Reaction telegraph equations and random walk systems

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Abstract

In reaction diffusion equations spatial spread is modeled by Brownian motion. If Brownian motion is replaced by a correlated random walk or related processes then semilinear random walk systems and reaction telegraph equations are obtained. We investigate these systems and discuss their relevance for modeling and applications in comparison with reaction diffusion equations.

1 Introduction

In all fields of science, economics and engineering, the interactions between several types or species have to be modeled. If the number of particles is large then one will describe the different types by their population sizes or densities and the reactions between them by (systems of) ordinary differential equations $\dot{u} = f(u)$. The right hand sides of these equations often assume the form of polynomials or rational functions (e.g. derived from stoichiometric laws) but in principle any type of nonlinearity can occur. It should be underlined that the description by differential equations is a most natural one since these equations describe nothing else than the law that connects the change of the system "in the immediate future" to the present state. Thus the concept of differential equation arises naturally from the proper choice of the state and the assumption of causality. The usefulness of the differential equations approach is thus limited only by the assumption of large numbers. Of course it is tacitly assumed that the nonlinearity of the equation describes the underlying process correctly. That may not always be taken for granted if one thinks of complicated chemical molecules that react only in specified geometric configurations.

Modelling spatial spread seems less straightforward. In general one will either start from deterministic balance considerations or from stochastic processes such as Brownian motion. Both approaches lead, with certain assumptions that are both intuitively plausible and supported by experiment, to the diffusion equation or, in another physical context, to the heat equation.

Assuming that reaction and diffusion act together one arrives at a type of system that has become known as reaction diffusion equation ([17, 38, 62, 68]). These systems have the form

$$\frac{\partial u_i}{\partial t} = D_i \Delta u_i + f_i(u_1, \ldots, u_m), \quad i = 1, \ldots, m, \quad x \in \Omega \subset \mathbb{R}^n,$$

or, in condensed notation,

$$u_t = D\Delta u + f(u). \tag{1}$$

The fact that the Laplacian and the reaction term appear as a sum is worth noticing. The most important example is the scalar reaction diffusion equation, $m = n = 1$,

$$u_t = Du_{xx} + f(u). \tag{2}$$

The diffusion equation can be justified *a posteriori* since many stochastic processes can be approximated by diffusion processes. Nevertheless, the diffusion equation has several deficiencies. The Laplace operator is strictly local, i.e. long distance effects are not taken into account. The diffusion equation shows the phenomenon of infinite propagation, i.e. particles can move with arbitrarily high speed, quite in contrast to common laws of physics.

The question arises what type of equation is obtained if the underlying stochastic process is not Brownian motion and if a diffusion approximation is not made. There is a wide variety of stochastic processes which can be used to model spatial spread and that lead to other types of evolution equations.

It turns out that these equations are quite interesting and sometimes show features different from those of reaction diffusion equations. On the other hand they appear mathematically more difficult. Probably they are not difficult *per se* but only in comparison with the large body of analytical tools that are available for the diffusion equation.

A first candidate for a stochastic process different from Brownian motion is a correlated random walk. Whereas in Brownian motion the directions of motion in successive time intervals are uncorrelated, in the correlated random walk the particles have some inertia. The simplest case is a walk in one space dimension with constant speed. This process leads to a hyperbolic system on the line. Other processes lead to integral equations or again to hyperbolic systems in higher dimensions.

Even in the scalar case the reaction diffusion equation (2) can be and has been interpreted in various ways. In a probabilistic setting, solutions to the linear diffusion equation are seen as probability densities ($u(t,x) \geq 0$, $\int_{-\infty}^{\infty} u(t,x) = 1$), or probability distributions ($\lim_{x\to-\infty} u(t,x) = 0$, $\lim_{x\to\infty} u(t,x) \to 1$, $u(t,x)$ nondecreasing in x). In the latter case $u(t,x)$ is the probability that the position $x_1(t)$ of the moving particle satisfies $x_1(t) < x$.

Fisher [18], in what is probably the first paper on a reaction diffusion equation, considered the spread of a genetic trait. In his interpretation $u(t,x)$ is the proportion of individuals that carry the trait at the point x at time t, in other words, the value $u(t,x)$ is a probability and thus $u(t,x) \in [0,1]$. Fisher studied the propagation of the trait in the form of a travelling wave solution of Eq. (2) which develops from an initial datum $u(0,x) = 1$ for $x < 0$, $u(0,x) = 0$ for $x > 0$. Shortly later, Kolmogorov, Petrovskij, Piskunov [48] see $u(t,x)$ as the density of matter at the position x at time t. Hence $u(t,x) \geq 0$ but no upper bound for u is required. In McKean's [51] model for particles that undergo a branching process and perform Brownian motion, the function $u(t,x)$ is again a probability distribution. We shall see that these distinctions carry over to random walk systems and reaction telegraph equations. The theory of vector valued reaction diffusion equations has been stimulated by biological models, e.g. for pattern formation (Turing models), nerve axon models (Hodgkin-Huxley

equations) and various problems in ecology. We shall discuss whether it is meaningful to consider extensions of these problems.

