# Dynamics of networks: Features which persist from the uncoupled limit

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### Abstract

The theme of these notes is to look for aspects of the dynamics of a network of units which can be continued from the uncoupled case, uniformly in the size of the system. Some answers are given, and particularly interesting answers are found in the case of conservative systems.

## 1 Introduction

Many systems are well described as a network of dynamically interacting units. As examples, consider:

- electricity production, distribution and consumption networks

- Josephson junction arrays

- optical computer memory

- crystals and quasicrystals

- multicellular organisms

- nervous systems

- economies

The issue I wish to address is how much can be learned from the uncoupled case. One might think the answer is "very little", and indeed there are many phenomena, like waves, pattern formation, and synchronisation, which depend heavily on the coupling. Nonetheless, a lot can be learned from the uncoupled limit. In particular, many results can be obtained which are uniform in the size of the network. In fact they also apply to infinite networks; from the physical point of view, infinite networks do not exist, but they are convenient mathematical idealisations.

By a *network* let us understand for present purposes a system of ordinary differential equations (ODEs) of the form

$$\dot{x}_s = F_s(x_s) + K_s(x, \epsilon), \tag{1}$$

where $s$ belongs to a countable index set $S$ labelling units of the network, $x_s$ belongs to some finite-dimensional manifold $M_s$, $x$ denotes the whole configuration $(x_s)_{s \in S}$, and $\epsilon$ belongs to a finite-dimensional manifold $P$ (parameter space). We might as well think of $P$ as $\mathbb{R}^p$, some $p \in \mathbb{N}$, because my analysis will be local to a neighbourhood of a point 0 of $P$, where

$$K_s(x, 0) = 0, \forall s \in S. \tag{2}$$

Thus $x$ is a point of $M = \times_{s \in S} M_s$, and the equations can be written as

$$\dot{x} = F(x) + K(x, \epsilon). \tag{3}$$

Let us call $F_s$ the *local dynamics of unit s* and $K$ the *coupling*. Suppose that each $M_s$ and $P$ are endowed with Finsler metrics, meaning a metric induced by a norm $|.|$ on tangent vectors, and for the norm of a tangent vector to $M$ we take the supremum over $s \in S$ of the norms of its components on the $M_s$. Furthermore, I suppose that $F$ is $C^1$ (with respect to the sup norm) and $K$ is jointly $C^1$ in $(x, \epsilon)$.

A prime example of coupling is *diffusive coupling on a graph*. Here $S$ is the set of nodes of a graph, $M_s = \mathbb{R}^N$ for some $N$, $P = \mathbb{R}$, and

$$K_s(x, \epsilon) = \frac{\epsilon}{k_s} \sum_{r \in N_s} (x_r - x_s), \tag{4}$$

where $N_s$ is the set of nodes neighbouring to $s$ in the graph, and $k_s$ is the number of neighbours. Note that our definition of a network does not require the coupling to be local in any sense, though we will in due course examine the effects of locality of the coupling if it exists.

Extensions of the definition of network, and variations on the theme, are possible. For example, there is the much studied discrete-time analogue, *coupled map lattices*. Many of our considerations can be translated directly to that context, but I prefer to concentrate on the continuous-time case as being closer to most applications, using coupled map lattices only as a simplifying pedagogical tool in Section 6. An extension which deserves serious attention is to couplings that are not described as perturbations to the vector field on the product manifold, for example time-delayed coupling, or coupling via a convolution in time, or via a capacitor or inductor or a bistable system (as in some cell membranes), or even via a partial differential equation (PDE) representing a transmission line. I have not yet addressed these generalisations, though I learnt recently that Jack Hale has looked at the transmission line case. Indeed, I learnt from Hale that in addition to the references that I cite here there are some other precursors to several of the issues I will address. I have not yet had the time to study them, however, and I apologise to both the authors and the readers for this incompleteness.

The lectures were given in two parts. In the first lecture, persistence of equilibria, stability, the effects of locality of the coupling, periodic orbits, uniformly hyperbolic sets, normally hyperbolic manifolds, and structural stability for networks were discussed. In the second lecture, I specialised to the Hamiltonian and time-reversible cases, which I refer to collectively as *conservative*. The notes follow the same outline. I will not touch on the important area of ergodic theory of networks (interested readers are referred to [30], for example), though recently I think I have found a beautiful approach to understanding the statistical behaviour of networks of chaotic units. Neither will I touch on the interesting and I believe potentially very useful concept of the rotation set for the chain-recurrent set; for information about that see [22]. I would like the notes to be seen as partly a survey of established results, and partly as an invitation to join an interesting ongoing research programme.

# 2 Equilibria

Let us begin by studying to what extent equilibria of the uncoupled network persist to the coupled case. This section is based on [27]. From Eq. (1), it is a question of solving

$$G(x, \epsilon) := F(x) + K(x, \epsilon) = 0. \tag{5}$$

Suppose $x^0$ is an equilibrium for $\epsilon = 0$, i.e. for all $s \in S$

$$F_s(x_s^0) = 0, \tag{6}$$

and suppose it is *uniformly non-degenerate*[1], meaning that there exists $B \in \mathbb{R}$ such that for all $s \in S$

$$|DF_s(x_s^0)^{-1}| \le B. \tag{7}$$

**Theorem 1** *There exists $\epsilon_0 > 0$, depending only on the suprema of $\|DF\|$, $\|DF^{-1}\|$, $\|\frac{\partial K}{\partial \epsilon}\|$ and $\|DK\|/|\epsilon|$ in a neighbourhood $U$ of $(x^0, 0) \in M \times P$, such that $x^0$ has a unique continuation $X(\epsilon)$ for $|\epsilon| < \epsilon_0$ (with $X(0) = x^0$).*

Here and elsewhere, when applied to functions on $M \times P$, $D$ denotes derivative with respect to $x \in M$ and $\frac{\partial}{\partial \epsilon}$ denotes derivative with respect to $\epsilon \in P$.

Theorem 1 is proved by application of the implicit function theorem (e.g. [12]) to Eq. (5). Furthermore, it follows from the proof that the function $X$ is $C^1$ and satisfies

$$\frac{\partial X}{\partial \epsilon} = -DG(X(\epsilon), \epsilon))^{-1} \frac{\partial K}{\partial \epsilon}(X(\epsilon), \epsilon), \tag{8}$$

as long as $DG$ remains invertible.

We can use this to estimate an explicit $\epsilon_0$ for Theorem 1. For example, taking norms of both sides of Eq. (8) and using Eq. (5),

$$|\frac{\partial X}{\partial \epsilon}| \le \frac{\|\frac{\partial K}{\partial \epsilon}(X(\epsilon), \epsilon)\|}{\|DF^{-1}(X(\epsilon))\|^{-1} - \|DK(X(\epsilon), \epsilon)\|}. \tag{9}$$

Suppose

$$\|DF(x)^{-1}\| \ge a - b\tau, \tag{10}$$

$$\|DK(x, \epsilon)\| \le c\eta, \tag{11}$$

$$\|\frac{\partial K}{\partial \epsilon}(x, \epsilon)\| \le d + e\tau + f\eta, \tag{12}$$

in some neighbourhood $U$ of $(x^0, 0)$, where

$$\tau = |x - x^0|, \tag{13}$$

$$\eta = |\epsilon|. \tag{14}$$

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[1] Actually it is not necessary to insist on uniformity as this is automatic under the $C^1$ hypothesis on $F$: if $DF$ is bounded and invertible then the inverse is bounded. See [13], for example.

