuous, the integers n_λ, m_λ must vary wildly as $\lambda \to 0$.

Now, the integers n_λ, m_λ are winding numbers, and it turns out that they reflect the way in which $Z^-(\lambda)$ is embedded in $Y \times \mathbb{R}^2$. It is in this sense that the sets $Z^-(\lambda)$ vary discontinuously as $\lambda \to 0$.

Acknowledgements

This research was supported in part by grants from the M.U.R.S.T. (Italy).

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