The organization of the paper is as follows. In Section 2 we introduce a general transport equation and derive as special cases or approximations the equations of a correlated random walk in one dimension and general random walk systems. We relate these equations to the telegraph equation for electromagnetic signals and to generalized heat equations. In Section 3 we couple these equations to reaction processes and thus derive semilinear hyperbolic systems that replace reaction diffusion equations. In Section 4 we study these systems on bounded domains, we find the appropriate boundary conditions and we study the corresponding eigenvalue problems, in particular the connection between domain size and stability. In Section 5 we recall the connection between one-dimensional problems and branching random walks and the travelling front problem. In Section 6 we quote results on pattern formation, space dependent diffusion, free boundary value problems, and nerve axon equations.

2 Models for spatial spread

In Brownian motion the state of the particle is given by its position in space. The particle does not have an assigned speed. Brownian motion is well suited to describe processes where the position of the particle is determined by many independent effects. When the particle has some memory or, in particular, if it has a well-defined velocity, then Brownian motion may not be an appropriate model. Of course these considerations are subject to choice of scale, and Brownian motion will usually appear as a limit case. Selecting velocity, in addition to location, as the most important variable, is motivated by various applications from physics, chemistry, and biology.

The state of the particle is given by its position $x \in \mathbb{R}^n$ and its velocity $s \in \mathbb{R}^n$. Let $u(t, x, s)$ be the density of particles at time t. If particles just move and do not change velocity then the evolution of this density is described by the equation

$$\frac{\partial u}{\partial t} + \nabla_x \cdot su = 0.$$

Here $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, the $\cdot$ denotes the formal inner product. In coordinate notation the equation reads

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} s_j \frac{\partial u}{\partial x_j} = 0.$$

This equation says that each particle moves with "its" velocity along a straight line. Now assume that particles stop their motion at random times determined by a Poisson process with parameter $\tilde{\mu}$, and then select a new velocity. Let $K(\cdot, \tilde{s})$ be the density of the new velocity, given the previous velocity is $\tilde{s}$. Then K must have the properties $K(s, \tilde{s}) \geq 0$, $\int_{\mathbb{R}^n} K(s, \tilde{s}) ds = 1$, and the equation of motion is

$$\frac{\partial u}{\partial t} + \nabla_x \cdot su = -\tilde{\mu} u + \tilde{\mu} \int_{\mathbb{R}^n} K(s, \tilde{s}) u(t, x, \tilde{s}) d\tilde{s}. \tag{3}$$

This equation describes pure motion. No particle is produced or deleted. The total number of particles $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(t,x,s)dsdx$ (in case it is bounded) is an invariant of motion. With respect to the space variable x the system is fully symmetric. So far no symmetry with respect to the variable s has been assumed. Eq. (3) and some nonlinear versions have been introduced by Othmer, Dunbar and Alt [58] as the "velocity jump process" (as opposed to a "position jump process"). The equation is formally close to a Boltzmann equation [9] or a Broadwell system [5], [66] with the difference that here particles turn spontaneously and not in response to collisions.

There are various special cases that are perhaps easier to study than the full problem. In those cases where the velocity ranges over a manifold of lower dimension it is practical to formulate these problems *ab initio* rather than as special cases or as limit cases of Eq. (3). If the speed is constant then $u(t,x,s)$ is defined on $\mathbb{R}\times\mathbb{R}^n\times S^{n-1}$ where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

One can specialize to a symmetric convolution kernel K on S^{n-1}. With the convolution kernel one can apply a Taylor expansion and replace the integral operator by the second moment term. Then one obtains the equation

$$\frac{\partial u}{\partial t} + \nabla_x \cdot su = \tilde{\mu}\Delta_s u \tag{4}$$

where Δ_s is the Laplace-Beltrami-Operator on S^{n-1}. If the velocity assumes only finitely many values s_i, $i=1,\dots,m$, then Eq. (3) is conveniently written as a system of equations for the variables $u^i(t,x)=u(t,x,s_i)$ (as in [60]),

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^{n} s_{ij}\frac{\partial u^i}{x_j} = -\tilde{\mu}u^i + \tilde{\mu}\sum_{l=1}^{m} K_{il}u^l, \quad i=1,\dots,m, \tag{5}$$

where the kernel in Eq. (3) is replaced by a matrix with $K_{il}\geq 0$, $\sum_{i=1}^m K_{il}=1$, $l=1,\dots,m$. We shall further discuss the general case at the end of this section. The simplest case arises for one space dimension, $n=1$, and only two velocities, $m=2$. If the velocities and the matrix K are chosen in a symmetric way, $s_1=\gamma>0$, $s_2=-\gamma$, $K_{11}=K_{22}=1-\tau$, $K_{12}=K_{21}=\tau$, and if the dependent variables are denoted by $u^1=u^+$, $u^2=u^-$, then the system assumes the form

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= -\tilde{\mu}u^+ + \tilde{\mu}((1-\tau)u^+ + \tau u^-),\\ u_t^- - \gamma u_x^- &= -\tilde{\mu}u^- + \tilde{\mu}(\tau u^+ + (1-\tau)u^-). \end{aligned}$$

Redefining $\tilde{\mu}\tau$ as μ we arrive at the normalized problem

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+),\\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-). \end{aligned} \tag{6}$$

This problem has a probabilistic interpretation: The function (u^+,u^-) is the probability density of a particle performing a correlated random walk on the real line with speed $\gamma>0$ and turning rate $\mu>0$. Notice that the parameter μ in Eq. (6) is the

rate at which the particle changes direction whereas in Eq. (3) the parameter $\tilde{\mu}$ is the rate at which the particle chooses a new direction.