Then

$$|\frac{\partial \tau}{\partial \epsilon}| \le \frac{d + e\tau + f\eta}{a - b\tau - c\eta}, \tag{15}$$

as long as $x$ remains in $U$ and the denominator remains positive. It follows that the continuation can be performed at least up to $\epsilon_0$ given by the value of $\eta$ at the first time that the solution of

$$\frac{d\tau}{d\eta} = \frac{d + e\tau + j\eta}{a - b\tau - c\eta} \tag{16}$$

reaches either the boundary of $U$ or the line $a - b\tau - c\eta = 0$.

Here is a concrete example, which we call *coupled bistability*. Suppose all the $F_s = f$, a $C^1$ "bistable" function of one variable, possessing attracting zeroes at $x = 0$ and 1 with $f'(0), f'(1) < 0$, and $f'$ is monotone on intervals $[0, c_0]$ and $[c_1, 1]$, with $f'(c_0) = f'(c_1) = 0$. Suppose $K$ is diffusive coupling on a graph as in Eq. (4), and $\epsilon \in \mathbb{R}_+$. Then we can estimate $\|DG^{-1}\|$ as in Eq. (9), or better by

$$\|DG^{-1}\| \le \frac{1}{\|\Lambda^{-1}\|^{-1} - \|\Delta\|}, \tag{17}$$

where $\Lambda$ is the diagonal part of $DG$ and $\Delta$ its off-diagonal part. Hence

$$\|DG^{-1}\| \le \frac{1}{\inf_{s \in S} |f'(x_s) - \epsilon| - |\epsilon|}, \tag{18}$$

as long as the denominator remains positive. But the $f'(x_s)$ all start negative and $\epsilon \ge 0$, so it follows that

$$\|DG^{-1}\| \le \frac{1}{\inf_{s \in S}(-f'(x_s))}, \tag{19}$$

as long as the denominator remains positive.

We could estimate $|\frac{\partial K}{\partial \epsilon}|$ by a formula like Eq. (12) but we can do better. All solutions of Eq. (5) which are continuations of $x^0 \in \{0, 1\}^S$ have $x \in [0, 1]^S$ for $\epsilon \ge 0$. This is because if $x_s$ is the leftmost value then $K_s \ge 0$, so to solve Eq. (5) we require $f(x_s) \le 0$. Since $f(0) = 0$ and $f'(0) < 0$, the solution can not cross 0. Similarly, it can not cross $x_s = 1$. In the case of an infinite network the above argument needs modifying by taking the limit of $x_s$ near $\inf_{s \in S}(x_s)$, but the same conclusion holds. Hence

$$|\frac{\partial K}{\partial \epsilon}| \le \sup_{x \in [0,1]^S, s \in S} \frac{1}{k_s} |\sum_{r \in N_s} (x_r - x_s)| = 1. \tag{20}$$

Putting Eqs (8), (19) and (20) together,

$$\frac{d\tau}{d\epsilon} \le \frac{1}{\inf_{s \in S}(-f'(x_s))} \le \frac{1}{\min(-f'(\tau), -f'(1 - \tau))}, \tag{21}$$

by the monotonicity of $f'$. Hence, the equilibria continue at least up to

$$\epsilon_0 = \int_0^{\min(c_0, 1-c_1)} \min(-f'(\tau), -f'(1 - \tau))d\tau. \tag{22}$$

Note that if the minimum in the integrand is always attained by the first term then this gives simply $-f(c_0)$. If it is always attained by the second term then it gives $f(c_1)$. Consequently, if either one or the other is true, as for cubic $f$, then we obtain the very simple explicit bound

$$\epsilon_0 = \min(-f(c_0), f(c_1)). \tag{23}$$

In conclusion, for $0 \leq \epsilon < \epsilon_0$ there is an injection from $\{0, 1\}^S$ into the set of equilibria. We can say that "computer memory survives weak coupling".

By the same method, we can continue unstable equilibria of the uncoupled system, using for one or more $s \in S$, $x_s^0 = u$, an unstable zero of $f$, provided $f'(u) > 0$. By continuity of $f$ at least one unstable zero must exist between 0 and 1, and generically it satisfies $f'(u) > 0$. But note that we will not be able to obtain quite as good a bound $\epsilon_0$ for these unstable equilibria because the neat cancellation of $\epsilon$ in the denominator of Eq. (18) will no longer occur. Nonetheless the resulting differential inequality is easily solved explicitly, and gives an explicit $\epsilon_0$.

The method can even be applied to problems where the uncoupled system possesses only one equilibrium, but on sufficient coupling gains many equilibria. An example is the type of system with negative diffusion discussed by Mallet-Paret in this Colloquium. The way the method can be applied is to start from a different uncoupled limit, which includes the diagonal part of the negative diffusion and hence exhibits bistability at each site once the diffusion parameter exceeds a certain value, and then show that in suitable parameter regimes one can continue the resulting stable equilibria with respect to the off-diagonal part of the diffusion. I will write up the details separately.

The coupled bistability example raises two questions. Firstly, do the equilibria keep the same stability-type as $\epsilon$ grows? Secondly, how do we know that no new equilibria are created as $\epsilon$ grows? We will address the first question immediately, and the second question later.

# 3 Stability

In this section it is proved (following [27] again, but improving on it) that subject to a strengthening of the condition of uniform non-degeneracy to uniform hyperbolicity, the equilibria obtained in Theorem 1 preserve their stability-type for at least $|\epsilon| < \epsilon_1$, for some $\epsilon_1 > 0$ but not necessarily as large as $\epsilon_0$. By *preserve their stability-type* I mean that there is a continuous (with respect to $\epsilon$) splitting of the tangent space into contracting and expanding subspaces for the linearised dynamics about the equilibrium $X(\epsilon)$; in particular their dimensions do not change. Here, "expanding" means "contracting in negative time". Let us say that an equilibrium $x^0$ of the uncoupled system Eq. (1) is *uniformly hyperbolic*[2] if there exists $B \in \mathbb{R}$ such that for

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[2] As in Section 2, one does not need to insist on the uniformity: uniformity with respect to $s \in S$ follows from $F \in C^1$, and uniformity with respect to $\lambda$ is automatic because for $|\lambda| > ||DF||$ we have $||(\lambda I - DF)^{-1}|| \leq \frac{1}{|\lambda| - ||DF||}$, and it is continuous with respect to $\lambda \notin \text{spec } DF$.

all $s \in S$ and $\lambda$ on the imaginary axis,

$$||(\lambda I - DF_s(x_s^0))^{-1}|| \le B, \tag{24}$$

where $I$ denotes the identity matrix.

**Theorem 2** *There exists $\epsilon_1 > 0$, depending only on the suprema of $\sup_{\Re\lambda=0} ||(\lambda I - DF)^{-1}||$, $||DF||$, $||\frac{\partial K}{\partial \epsilon}||$ and $||DK||/|\epsilon|$ in a neighbourhood $U$ of $(x^0, 0) \in M \times P$, such that the continuation $X(\epsilon)$ of $(x^0, 0)$ keeps the same stability type for $|\epsilon| < \epsilon_1$.*

To prove this, the linearised dynamics is given by

$$\dot{\xi} = DG(X(\epsilon), \epsilon)\xi := M(\epsilon)\xi. \tag{25}$$

For $\lambda \notin \text{spec } M$, define the *resolvent operator*

$$R_\lambda(M) = (\lambda I - M)^{-1}. \tag{26}$$

Then if there is no spectrum on the imaginary axis, the desired splitting is given by the complementary projections

$$\Pi_c = \frac{1}{2\pi i} \int_{\Gamma_c} R_\lambda d\lambda, \tag{27}$$

$$\Pi_e = \frac{1}{2\pi i} \int_{\Gamma_e} R_\lambda d\lambda, \tag{28}$$

where $\Gamma_c$ and $\Gamma_e$ are closed anticlockwise contours in the complex $\lambda$-plane enclosing respectively all the spectrum in the left half-plane and all the spectrum in the right half-plane.