In Eq. (6) introduce the total particle number and the flow

$$u = u^+ + u^-, \quad v = u^+ - u^-. \tag{7}$$

In terms of these variables the system (6) can be written

$$\begin{aligned} u_t + \gamma v_x &= 0, \\ v_t + \gamma u_x &= -2\mu v. \end{aligned} \tag{8}$$

Of course u^+, u^- can be recovered as $u^+ = (u+v)/2$, $u^- = (u-v)/2$.

For later use we introduce one further reformulation. Define $\gamma v = \tilde{v}$. Then Eq. (8) becomes

$$\begin{aligned} u_t + \tilde{v}_x &= 0, \\ \frac{1}{2\mu}\tilde{v}_t + \frac{\gamma^2}{2\mu}u_x + \tilde{v} &= 0. \end{aligned} \tag{9}$$

The system (6) or (8) is the starting point of the present paper. As we shall see in the sequel, this *random walk system* assumes the role of the one-dimensional diffusion equation when finite speed of propagation is required.

It seems that Taylor [69] and Fürth [20] were the first who considered a correlated random walk, and S.Goldstein [25] performed a detailed analysis of such systems in the discrete and in the continuous case, see also [47], [12] and the exposition in [74]. A particle moves on the real line with constant speed $\gamma > 0$. The direction of motion is governed by a Poisson process with parameter μ. Thus at any time (except when direction is changed) the state of the particle is given by its location $x \in \mathbb{R}$ and its direction of motion. The state space is $\mathbb{R} \cup \{\pm\}$. The probability density for the state of the particle at time t is a function on the state space which we write $(u^+(t,\cdot), u^-(t,\cdot))$. The evolution of this density is governed by the hyperbolic system (6). For smooth initial data the functions (u^+, u^-) form a classical solution. For non-smooth data Eq. (6) must be seen as shorthand notation for a related integral equation. For the stochastic interpretation of Eq. (6) or Eq. (8) the density satisfies $u^+(t,x) \geq 0$, $u^-(t,x) \geq 0$, $\int_{-\infty}^{\infty}(u^+(t,x) + u^-(t,x))dx = 1$. If (u^+, u^-) is interpreted as particle density then it makes sense to consider non-integrable solutions.

The systems (6) and (8) are equivalent. If (u,v) is a C^2 solution of the system (8) then

$$u_{tt} + \gamma v_{xt} = 0, \quad \gamma v_{tx} + \gamma^2 u_{xx} = -2\mu\gamma v_x = 2\mu u_t,$$

and thus

$$u_{tt} + 2\mu u_t = \gamma^2 u_{xx}. \tag{10}$$

This transition has been found by Kac [45]: For any smooth solution of Eq. (8) the particle density u satisfies the telegraph equation (10). However, in this transition a constant is lost. If (u,v) is a solution of (8) then $(u, v + ce^{-2\mu t})$, c any constant, is

a one-parameter family of solutions of (8), and all these are mapped into the same solution of Eq. (10). Assume two solutions (u, v) and $(u, \tilde{v})$ of (8) differ only in the second component. Then, by the first equation of (8), the difference $\tilde{v} - v$ is a function of t only, and by the second equation, this function is $ce^{-2\mu t}$.

On the other hand, assume u is a C^2 solution of (10). Then define, with some constant c,

$$v_0(x) = -\gamma \int_0^x u_t(0, y)dy + c, \tag{11}$$

$$v(t, x) = v_0(x)e^{-2\mu t} - \gamma \int_0^t e^{-2\mu(t-s)} u_x(s, x)ds. \tag{12}$$

Then the second equation of (8) is satisfied, and with assistance of Eq. (10) it follows that

$$\begin{aligned} e^{2\mu t}\gamma v_x(t, x) &= \gamma v_x(0, x) - \gamma^2 \int_0^t e^{2\mu s} u_{xx}(s, x)ds \\ &= \gamma v_x(0, x) - \int_0^t e^{2\mu s}[u_{tt}(s, x) + 2\mu u_t(s, x)]ds \\ &= \gamma v_x(0, x) - e^{2\mu t} u_t(t, x) + u_t(0, x). \end{aligned}$$

Thus

$$[u_t + \gamma v_x](t, x) = e^{-2\mu t}[u_t + \gamma v_x](0, x) = 0.$$

Hence every C^2 solution u of (10) is part of a C^2 solution (u, v) of (8). Thus there is a one-to-one correspondence between the solutions of Eq. (10) and one-parameter families of solutions of Eq. (8). The true nature of the constant c will show up in the case of several space dimensions.