To prove that no spectrum reaches the imaginary axis for $\epsilon$ small enough and that the projections are continuous with respect to $\epsilon$, we note firstly that if

$$||M' - M|| < ||R_\lambda(M)||^{-1}, \tag{29}$$

then $R_\lambda(M')$ exists, so $\lambda \notin \text{spec } M'$. Let

$$r = \sup_{\Re\lambda=0} ||R_\lambda(M(0))||, \tag{30}$$

which is finite, by the hypothesis Eq. (24). Then there is no spectrum on the imaginary axis as long as

$$||M(\epsilon) - M(0)|| < \frac{1}{r}, \tag{31}$$

and by continuity of $M$ on $\epsilon$ (in fact uniformly in the system size), we deduce that there is $\epsilon_1 > 0$ such that no spectrum reaches the imaginary axis for $|\epsilon| < \epsilon_1$. Secondly, we note that $R_\lambda(M)$ is continuous with respect to $M$, for $\lambda \notin \text{spec } M$. Indeed, if Eq.(29) holds then

$$R_\lambda(M') - R_\lambda(M) = R_\lambda(M)(M - M')R_\lambda(M'). \tag{32}$$

So in particular we have

$$\|R_\lambda(M') - R_\lambda(M)\| \leq \frac{\|R_\lambda(M)\|^2 \|M' - M\|}{1 - \|R_\lambda(M)\| \|M' - M\|}. \tag{33}$$

The continuity of the projections with respect to $\epsilon$ (as long as no spectrum reaches the imaginary axis) follows from this.

As an illustration, for the equilibria of the example of the previous section, obtained by continuation from $\{0, 1\}^S$,

$$r = \frac{1}{\min(|f'(0)|, |f'(1)|)}, \tag{34}$$

and

$$\|M(\epsilon) - M(0)\| \leq \sup_{s \in S} |f'(x_s) - f'(x_s^0) - \epsilon| + \epsilon. \tag{35}$$

So we deduce that $X(\epsilon)$ is stable as long as

$$\max(2\epsilon, |f'(0) - f'(\tau) + \epsilon| + \epsilon, |f'(1) - f'(1 - \tau) + \epsilon| + \epsilon) < \min(|f'(0)|, |f'(1)|). \tag{36}$$

This can be translated into a bound on $\epsilon$ by using Eq. (21).

In fact, for this example if every unit has the same number of neighbours then we can do better, because it becomes a gradient flow. *Gradient flows* are dynamical systems of the form:

$$\dot{x} = -\nabla W(x) \tag{37}$$

for some "energy function" $W$, where the gradient is taken with respect to some inner product. For gradient flows, the stability type of an equilibrium is preserved as long as it continues to be *non-degenerate*, meaning $DG$ (in this case $D^2W$) remains invertible. Thus in this case, the bounds for continuation and preservation of stability-type can be taken identical. This is because the spectrum is purely real, so the only point of the imaginary axis it can approach is 0, and that corresponds to losing non-degeneracy, whereas in the general case, Poincaré-Andronov-Hopf bifurcation can also occur. So the equilibria of the coupled bistability example remain stable as long as they remain non-degenerate, in particular at least up to $\epsilon_0$.

In the case of *attracting* stability type (i.e. hyperbolic with trivial expanding subspace) one could ask the question whether the equilibrium is attracting for the full (nonlinear) system (with respect to sup-norm). This is almost certainly true, but I did not yet check it. If so, one could further ask whether uniform stability estimates can be found with respect to the size of the system. For example, are there $\kappa < 1, \delta_0 > 0, \mu > 0$ independent of system size, such that for all $\delta < \delta_0$, the orbits of all initial conditions within $\kappa\delta$ of the equilibrium lie within $\delta e^{-\mu t}$ for all positive time $t$? Again, this is almost certainly true, but I have not yet done the estimates.

# 4  Spatial structure

Next, we address the question of what additional statements we can deduce about the equilibria we have continued, if the coupling is local in some sense. Let us suppose

a metric $d$ on $S$. For example, for a connected graph, $d(r,s)$ could be the shortest number of edges joining the nodes $r$ and $s$, and this is our default metric if we refer to the network as a graph.

There are several interesting types of local coupling. The conditions all apply to the combination

$$L_{rs} = \frac{1}{\epsilon} \frac{\partial K_s}{\partial x_r}. \tag{38}$$

The simplest is *nearest neighbour coupling* in a graph:

$$|L_{rs}| = 0 \text{ for } d(r,s) > 1. \tag{39}$$

Another is *exponentially decaying coupling*, though this needs careful formulation in cases where the number of units at distance $\rho$ grows exponentially with $\rho$.

If the coupling is local we expect the resulting solutions $X(\epsilon)$ to have some special spatial features. For instance, in the example of Section 2, if $x_s^0 = 1$ for one unit, say $s = o$, and 0 elsewhere, then does $X_s(\epsilon)$ decay exponentially with $d(s,o)$? More generally in this example, if two equilibria $x^0$ and $\tilde{x}^0$ at $\epsilon = 0$ differ only at one site, $o$, then does the difference $X_s(\epsilon) - \tilde{X}_s(\epsilon)$ decay exponentially with $d(s,o)$?

We have proved this for the example of Section 2 and in a number of other situations. The key step is to derive exponential decay estimates for the norms of the matrix elements of $DG^{-1}$. We call this property *finite coherence length*. First we describe some finite coherence length results and then we will return to how to deduce exponential decay results of the type of the previous paragraph. These results were derived in a different context but apply here nonetheless. Note that cases 1 and 3 do not require any smallness assumption on the off-diagonal part of $DG$.

**Case 1:** One-dimensional chains with nearest neighbour coupling only, though not necessarily symmetric nor translation invariant. This was done in [24]. Write

$$DG = \Lambda + L_+ + L_-, \tag{40}$$

where $\Lambda$ is the (block-)diagonal part of DG, and $L_{\pm}$ are the super- and sub- (block-) diagonals. Supposing $DG$ is invertible, let

$$\eta = ||DG^{-1}|| \max(||L_+||, ||L_-||), \tag{41}$$

and

$$\kappa = \frac{1}{2\eta} + \sqrt{\frac{1}{4\eta^2} + 1} > 1. \tag{42}$$

Then for all $\lambda \in (\kappa^{-1}, 1)$ there exists $K \in \mathbb{R}$ (see [24] for an explicit way to obtain $K$) such that

$$|(DG^{-1})_{rs}| \leq K\lambda^{d(r,s)}. \tag{43}$$

**Case 2:** One-dimensional chains with not too large exponentially decaying coupling, again not necessarily symmetric nor translation invariant. This was done in [25]. Write

$$DG = \Lambda + \Delta, \tag{44}$$

with $\Lambda$ being the (block-)diagonal part and $\Delta$ the rest. Supposing $\Lambda$ is invertible, put

$$J = -\Lambda^{-1}\Delta, \tag{45}$$

so

$$DG = \Lambda(I - J). \tag{46}$$

Suppose that $\Delta$ decays exponentially in such a way that

$$|J_{rs}| \leq C_{d(r,s)}, \tag{47}$$

for some sequence $C_k$ with

$$0 \leq C_k \leq K\lambda^k, \lambda < 1, \tag{48}$$

and

$$\sum_{k \geq 1} C_k < 1. \tag{49}$$

Then the action of $I - C$ decomposes into Bloch spaces $B_\theta$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, where the dependence on site number $s \in \mathbb{Z}$ is proportional to $e^{is\theta}$. Let

$$A(\theta) = (I - C)|_{B(\theta)}, \tag{50}$$

which is analytic in the strip $|\Im\theta| < \log \lambda^{-1}$ and invertible on the real axis. So $A(\theta)^{-1}$ is analytic in the strip

$$|\Im\theta| < \mu := \min(\log \lambda^{-1}, \Im\hat\theta), \tag{51}$$

where $\hat\theta$ is the nearest degeneracy of $A$ to the real axis. Thus

$$\limsup_{|r-s| \to \infty} \frac{\log |((I - C)^{-1})_{rs}|}{|r - s|} \leq -\mu. \tag{52}$$

Using the formula

$$((I - J)^{-1})_{rs} = \sum_{\gamma \in \Gamma} \prod_i J_{s_i, s_{i+1}}, \tag{53}$$

where $\Gamma$ is the set of paths $\gamma = (s_i)$ from $r$ to $s$, and Eq. (47), we deduce that Eq. (52) also applies to $(I - J)^{-1}$.

This case has a generalisation (though with less explicit decay rate) to exponentially decaying coupling on all "non-exponential" graphs under hypotheses Eq. (47) and Eq. (49); see section IV-3 of [5].

**Case 3:** Graphs for which $DG$ is symmetric. This is satisfied for gradient flows, or if the coupling is symmetric and the on-site dynamics is locally a gradient flow (in particular, one-dimensional on-site dynamics suffices). We suppose $DG$ is invertible, but we will use the $\ell_2$ norm, denoted by subscript 2, in order to establish exponential decay estimates. By the hypothesis of symmetry,

$$\|DG^{-1}\|_2 \leq \|DG^{-1}\|. \tag{54}$$

Choose a site $o \in S$, and for $n \geq 1$, let

$$P_n = \sum_{s:d(o,s)\geq n} |(DG^{-1})_{os}|^2. \tag{55}$$

Let

$$A_{pq}^{(n)} = \sum_{s:d(s,o)=n-1} DG_{ps}^* DG_{sq} \text{ for } d(p,o) = d(q,o) = n, \tag{56}$$

$$B_{pq}^{(n)} = \sum_{s:d(s,o)=n} DG_{ps}^* DG_{sq} \text{ for } d(p,o) = d(q,o) = n-1, \tag{57}$$

and

$$T = \sup_{n\geq 1}(\|A^{(n)}\|_2, \|B^{(n)}\|_2) \leq \|DK\|_2^2 \leq \|DK\|^2. \tag{58}$$

Then it is proved in [8], using a result from [6], that

$$P_n \leq C\lambda^{n-1}\|DG^{-1}\|^2, \tag{59}$$

where

$$C = \|DG^{-1}\|^2 T, \tag{60}$$

$$\lambda = \frac{2K}{1 + \sqrt{1 + 4K^2}} < 1. \tag{61}$$

I expect that there is a general result which would encompass all three of these cases and many more. It would have the form "$A_{rs}$ exponentially decaying and $A$ invertible implies $(A^{-1})_{rs}$ exponentially decaying", but I have not yet found it. Note that many people (e.g. [30, 11]) consider weighted spaces which force exponential decay, and then prove that the solution lies in these spaces. I prefer not to do this.

Finally, I discuss some of the consequences of finite coherence length results. Firstly, the reason for the name is that the matrix $DG^{-1}$ governs the response to a small external force distribution $h = (h_s)_{s\in S}$. The problem

$$\dot{x} = G(x, \epsilon) + \eta h \tag{62}$$

has a unique equilibrium solution near each non-degenerate equilibrium $X(\epsilon)$ of the undisturbed system, given by integrating

$$\frac{\partial x}{\partial \eta} = -DG^{-1}h \tag{63}$$

from $\eta = 0$ to 1 starting at $x = X(\epsilon)$. So if the matrix elements of $DG^{-1}$ decay exponentially with distance between units then the response to a localised external force $h$ decays exponentially from that location.

Secondly, if two equilibria at the uncoupled limit differ only at one site, say $o \in S$, then the difference between their continuations $X(\epsilon)$ and $\tilde{X}(\epsilon)$ decays exponentially

from that site, at least for $\epsilon$ small enough. To prove this let us adapt an idea of [5]. Replace the equation $G_o(x) = 0$ for the unit $o$ by the equation

$$x_o = \lambda \tilde{x}_o(\epsilon) + (1 - \lambda)x_o(\epsilon), \tag{64}$$

where $\lambda$ will range from 0 to 1. Denote the corresponding system of equations by

$$\bar{G}(x, \lambda) = 0. \tag{65}$$

Now for $\lambda = 0$ we have a solution $X(\epsilon)$ and for $\lambda = 1$ we have a solution $\tilde{X}(\epsilon)$. For $\epsilon$ small these are continuations of each other with respect to $\lambda$ via a solution $X(\epsilon, \lambda)$ of Eq. (65), because this is true at $\epsilon = 0$. As long as they remain continuations of each other with respect to $\lambda$, we have

$$\frac{\partial X}{\partial \lambda} = -D\bar{G}^{-1} \frac{\partial \bar{G}}{\partial \lambda}, \tag{66}$$

and exponential decay of the elements of $D\bar{G}^{-1}$. By integrating this equation from $\lambda = 0$ to 1, we deduce exponential decay for $\tilde{X}_s(\epsilon) - X_s(\epsilon)$ with respect to $d(s, o)$. The argument can be extended to initial equilibria differing at more than one site.

## 5 Periodic Orbits

So far we have addressed only the simplest type of dynamics: equilibria. What can be said about persistence of periodic orbits of the uncoupled network?

Aubry and I tackled this problem in the conservative context [24], and I will discuss that case in Section 10, but it can also be studied in the general context, as I will now outline. The details are being written up in a paper with Sepulchre.

Periodic orbits of a dynamical system

$$\dot{x} = G(x, \epsilon) \tag{67}$$

on a manifold $M$ can be seen as zeroes of the following map:

$$\Phi : (\hat{x}, T) \mapsto T\dot{\hat{x}} - G(\hat{x}, \epsilon), \tag{68}$$

where $\hat{x}$ belongs to a space of periodic functions of period 1, and $T \in \mathbb{R}_+$ represents the period. The object on the right is then also a periodic function of period 1. The aim is to make it be the zero function, because then $x(t) = \hat{x}(t/T)$ is a periodic orbit of Eq. (67). There are many choices for the spaces of periodic functions, which we call *loop spaces*. Perhaps the simplest is that $\hat{x} \in C^1(S^1, M)$ and the righthand side belongs to $C^0(S^1, TM)$, where $S^1$ denotes the circle of length 1 and $TM$ the tangent bundle to $M$. If $G$ is $C^1$ then so is $\Phi$.

Then, with a view to applying the implicit function theorem again, we can ask for which periodic orbits of the uncoupled system is $D\Phi$ invertible? Unfortunately, the answer is "none". The problem is that (for an autonomous system) the infinitesimal phase shift $(\dot{\hat{x}}, 0)$ is always in the kernel of $D\Phi$. But $\Phi$ is covariant with respect to

phase shifts, so this is not a real problem. It can be tackled by using Fredholm theory, which is the extension of the implicit function theorem when some non-invertibility is inevitable.