If the telegraph equation (10) is written in the form

$$\frac{1}{2\mu} u_{tt} + u_t = \frac{\gamma^2}{2\mu} u_{xx} \tag{13}$$

then one sees immediately its connection to the diffusion equation. For $\gamma \to \infty$, $\mu \to \infty$ such that $\gamma^2/(2\mu) \to D > 0$, Eq. (13) becomes formally the diffusion equation (2). Thus Brownian motion is obtained as the limit of a correlated random walk if the speed becomes large and the turning rate becomes large (the free path length becomes short) in such a way that the limit $\gamma^2/(2\mu)$ exists.

Here the telegraph equation has been compared to the extreme situation where the free path length becomes very small. One can also compare to the other extreme of very large free path length, i.e. to the wave equation

$$U_{tt} = \gamma^2 U_{xx}. \tag{14}$$

Kac [45] has found a principle to produce solutions of the telegraph equation from those of the wave equation. Kaplan [46] has given an elegant proof and he has outlined the scope of this principle. A Poisson process with intensity $\mu > 0$ is a random variable $N(t)$ that counts events. Let $N(0) = 0$. The probability that an event occurs in $(t, t + \Delta t)$ is $\mu\Delta t + o(\Delta t)$. Then $Prob\{N(t) = k\} = (\mu t)^k e^{-\mu t}/k!$. Introduce the random variable $T(t)$ by

$$T(t) = \int_0^t (-1)^{N(s)} ds. \tag{15}$$

Then $-t \leq T(t) \leq t$. Kaplan shows the following. Let $U \in C^2(\mathbb{R})$, $U = U(t)$. Define

$$u(t) = Exp\{U(T(t))\}. \tag{16}$$

Then

$$\ddot{u}(t) + 2\mu\,\dot{u}(t) = Exp\{\ddot{U}(T(t))\}, \tag{17}$$

in particular, for $U(t) = t$, $Exp\{T(t)\} = (1 - e^{-2\mu t})/(2\mu)$.

Introduce a space variable x. Let $U = U(t,x)$ be in $C^2(\mathbb{R}^2)$ and satisfy Eq. (14). Then, since ∂_x^2 and Exp commute, the function

$$u(t,x) = Exp\{U(T(t),x)\} \tag{18}$$

satisfies Eq. (10) and $\lim_{t\to 0} u(t,x) = U(0,x)$, $\lim_{t\to 0} u_t(t,x) = U_t(0,x)$. Eq. (18) says that, for any fixed x, the value $u(t,x)$ is an average of the values $U(t,x)$ where the weight is independent of the solution. Furthermore Kaplan shows that $A(t,x) = Prob\{T(t) \leq x/\gamma\}$ is a distribution function, and

$$u(t,x) = \int_{-\infty}^{\infty} U(s,x) d_s A(t,s). \tag{19}$$

The function $A = A(t,x)$ is itself a solution of Eq. (10), with $A(0,x) = H(x)$, $A_t(0,x) = -\delta(x)$, H and δ being the Heaviside and delta function, respectively. These ideas have been generalized to operator equations in [43], [42].

The telegraph equation appears in the work of Kirchhoff (1857) (according to [49]), it has been derived from Maxwell's equations by Lord Kelvin and O.Heaviside (1876), it has been studied by Heaviside, Du Bois-Reymond [13], Poincaré [61], Picard [59], and many others. The book by Lieberstein [49] gives an excellent historical account of the physical applications, in particular on the idea to balance large leakage in long cables by introducing additional self-induction. Many insights into the mathematical developments can be gained from Riemann-Weber [72], § 125. Consider a transmission line of two parallel wires of length l. Let R be the Ohm resistance, L the self-induction, C the capacity, and A the loss of isolation (leakage), each per unit of length. Let v be the voltage and i the cross current. These variables satisfy the equations

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} + Ri = 0, \quad \frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Av = 0.$$

Differentiate the first equation with respect to x and the second with respect to t, then eliminate the mixed derivatives and obtain the second order equation

$$LCv_{tt} + (AL + RC)v_t = v_{xx} - RAv. \tag{20}$$

This equation is the telegraph equation or, as it is called in the earlier literature, the telegrapher's equation or telegraphist's equation. The function i satisfies the same equation.

As has been indicated above, the telegraph equation is in some sense between the wave equation and the diffusion equation. Assume R, C, A are given. If L is small

then the equation is essentially a heat equation with a decay term. Signals become flattened out. If L is large then (although $AL + RC$ also increases) the equation is truly hyperbolic, sharp signals can be transmitted. For the corresponding initial value problems on the real axis, there is d'Alembert's formula for the wave equation and Poisson's formula for the diffusion equation. There is a similar well known formula for the telegraph equation where the kernel is a Bessel function with variable argument, see [67] for a detailed discussion.