Now let us apply the above to a network system. Suppose the local dynamics on one site $o \in S$ has a non-degenerate periodic orbit (i.e. no normal Floquet multiplier $+1$), and a hyperbolic equilibrium at all the other sites. Then the resulting periodic orbit for the network is non-degenerate in the above sense and hence persists for some range of $\epsilon$, uniformly in the size of the network. Again one can address the questions of stability and finite coherence length. We find analogous results to those for equilibria.

What happens if we start from periodic orbits on two or more sites, with equilibria at the others? If the periods are in rational ratio, i.e. integer multiples of some common super-period $T$, then the full system has periodic orbits of period $T$, and one might hope to continue them to non-zero $\epsilon$. But this is false, because in addition to (overall) phase-shift degeneracy there are relative phase shift degeneracies in this case. In fact, the product of $N$ periodic orbits and arbitrarily many equilibria is an $N$-torus. If the periods are in rational ratio, the $N$-torus is foliated into an $(N-1)$-torus of periodic orbits of the same period. Hence there are automatically $N-1$ relative phase-shift degeneracies. It is easy to create couplings which destroy all these periodic orbits, for example by making the frequency ratio incommensurate. Nonetheless, we do expect persistence of the invariant $N$-torus, though the dynamics on the torus may become more complicated than periodic. This will be addressed in Section 8.

# 6    Uniformly Hyperbolic Sets

The persistence results for equilibria and periodic orbits have a powerful generalisation to aperiodic orbits, under an analogous condition of uniform hyperbolicity. It is simplest here to make an excursion into discrete-time, which avoids technical problems due to the continuous time-translation symmetry of autonomous ODEs. Thus we consider *coupled map lattices*[3] of the general form:

$$x_s^{t+1} = F_s(x_s^t) + K_s(x^t, \epsilon), s \in S, t \in \mathbb{Z}, \tag{69}$$

which we condense to the general form for a parametrised family of discrete-time dynamical systems:

$$x^{t+1} = G(x^t, \epsilon), x^t \in M. \tag{70}$$

It is clear that the problem of persistence of fixed points of Eq. (70) is virtually identical to the problem of persistence of equilibria of networks, with the only change being from a zero-finding problem to a fixed point problem. Thus the key condition of invertibility of $DG$ should be replaced by invertibility of $I - DG$. The problem of persistence of periodic orbits, of period $q$ say, of a coupled map lattice also reduces to

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[3]Note that the word "lattice" is used in a weaker sense than the pure mathematicians' sense: there is no need for $S$ to be closed under an operation of subtraction.

the same sort of problem, either by considering the $q^{th}$ iterate of $G$ or more simply by writing a system of $q$ equations for $x^0, \ldots, x^{q-1}$.

The standard definition of uniform hyperbolicity for dynamical systems involves a splitting of the tangent space into the direct sum of invariant expanding and contracting bundles, with uniform exponential estimates. I prefer to give a functional-analytic definition of uniform hyperbolicity, which can be proved to be equivalent to the usual one (see below), but is much more useful and allows generalisation to non-dynamical problems like elliptic PDEs[4], for example [1]. To lead into the definition, note that orbits of Eq. (70) (fixing $\epsilon$ for the moment) are fixed points of the operator $H : M^{\mathbf{Z}} \to M^{\mathbf{Z}}$ defined by

$$H(x)^t = G(x^{t-1}). \tag{71}$$

In the case of a coupled map lattice, $M$ is already a product of manifolds, one for each unit of the lattice, so $M^{\mathbf{Z}}$ is a product of manifolds, one for each point in space-time. For norm on $M^{\mathbf{Z}}$, use the supremum over the $\mathbf{Z}$-direction of the norms on $M$.

An orbit $x \in M^{\mathbf{Z}}$ of Eq. (70) is said to be *uniformly hyperbolic* if $I - DH$ is invertible. An invariant set $\Sigma$ is uniformly hyperbolic if its orbits are uniformly hyperbolic with a common bound on $\|(I - DH)^{-1}\|$ (and if $\Sigma$ is non-compact, a common module of continuity for $DH$ is required). This definition is not so new, in fact, as it is essentially Mather's characterisation of Anosov systems, presented to the Dutch Academy of Sciences in 1968, perhaps in this very room. Furthermore, its continuous-time analogue, named *exponential dichotomy*, goes back to 1958 or so (see [28])! For an introductory article which explains many aspects of the equivalence between this definition and the usual one, see [20], and for more see [32]. For the particular case of symplectic twist maps, see [6].

With this definition, persistence of uniformly hyperbolic orbits, uniformly in the size of the lattice, becomes immediate. It is just the same as for non-degenerate fixed points of a coupled map lattice, but with the lattice replaced by the grand lattice, and $G$ replaced by $H$. The mapping between the perturbed and the unperturbed orbits of uniformly hyperbolic sets is easily shown to be a homeomorphism. Alternative, but closely related, proofs were given for one-dimensional chains by [30] and [11].

The way the persistence results are applied to coupled map lattices differ from that in Section 2, however, as one would not usually start from a grand lattice with no coupling in time. Mind you, this latter concept, named the "anti-integrable limit" by Aubry [4], has proved extremely fruitful, and I will say a few words about it in Section 9. Instead one would start from the limit with no coupling in "space". If the dynamics of the individual units is uniformly hyperbolic, and uniformly so over the whole lattice, then the resulting orbits are uniformly hyperbolic and hence persist uniformly with respect to the coupling. For example, one could take a coupled map lattice with a Plykin attractor at one site and attracting fixed points at the others; this would result in an attractor with spatially localised chaos for weak coupling.

The continuous-time case is analogous, but complicated slightly by the fact that there is always time-translation degeneracy. This means that the definition of uniform

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[4]Note the unfortunate clash of terminology which renders many elliptic PDE problems uniformly hyperbolic!

hyperbolicity needs to be adapted and to establish a persistence result we have to allow the time-parametrisation of the orbits to change with $\epsilon$.

One aspect of uniform hyperbolicity which is important for applications is the robustness of uniformly hyperbolic orbits to small time-dependent perturbations. In fact, the notion of uniform hyperbolicity (using the above definition) is not at all limited to autonomous dynamical systems. In particular, if a system with a uniformly hyperbolic orbit is subjected to small forcing then the orbit has a unique continuation to a solution of the forced problem. In the most interesting case for applications, namely where the orbit is attracting, this solution represents the response for all initial conditions close to the original orbit. The solution can be found by continuation, as throughout these notes, and bounds can be deduced on the size of forcing for which one can be sure that the response remains within desired safety limits. Bishnani and I are in the process of developing such estimates, in continuous-time. We are particularly interested in adapting the measures of size of the forcing to the system in hand in order not to unnecessarily restrict the size of the forcing function. This technique should have a uniform extension to network problems.

Next I give the promised construction of the stable-unstable splitting for the tangent bundle on uniformly hyperbolic sets (in the discrete-time case). The construction of the splitting is the same for networks as for single dynamical systems; the only difference is that to deduce exponential decay estimates in time for a network we need to suppose suitable spatial structure for the coupling. Under this condition, we also obtain a finite coherence length property for the splitting.

If $I - DH$ is invertible then given a tangent vector $\xi$ at a point $x \in M$, define the tangent vector $\Xi \in TM^{\mathbf{Z}}$ along the orbit of $x$ by

$$\Xi^0 = \xi, \tag{72}$$

and all other components zero. Define the tangent vectors

$$\xi^- = -DG_{G^{-1}x}\pi^{(-1)}(I - DH)^{-1}\Xi \tag{73}$$
$$\xi^+ = \pi^0(I - DH)^{-1}\Xi, \tag{74}$$

at $x \in M$, where $\pi^t : M^{\mathbf{Z}} \to M$ is the component at time $t \in \mathbb{Z}$. Then $\xi^-$ has bounded backwards orbit, $\xi^+$ has bounded forward orbit, and

$$\xi^+ + \xi^- = \xi. \tag{75}$$

This is the desired splitting of $\xi$.

To show that the forwards, respectively backwards, orbits of $\xi^{\pm}$ decay exponentially, note that for a single dynamical system or finite network we can use Case 1 of Section 4, where the one dimension is taken to be time. Thus $(I - DH)^{-1}\Xi$ decays exponentially in both directions of time, and its forward and backward parts are in fact the forward and backward orbits of $\xi^{\pm}$ respectively. For an infinite network, or to obtain uniform results with respect to the size of the network, one has to impose hypotheses on the coupling which guarantee exponential decay of the matrix elements of $(I - DH)^{-1}$ in space-time. For example, the hypotheses Eq. (47) and (49) suffice

for an arbitrary graph, as explained at the end of Case 2. Then summing over the $t = $ constant sets in the space-time graph gives the desired exponential decay in time.

To complete the treatment of the splitting, we must show that the union $E_x^+$ of the $\xi^+$ over all tangent vectors $\xi$ at a given point $x \in M$, and the union $E_x^-$ of the $\xi^-$, are linear spaces forming a direct sum decomposition of $TM_x$. They are linear spaces because if $\xi_1^+$ and $\xi_2^+ \in E_x^+$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then let $\xi_\alpha = \alpha_1 \xi_1^+ + \alpha_2 \xi_2^+$ and apply Eq. (74), to deduce that $\xi_\alpha \in E_x^+$. Similarly $E_x^-$ is a linear subspace. We already proved that every $\xi \in TM_x$ can be split into a sum of vectors in $E_x^\pm$. It is clear that $E_x^+ \cap E_x^- = 0$, because the orbit $\Xi$ of any tangent vector $\xi$ in the intersection is bounded in both directions of time and hence is a solution of $(I - DH)\Xi = 0$, so $\Xi = 0$ by the invertibility of $I - DH$.

Continuity of the splitting follows from the exponential decay. In particular, the dimensions (in the finite dimensional case) of $E_x^\pm$ are constant over chain-transitive components of uniformly hyperbolic sets.

The spaces $E^\pm$ are traditionally called "stable" and "unstable" subspaces, respectively, but this is confusing terminology, as the stable subspace is unstable with respect to the dynamics and the unstable one is stable. I prefer to call $E_x^\pm$ the forward and backward contracting subspaces, respectively. Note again that some people use "expanding" for $E_x^-$, but the forward orbits of all vectors of $TM_x \backslash E_x^+$ are also eventually expanding.

For coupling with suitable spatial structure, the splitting can be shown to have a finite coherence length property, i.e. the matrix elements of the induced projections depends exponentially weakly on the distance between sites.

Forward and backward contracting manifolds can also be constructed from this point of view. They are the sets of points in $M$ whose forward, respectively backwards, orbits converge together, and the content of the "stable manifold theorem" is that they are the images of injective differentiable immersions from the forward and backward contracting subspaces, respectively into $M$. Note again that I avoid the terms "stable", "unstable" and "expanding". "Expanding" is additionally dangerous here because it can happen (e.g. the separatrix for a frictionless pendulum, and in general any homoclinic tangency) that tangent vectors to a backward contracting manifold also contract in forwards time! It would be interesting to obtain uniform estimates on the sizes of the contracting manifolds, especially in the case of attractors, where the union of the stable manifolds is the basin of attraction.

# 7  Structural Stability

The persistence results of the previous Sections are very nice, but leave open an important question: how do we know that new equilibria, or periodic orbits, or other forms of recurrent motion are not created on adding small coupling? More strongly, is the uncoupled system structurally stable, uniformly in the size of the network?

A $C^r$-dynamical system ($r \geq 1$ throughout this section) is said to be $C^r$-*structurally stable* if all $C^r$-small enough perturbations are topologically equivalent to it. Two flows $\phi_i : M_i \times \mathbb{R} \to M_i, i = 1, 2$ are *topologically equivalent* if there is a home-

omorphism $\Theta : M_1 \to M_2$ and a time-reparametrisation map $\tau : M_1 \times \mathbb{R} \to \mathbb{R}$ which is an orientation-preserving homeomorphism of $\mathbb{R}$ for each $x \in M_1$, such that

$$\tau(\phi_1(x,t),t') = \tau(x,t+t') - \tau(x,t), \tag{76}$$

$$\phi_2(\Theta(x),\tau(x,t)) = \Theta(\phi_1(x,t)). \tag{77}$$

The broadest notion of recurrence is "chain recurrence". The *chain-recurrent set* $R$ for a dynamical system $\dot{x} = G(x)$ is the set of points $x$ which, for all $\eta > 0$, lie on a periodic solution of the differential inclusion

$$\dot{x} \in B(G(x),\eta), \tag{78}$$

the ball of radius $\eta$ around $G(x)$.

The basic result of finite-dimensional structural stability theory is that the chain-recurrent set is $C^r$-structurally stable if it is uniformly hyperbolic (and the converse is proved for $r = 1$). So under this condition, no new chain recurrent behaviour is generated under perturbation.

If one wants structural stability of transient behaviour too, it is necessary to strengthen the hypotheses. An *AS system*[5] is one for which the chain recurrent set $R$ is uniformly hyperbolic and for all $x, y \in R$, the stable manifold of $x$ is transverse to the unstable manifold of $y$. Every AS system is $C^r$-structurally stable, and this is also proved to go both ways for $r = 1$. For an up-to-date introduction, see [31].

The question to ask here is whether structural stability can be proved with uniform estimates on the size of the network. This is almost certainly true. Hence we would deduce that in the example of Section 2, for instance, no new equilibria are created for $|\epsilon| < \epsilon_2$, some $\epsilon_2 > 0$, and in this range the dynamics is topologically equivalent to the uncoupled case. In particular, for this range of $\epsilon$, no travelling front solutions would exist, i.e. solutions with a region of units essentially in the 0-state and a region of units essentially in the 1-state, separated by a front which moves. This is known as "propagation failure" [16].

# 8   Normally Hyperbolic Sets

Let us now return to the question of what happens if several (say $N$) units, each with an attracting periodic orbit are coupled together weakly, the remaining units having attracting equilibria. The product system then has an attracting invariant torus of dimension $N$. I believe that this $N$-torus persists for small coupling, uniformly in the system size, subject to some uniformity in the attraction rates and periods of the periodic orbits. The dynamics on the $N$-torus, however, can and indeed typically will, become more complicated than the uncoupled case (for which the flow is conjugate to a uniform translation).