Poincaré [61] derived this formula and considered it for initial data with compact support. He writes: " ... These results lead to various observations. First one sees that the head of the perturbation moves with a certain speed in such a way that the perturbation is zero in front of the head, in contrast to what happens in Fourier's theory and in agreement with the laws of propagation of light and sound by plane waves, derived from the equation of the vibrating string. But, with respect to the latter case, there is an important difference, because the perturbation, while propagating, leaves a nonzero remainder ... "

Poincaré's comments are interesting for several reasons. First he finds it remarkable that a model for the propagation of electric signals where no wave speed or particle speed has been introduced *a priori*, nevertheless produces a well-defined wave speed. This observation is trivial from what we presently know about hyperbolic systems, but it is not trivial looking at the physical problem. The second remarkable fact is that Poincaré relates the telegraph equation to Fourier's theory of heat. We shall follow this line of thought later in this section.

In the appropriate initial value problem for the system (6) the values of the functions u^+, u^- are prescribed for $t = 0$,

$$u^+(0,x) = u_0^+(x), \quad u^-(0,x) = u_0^-(x). \tag{21}$$

Then the initial data for Eq. (8) are

$$u(0,x) = u_0(x) = u_0^+(x) + u_0^-(x), \quad v(0,x) = v_0(x) = u_0^+(x) - u_0^-(x). \tag{22}$$

The usual initial data for the telegraph equation (10) are

$$u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x). \tag{23}$$

We follow the initial conditions in the transition from the random walk system (8) to the telegraph equation (10). Let (u,v) be a solution of Eq. (8) with initial data (22). Then u satisfies Eqs. (10) and (23) with

$$\phi(x) = u_0(x), \quad \psi(x) = -\gamma v_0'(x). \tag{24}$$

Let u be a solution of Eq. (10) with initial data (23). Then define v by Eqs. (11)–(12) and (u,v) satisfies Eqs. (8), (22) with $v(0,x) = -\gamma \int_0^x \psi(y)dy + c$.

The classical formula (see, e.g. [67]) for the solution to the initial value problem (10) and (23) in terms of the Riemann function can be used to find a "Poisson formula" for the random walk problem (6) and (21).

Let $I_\nu(x) = e^{\nu\pi i} J_\nu(ix)$ be the Bessel function of purely imaginary argument, in particular

$$I_0(x) = J_0(ix) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} (\frac{x}{2})^{2k},$$

$I_0'(x) = I_1(x)$. The functions $I_0(x)$, $I_1(x)$ are positive for $x > 0$, and $I_1(x)/x$ is analytic. Define the kernels, for $x - \gamma t \leq y \leq x + \gamma t$,

$$K(t,x,y) = \frac{\mu e^{-\mu t}}{2\gamma} I_0(\frac{\mu}{\gamma}\sqrt{\gamma^2 t^2 - (y-x)^2}), \tag{25}$$

$$K_\pm(t,x,y) = \frac{\mu e^{-\mu t}}{2\gamma} \frac{I_1(\frac{\mu}{\gamma}\sqrt{\gamma^2 t^2 - (y-x)^2})}{\sqrt{\gamma^2 t^2 - (y-x)^2}} (\gamma t \mp (y-x)). \tag{26}$$

The solution to the initial value problem (6)and (21) reads

$$\begin{aligned} u^+(t,x) &= u_0^+(x-\gamma t)e^{-\mu t} + \int_{x-\gamma t}^{x+\gamma t} K_+(t,x,y) u_0^+(y) dy \\ &\quad + \int_{x-\gamma t}^{x+\gamma t} K(t,x,y) u_0^-(y) dy, \\ u^-(t,x) &= u_0^-(x+\gamma t)e^{-\mu t} + \int_{x-\gamma t}^{x+\gamma t} K_-(t,x,y) u_0^-(y) dy \\ &\quad + \int_{x-\gamma t}^{x+\gamma t} K(t,x,y) u_0^+(y) dy. \end{aligned} \tag{27}$$

One sees immediately that the solution depends in a monotone way on the initial data and that $u(t,x)$ depends only on data in $[x - \gamma t, x + \gamma t]$. Furthermore one sees how discontinuities of the initial data decay. In each equation the first term contains the discontinuity whereas the integral terms represent continuous functions. This formula is more symmetric than the corresponding formula for the telegraph equation. Of course it makes sense for any locally bounded and measurable initial data, thus extending the differential equation to an evolutionary system.

It is not evident how the concept of a random walk system (9) should be generalized to several space dimensions. One possible approach is guided by the theory of heat. The heat equation or Fourier's law (Fourier 1822) is the classical model for heat conduction. It is equivalent to the following two assumptions: A conservation law for the temperature u and the heat flow v,

$$\kappa\rho\, u_t + \operatorname{div} v = 0 \tag{28}$$

where ρ is the density and κ is the heat capacity. The heat flow v is proportional to the negative gradient of the temperature,

$$v = -k \operatorname{grad} u \tag{29}$$

where k is the heat conductivity. If Eq. (29) is introduced into Eq. (28) then the heat equation

$$u_t = \frac{k}{\kappa\rho}\Delta u \tag{30}$$

results.