To justify this, consider the more general problem of persistence of normally hyperbolic sets. Roughly speaking, these are invariant subsets with a tangent bundle $T$ (e.g. a submanifold, but solenoids are also allowed, for instance), such that expansion

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[5]Following [32], "AS" stands for "Axiom A and Strong Transversality Condition".
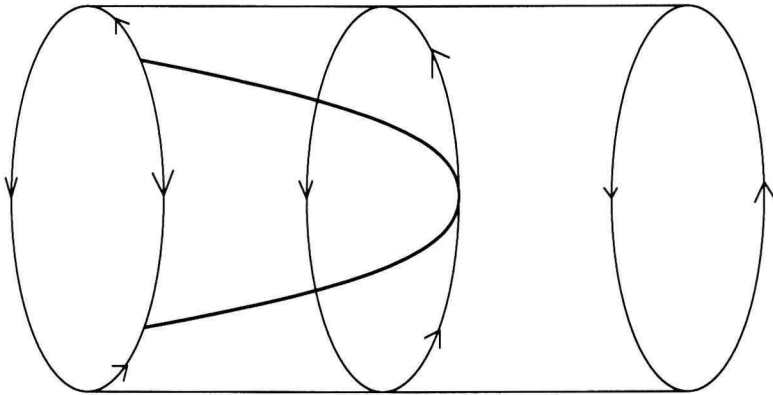
Figure 1: The attracting invariant cylinder for an uncoupled neuron.

and contraction in $T$ is weaker than expansion and contraction in the normal bundle $N$ (i.e. the total tangent space modulo $T$). There is a well established theory of persistence of normally hyperbolic sets for finite dimensional systems (e.g. [36]). The issue I wish to address is whether for a network the persistence results can be made uniform in its size.

The discrete-time case of this question was investigated by [11]. It would be interesting to check that everything works for the continuous-time case also. This would, for example, justify the reduction of a system of weakly coupled oscillators with attracting periodic orbits to a system of phase equations

$$\dot{\theta} = F(\theta), \theta \in (S^1)^S. \tag{79}$$

In the case that the motion is close to a rational rotation, averaging could then be used to reduce to phase-difference equations as proposed by [19], up to a small remainder term.

Another very interesting problem to study, which would be greatly helped by such a result on persistence of normally hyperbolic sets for networks, is a model that I wish to propose for (physiological) neural networks. Think of a neuron as a dynamical system given by the unfolding of a saddle-node on a cycle together with slow evolution in the "parameter", depending on the inputs (and its own state, but I will regard that as a coupling effect too). Thus the uncoupled neuron has a normally hyperbolic attracting invariant cylinder in its phase space, as sketched in Figure 1.

An uncoupled network of $N$ such units has a normally hyperbolic $N$-cylinder (i.e. product of $N$ cylinders). Assume that the effect of neuron $s$ going round the cylinder is to cause a slight parameter increase or decrease (depending whether the coupling is excitatory or inhibitory) on all neurons to which it outputs, including possibly itself. Then, provided the coupling is integrable in some sense, we expect the product system to continue to have a normally hyperbolic $N$-cylinder on which the motion is close to the uncoupled case, but with more interesting dynamics. The question is what sort of dynamics can it exhibit? In particular, can it exhibit "intelligence"?

# 9  Conservative case: equilibria

For Sections 9 and 10, I specialise to the case of Hamiltonian systems and their close relatives: time-reversible systems. A system is *Hamiltonian* if it can be written in the form

$$\dot{z} = JDH(z), \tag{80}$$

where $J$ is the isomorphism from 1-forms to tangent vectors induced by a symplectic form $\omega$, by

$$\omega(J\eta, \xi) = \eta(\xi), \forall \xi. \tag{81}$$

It is *time-reversible* if

$$\dot{z} = F(z) = -DR^{-1}F \circ R(z), \tag{82}$$

for some involution $R$, which shall be required to reverse precisely half the dimensions. The intersection of the two categories is especially relevant, in particular with the added condition that $R$ be anti-symplectic, i.e.

$$DR^{-1}JDR = -J. \tag{83}$$

A basic example which we shall treat is

$$\ddot{x}_s + V'(x_s) = \epsilon \sum_{r \in N_s} g_{rs}(x_r - x_s), \tag{84}$$

where $N_s$ is the set of neighbours of site $s$ in a graph $S$, $g_{rs}$ is symmetric, and $x_s \in \mathbb{R}$. This is Hamiltonian with

$$H(x,p) = \sum_{s \in S}(\frac{1}{2}p_s^2 + V(x_s)) + \frac{\epsilon}{2} \sum_{d(r,s)=1} g_{rs}(x_r - x_s)^2, \tag{85}$$

where the second sum is over unordered pairs $(r, s)$, and time-reversible with (anti-symplectic) involution

$$R(x,p) = (x, -p). \tag{86}$$

Persistence of equilibria for such systems is a special case of persistence of equilibria for general networks, so does not require special treatment. Nonetheless, there is a slight improvement which can be made using the Hamiltonian structure. Instead of solving $JDH(z) = 0$ one can solve $G(z) := DH(z) = 0$, and so $DG$ is symmetric, which can improve estimates, particularly on the exponential decay. This approach has been powerful in deducing interesting results for symplectic twist maps, because their orbits are equivalent to the equilibria of an associated one-dimensional translation-invariant nearest neighbour chain. The uncoupled case is a singular limit from the symplectic map point of view, christened the "anti-integrable limit" by Aubry. The ideas were first developed by [4] and [35], and subsequently used to construct many interesting types of orbit for symplectic twist maps, e.g. cantori of various types [7, 26], and bifurcations [3, 17].

Note that one special feature of the Hamiltonian case is that the signature of the second variation $D^2H$ of the Hamiltonian can not change as long as $D^2H$ remains

non-degenerate. This does not prevent stability change, but limits the possibilities. In particular, if it starts definite it must remain definite and the equilibrium remains stable as long as it can be continued (in fact, fully so with respect to $\ell_2$-norm, not just linearly, though in general not with respect to sup-norm). If $D^2H$ starts indefinite then we can nonetheless deduce a range of $\epsilon$ for which the stability type of the equilibrium remains unchanged, as in Section 3, provided some additional hypotheses are satisfied (separation of spectrum with opposite signatures). The notion of stability type has to be refined in the conservative case to allow a central component in the splitting and to take account of the Krein signature of pure imaginary spectrum (e.g. [21]).

# 10   Conservative case: periodic orbits

In contrast to the case of equilibria, continuation of periodic orbits of conservative systems requires special treatment. This is because for an autonomous Hamiltonian system every periodic orbit is degenerate, owing to energy conservation. Of course, this is easily dealt with by restricting the vector field to an energy surface, giving a family of systems with one extra parameter, namely the energy. But it can be dealt with in other ways and I shall discuss one.

Similarly, symmetric periodic orbits of time-reversible systems, that is orbits which are sent to their time-reverse by the involution $R$, are automatically degenerate, because denoting the total dimension by $2N$, symmetric periodic orbits correspond to intersections of the orbit ($(N+1)$-dimensional) of a reflection surface with the reflection surface ($N$-dimensional), which is 1-dimensional if transverse. This problem is not as easily dealt with as the autonomous Hamiltonian case, and thus the method I will sketch is particularly useful.

The method that Aubry and I proposed for these two classes of problems is to assume anharmonicity and to continue at constant period [24]. Anharmonicity means that the period of the orbit varies non-trivially with respect to energy (or appropriate parameter) along the family of periodic orbits. Of course, anharmonicity is not always present, and then other approaches are required, but it is a common case.

This method allows us to construct "self-localised vibrations" [24, 23], discovered numerically by [33], and named "discrete breathers" by [10] and "nonlinear localised excitations" by [15].

As I ran out of time to write up these notes, my treatment of this problem will be regrettably brief. The reader is referred to [24] for more details and to [23] for some suggestions for directions for future work.