The heat equation has the property that, contradicting other established physical laws, small amounts of heat are propagated with arbitrarily high speed. In their account of the history of the problem, Joseph and Preziosi [44] describe how, from the times of Maxwell, it was well understood, at least by some scientists in each generation, that the problem of infinite propagation is intimately connected with the fact that in Fourier's model the flow adapts instantaneously to the gradient whereas in real physical systems as well as in an appropriate microscopic description some time would be needed to observe the gradient and to adapt to it. The simplest model for an adaptation process is a linear feedback loop. Then Fourier's law is replaced by the equation

$$\tau v_t = -k\mathrm{grad}\, u - v. \tag{31}$$

Then Eqs. (28)and (31), i.e.

$$\begin{aligned} u_t + \frac{1}{\kappa\rho}\mathrm{div}\, v &= 0, \\ \tau v_t + k\mathrm{grad}\, u + v &= 0. \end{aligned} \tag{32}$$

becomes a hyperbolic system of $n+1$ equations that replaces the heat equation. There is a well defined finite propagation speed. In a physical context the time constant τ is so small that it can be neglected in most practical situations.

According to [44], the first formulation of the law (32) appears in a paper by Cattaneo [8]. Already the title of that paper indicates that the real problem is instantaneous propagation whereas infinite speed is only a consequence of Fourier's law. In [57] the system (32) appears as the linearization of some reaction equation.

As in Eq. (10) one can eliminate the function v and arrive at a damped wave equation or telegraph equation

$$\tau u_{tt} + u_t = \frac{k}{\kappa\rho}\Delta u. \tag{33}$$

In the formal limit $\tau \to 0$ we obtain the heat equation.

In the theory of diffusion we have an almost identical situation. The conservation law $u_t + \mathrm{div}\, v = 0$ for the concentration and the flow of some substance, and the first Fickian law (Fick 1855, see [10]) $v = -D\mathrm{grad}\, u$, with D being the diffusion coefficient, lead to the diffusion equation (the second Fickian law) $u_t = D\Delta u$. The same arguments as before lead to the system

$$\begin{aligned} u_t + \mathrm{div}\, v &= 0, \\ \tau v_t + D\mathrm{grad}\, u + v &= 0. \end{aligned} \tag{34}$$

Replacing the diffusion equation by the linear system (34) is one way to avoid the difficulties of infinite propagation. Quite another way leads to the so-called porous medium equation, i.e. a diffusion equation $u_t = \operatorname{div}(D(u)\operatorname{grad} u)$ where the diffusion coefficient depends on density and vanishes for $u = 0$, see e.g. [1].

Again the concentration u satisfies a telegraph equation

$$\tau\, u_{tt} + u_t = D\Delta u. \tag{35}$$

If w is any vector field with $\operatorname{div} w = 0$ then $(u, v) = (0, w\exp\{-t/\tau\})$ is a solution of (34). These solutions are mapped into the zero solution of (35). The system (34) is a generalization of Eq. (9). The constant c in Eq. (11) is a vector field with divergence 0. For $\tau \to 0$ we formally obtain Eq. (1). When comparing the system (34) to a one-dimensional random walk (9) or to Brownian motion we shall identify

$$\tau = \frac{1}{2\mu}, \quad D = \frac{\gamma^2}{2\mu}. \tag{36}$$

Plane wave solutions of Eq. (34) satisfy a system of the form (9).

The work of S.Goldstein [25] has been extended to motions on multidimensional grids by Gillis [23], see also [37]. However there are difficulties to design random walks in the plane (and in $\mathbb{R}^n$ for any $n \geq 2$) for which the probability density would satisfy a telegraph equation of the form (35) (see [56]).

We establish a connection between the velocity jump process (3) and the system (34). In Eq. (3) assume space dimension 2 and constant speed. Then, with φ being the coordinate on S^1, the equation is

$$\begin{aligned} u_t(t,x,\varphi) + \gamma\cos\varphi\, u_{x_1}(t,x,\varphi) + \gamma\sin\varphi\, u_{x_2}(t,x,\varphi) \\ = -\tilde{\mu}\, u(t,x,\varphi) + \tilde{\mu}\int_0^{2\pi} K(\varphi,\psi)u(t,x,\psi)d\psi. \end{aligned} \tag{37}$$

The total population size

$$U(t,x) = \int_0^{2\pi} u(t,x,\varphi)d\varphi \tag{38}$$

satisfies the conservation law

$$U_t(t,x) + \operatorname{div} W(t,x) = 0 \tag{39}$$

where $W = (W_1, W_2)^T$,

$$W_1(t,x) = \int_0^{2\pi} \gamma\cos\varphi\, u(t,x,\varphi)d\varphi, \quad W_2(t,x) = \int_0^{2\pi} \gamma\sin\varphi\, u(t,x,\varphi)d\varphi. \tag{40}$$

Assume $K \equiv 1/2\pi$. A simple calculation shows that

$$W_t + \gamma^2 \int_0^{2\pi} P(\varphi)\operatorname{grad}_x u(t,x,\varphi)d\varphi + \tilde{\mu}W = 0 \tag{41}$$

where

$$P(\varphi) = \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}, \tag{42}$$

for any given φ, is a projection that maps any given vector onto the span of $e_\varphi = (\cos\varphi, \sin\varphi)^T$. Eqs. (39)and (41) should be compared to Eq. (34) with $\tau = 1/\tilde{\mu}$ and $D = \gamma^2/\tilde{\mu}$. We see that formally the transition from the velocity jump process (37) to the system (34) is accomplished by replacing the projection by the identity. Thus the transition from Eq. (37), for functions $\mathbb{R}^2 \times S^1 \to \mathbb{R}$, to Eq. (34) for functions from $\mathbb{R}^2 \to \mathbb{R}^{n+1}$, relies on the assumption that, on the average, $\operatorname{grad}_x u(t,x,\varphi)$ is parallel to e_φ. Thus a smoothness property of the flow is assumed that should approximately be satisfied at least after long times. Notice, however, that in the heuristic argument no assumptions on γ or $\tilde{\mu}$ have been made.

3 Reaction random walk systems

The random walk system (8) replaces the diffusion equation in the case of a correlated random walk. The problem of defining meaningful generalizations of the scalar reaction diffusion equation can be approached in several ways. If we assume symmetry then the system should have the form

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + F(u^+, u^-), \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + F(u^-, u^+). \end{aligned} \tag{43}$$

If total production does not depend on the direction of motion then $F(u^+, u^-) + F(u^-, u^+)$ should be a function $f(u)$ of $u = u^+ + u^-$. This requirement leaves still many possibilities for the form of the function F. In order to define specific problems we assume again the particle view. If the net production $f(u)$ is distributed among the two directions then one has the simple system

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + \tfrac{1}{2} f(u), \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + \tfrac{1}{2} f(u). \end{aligned} \tag{44}$$

In many situations this assumption appears unrealistic: If particles disappear then they are removed from the appropriate class. Thus we introduce a birth rate m and a death rate g, both depending on the total population number u, and we assume that newly produced particles choose both directions with equal probability. Then we arrive at the following system,

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + \tfrac{1}{2} m(u)u - g(u)u^+, \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + \tfrac{1}{2} m(u)u - g(u)u^-. \end{aligned} \tag{45}$$

Finally we can assume that the velocity of a "daughter" is correlated with that of the "mother". If the parameter $\tau \in [0,1]$ determines the distribution of directions then

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + (\tau u^+ + (1-\tau)u^-)m(u) - g(u)u^+, \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + ((1-\tau)u^+ + \tau u^-)m(u) - g(u)u^-. \end{aligned} \tag{46}$$

The system (45) (the "uncorrelated case") is obtained for $\tau = 1/2$, and Eq. (44) is obtained with $g = 0$, $f = m$.

As in Eq. (8) we introduce the variables u and v. Then these three systems assume the form

$$\begin{aligned} u_t + \gamma v_x &= f(u), \\ v_t + \gamma u_x &= -h(u)v, \end{aligned} \tag{47}$$

where

$$f(u) = m(u)u - g(u)u \tag{48}$$

and

$$h(u) = 2\mu, \tag{49a}$$

$$h(u) = 2\mu + g(u), \tag{49b}$$

$$h(u) = 2\mu + (1-2\tau)m(u) + g(u) \tag{49c}$$

for Eqs. (44), (45), (46), respectively.

We call a system of the form (47) a *reaction random walk system* or simply a random walk system.

Since we do not have a microscopic description of the system (34), it is not obvious how to design specific models that would replace Eqs. (44-46) in the case of several space dimensions. We propose the system

$$\begin{aligned} u_t + \operatorname{div} v &= f(u), \\ \tau v_t + D\operatorname{grad} u + v &= 0 \end{aligned} \tag{50}$$

as the appropriate generalization of Eq. (44) or Eq. (47) and (49a), as a random walk system where the reaction is independent of the direction of motion or, more appropriate in the present context, where the feedback loop is independent of density. A generalization of Eqs. (47)and (49b,c) is

$$\begin{aligned} u_t + \operatorname{div} v &= f(u), \\ \tau v_t + D\operatorname{grad} u + h(u)v &= 0. \end{aligned} \tag{51}$$

Holmes [41] has observed that, similar to Eqs. (8) and (10), some reaction random walk systems can be transformed into (nonlinear) telegraph equations. In Eq. (47) form second derivatives, eliminate mixed derivatives, and obtain

$$u_{tt} - f'(u)u_t = \gamma^2 u_{xx} + h(u)[f(u) - u_t] + h'(u)u_x\gamma v.$$

There is no way to get rid of the term containing v unless h is constant. In the latter case we arrive at the nonlinear telegraph equation

$$u_{tt} + (h - f'(u))u_t = \gamma^2 u_{xx} + hf(u). \tag{52}$$

Thus Eq. (44) and also Eq. (45), with g constant, can be carried into telegraph equations, but not the general system (47). The standard example is $g = 0$ which gives

$$u_{tt} + (2\mu - f'(u))u_t = \gamma^2 u_{xx} + 2\mu f(u). \tag{53}$$

This equation we call a *reaction telegraph equation.* As in the linear case, one can take the limit to Brownian motion. In Eq. (53) divide by 2μ, let $\mu \to \infty$, $\gamma \to \infty$ such that $\gamma^2/(2\mu) \to D$. Then formally one obtains the reaction diffusion equation (2).