I begin with the simplest case, namely "1-site breathers". Each unit of the uncoupled network is assumed to be a Hamiltonian or time-reversible system and to possess a one-parameter family of periodic orbits. We parametrise the family by the *action* $I = \int p.dq$ in the Hamiltonian case, and some analogous measure of its size in the time-reversible case (if not also Hamiltonian). Denote its period by $T(I)$. For $\epsilon = 0$, choose a periodic orbit of the local system on one unit $o \in S$ and put all other units on equilibrium points. This gives a periodic orbit $\gamma$ of the product system.

**Hypotheses** 1. *Anharmonicity*: $dT/dI \neq 0$ for the chosen periodic orbit of unit $o$.
2. *Non-resonance*: $d(\{\omega_s : s \in S \setminus \{o\}\}, \omega\mathbb{Z}) > 0$, where $\omega = 2\pi/T$ and $\omega_s$ is the frequency of infinitesimal vibrations about the equilibrium on site $s$.
3. *Uniformity*: The vector field is $C^1$ with respect to sup norm over sites, with uniform bounds with respect to the size of the network.

**Theorem 3** *Under hypotheses 1, 2 and 3, $\gamma$ has a unique (up to phase shift) continuation $\gamma(\epsilon)$ as a periodic orbit of the same period $T$ for $|\epsilon| < \epsilon_0$, for some $\epsilon_0 > 0$, uniformly in the size of the network.*

Our proof is by the implicit function theorem for a $C^1$ function $\Phi$ from a space of $C^1$ loops to a space of $C^0$ loops, as in Section 5, but here we work with fixed period $T$ and impose some restrictions related to the Hamiltonian or time-reversible structure. The point of the above hypotheses is to make $D\Phi$ invertible; physically, they allow the breather to detune from the phonons and avoid harmonics of the frequency falling in the phonon band. It should be possible to do a proof in a space of Fourier coefficients too; indeed this was our original approach but we ran into technical problems making sure that $\Phi$ was $C^1$. I have subsequently found out how to prove this and intend to write up the proof soon.

In [24] we also prove finite coherence length for breathers in one-dimensional nearest neighbour chains. This can easily be generalised to other forms of network with suitable spatial structure, e.g. any which fit cases 1,2 or 3 of Section 4.

One question is what estimates on the continuation we can obtain. We did not yet obtain any explicit estimates, as one step in our proof was to transform to action-angle variables at the excited site, and the continuation estimates will depend on the size of the derivative of this transformation and its inverse, which in general are not explicitly calculable. However, they are calculable for certain potentials and estimable for many others, and it would be very interesting to do this and work out some explicit estimates for the continuation of breathers. Once the Fourier coefficient proof is available, I believe that it would be the best one to use, as I think it is better adapted to the problem.

Another question is about stability of the resulting breathers. One can not expect them to be linearly stable uniformly in the system size, because of the possibility of constructive interference of phonons. I believe, however, that the 1-site breathers obtained above are linearly stable in $\ell_2$ for $\epsilon < \epsilon_1$, for some $\epsilon_1 > 0$ uniform in the system size, provided that the non-resonance condition is strengthened to the following:

**Stability condition** There exists $r > 0$ such that $d(\{e^{i\omega_s T} : s \neq o\}, \{e^{-i\omega_s T} : s \neq o\}) \geq r$, where $\omega_s$ is the frequency of infinitesimal vibrations about the equilibrium for site $s$, with sign chosen according to the signature of the second variation of the Hamiltonian there (positive for the minima of $V$ in the example of Eq. (84)).

In particular, $\omega$ must avoid $2|\omega_s|/n$ for all $n \in \mathcal{N}$, $s \in S$. The reason for the stability condition is that it ensures that no collisions of Floquet multipliers of opposite Krein signature can occur, which implies spectral stability is preserved (e.g. [2]). As

before, it should be true that the spectrum moves controllably with respect to $\epsilon$ and hence the existence of $\epsilon_1$ will follow.

It is unlikely that the breathers would be fully stable, as the usual arguments about the set of KAM tori not dividing phase space apply as soon as the number of sites exceeds two. However, they are Nekhoroshev stable, meaning that if you start close (in $\ell_2$-norm) then you stay close for an exponentially long time [9].

The next issue is existence of multi-site breathers. Here again, the interested reader is referred to [24], where we construct them for time-reversible systems subject to two conditions on the solution in the uncoupled case: firstly, there must be a common period, satisfying the non-resonance condition, and secondly, there must be an origin of time with respect to which the solution is time-symmetric. In that paper, we also suggested a method for proving existence of multisite breathers without using time-reversal invariance. I have subsequently come up with what I believe will be a better approach, a Melnikov-type method, which I hope to write up soon. $N$-site breathers will correspond to critical points of a function on an $(N-1)$-torus. There should be relations between the spectral stability type of the breather and the index of the critical point. The $(N-1)$-torus should give a non-invariant but Nekhoroshev-stable $N$-torus in the phase space.

It should not be necessary to have an "uncoupled" case from which to continue breathers. Any starting point possessing non-degenerate breathers would suffice; this is probably the case for Flach's breathers in homogeneous Fermi-Pasta-Ulam chains [14], for example. Furthermore, the anharmonicity condition is probably not really required: one could continue at constant energy rather than constant period. This would allow one to prove existence of impurity modes for nonlinear systems, by continuation from the linear case.

Another question is about existence of quasiperiodic breathers. There are works (e.g.[18]) which prove existence of invariant $N$-tori of quasiperiodic motions for $N = 2, 3, \ldots$ in large and infinite-dimensional Hamiltonian systems, but it is not clear whether they can be applied here. Also, in some numerics (e.g. [34]) travelling breathers are observed, and it is a challenge to try to prove or disprove their existence.

The real issue with discrete breathers is to explain why typical initial conditions seem to be "attracted" to a distribution of breathers. My guess [23] is that it is a similar phenomenon to the stickiness of elliptic islands for area-preserving maps, but this merits much investigation. Then it would be very interesting to develop their physical significance and investigate their role for statistical mechanics.

# 11  Limits to continuation

I conclude by raising two questions which I am not yet in a position to answer: what are the limits to continuation, and what happens beyond?

Firstly, how are equilibria or periodic orbits or uniformly hyperbolic sets or normally hyperbolic sets of a network lost (if at all) as coupling increases? It is clear that they can undergo bifurcations just as for finite-dimensional systems, but are there new possibilities for infinite networks? Aubry and Marin are investigating this

numerically for the case of discrete breathers.

A simpler setting in which quite a lot can be understood is the bifurcations of the set of equilibria for 1D nearest neighbour chains of particles in a cubic potential, because this reduces to studying orbits of the area-preserving Hénon map. The analogues of the breathers are the orbits whose symbol sequence has all but finitely many 0s in the uncoupled case (0 labelling the potential well and 1 labelling the local maximum). In particular, the first bifurcation is known to be the annihilation of the symbol sequences $0^\infty 1010^\infty$ and $0^\infty 1110^\infty$ (Smillie), which occurs without loss of finite coherence length, but the symbol sequences $0^\infty 10^\infty$ and $0^\infty 110^\infty$ almost certainly are lost by annihilation together with $0^\infty$ and $1^\infty$, and the coherence length goes to infinity there.

Secondly, what new phenomena lie beyond the regime of continuation from the uncoupled limit? This is a particularly interesting question in the case of loss of a uniformly hyperbolic attractor.

# Acknowledgements

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