The same idea can be applied to the system (50). From the initial value problem

$$\begin{aligned} u_t + \operatorname{div} v &= f(u), && (54a)\\ \tau v_t + D\operatorname{grad} u + v &= 0, && (54b)\\ u(0,x) = u_0(x), &\qquad v(0,x) = v_0(x). && (54c) \end{aligned}$$

one proceeds to the initial value problem of a reaction telegraph equation for the function u

$$\begin{aligned} &\tau u_{tt} + (1 - \tau f'(u))u_t = D\Delta u + f(u), && (55a)\\ &u(0,x) = u_0(x), \quad u_t(0,x) = -\operatorname{div} v_0(x) + f(u_0(x)), && (55b) \end{aligned}$$

and for the flow v

$$\begin{aligned} &\tau v_{tt} + v_t = D\operatorname{grad}\operatorname{div} v - D\operatorname{grad} f(u), && (56a)\\ &v(0,x) = v_0(x), \quad \tau v_t(0,x) = -D\operatorname{grad} u_0(x) - v_0(x). && (56b) \end{aligned}$$

Eq. (56a) can also be written in a form that is similar to Eq. (55a)

$$\tau v_{tt} + (1 - \tau f'(u))v_t = D\operatorname{grad}\operatorname{div} v - f'(u)v. \tag{57}$$

Now we see what really happens in Kac's transition Eqs. (8)and (10). The density u and the flow v both satisfy telegraph equations. The equation for u is nonlinear and independent of v, but the equation for v is linear with coefficients depending on u. Thus the equation for u separates. However the initial conditions do not separate, the condition for u_t depends also on v_0. If a solution $(0, w\exp(-t/\tau))$, $\operatorname{div} w = 0$, is added in Eq. (54) then this solution drops out in Eq. (55), it appears only in Eq. (56).

For $\tau \to 0$ the telegraph equation (55a) becomes the diffusion equation (1). Since τ is small in many (but not all) applications, one can consider (55a) as a singular perturbation of (1). The distance between the solutions has been discussed for the linear random walk case Eq. (10) by Griego and Hersh [28] (see also [26]) in a stochastic setting, and by Hale [36], Milani [53].

4 Bounded domains

If any of the reaction random walk systems are considered on compact domains then one has to specify boundary conditions that reflect the properties of the (supposed) underlying stochastic process similar to the Dirichlet, Neumann, and Robin boundary conditions for the reaction diffusion equation. The boundary conditions must respect the hyperbolic structure of the problem; data can only be prescribed along characteristics that are directed inwards.

We first consider the case $n = 1$. Then the domain is an interval $[0, l]$. We consider the standard system (44) or (47) and (49a) and the telegraph equation (53).

The homogeneous Dirichlet condition requires that any particles arriving at the boundary are absorbed. At $x = 0$ only particles of type u^- arrive, no particles of type u^+ emerge; similarly at $x = l$. Hence the homogeneous Dirichlet condition is

$$u^+(t,0) = 0, \quad u^-(t,l) = 0. \tag{58}$$

In terms of the variables u and v this boundary condition reads

$$v(t,0) = -u(t,0), \quad v(t,l) = u(t,l). \tag{59}$$

In the transition to the telegraph equation the boundary condition (59) for (u, v) becomes a time-dependent boundary condition for u alone

$$u_t(t,0) = \gamma u_x(t,0) - 2\mu u(t,0), \quad u_t(t,l) = -\gamma u_x(t,l) - 2\mu u(t,l). \tag{60}$$

As an intermediate problem one can study Eq. (53) with the stationary Robin condition

$$u(t,0) = \frac{\gamma}{2\mu} u_x(t,0), \quad u(t,l) = -\frac{\gamma}{2\mu} u_x(t,l). \tag{61}$$

Again we can consider the limit to Brownian motion. Then $\gamma/(2\mu)$ goes to zero, and formally we arrive at the usual homogeneous Dirichlet condition for the reaction diffusion equation. However, this limit is purely formal, the solutions behave rather differently, see (82) and (85).

The homogeneous Neumann condition describes reflection of particles at the boundary, hence

$$u^+(t,0) = u^-(t,0), \quad u^-(t,l) = u^+(t,l). \tag{62}$$

In terms of the variables u and v the Neumann boundary condition is

$$v(t,0) = 0, \quad v(t,l) = 0. \tag{63}$$

Again, for the telegraph equation we obtain formally

$$u_x(t,0) = 0, \quad u_x(t,l) = 0. \tag{64}$$

Similarly one can formulate inhomogeneous boundary conditions.

Existence and uniqueness results for the Cauchy problems of the system (44) with boundary conditions (58), (62) are shown in [3], [40]. Of course there is a

vast literature on Cauchy problems for nonlinear wave equations (e.g. [52, 53, 55, 26]). It is an interesting question to what extent the solutions of the hyperbolic initial boundary value problems have properties similar to those of the corresponding reaction diffusion equations. Although there are maximum and comparison principles for hyperbolic equations (e.g. [73], [40]) such principles are not valid as generally as in the parabolic case. The system (6) preserves positivity, also with Dirichlet or Neumann conditions, and also (44) with $f(0) = 0$, $f'(0) > 0$. However, a comparison principle holds only under rather strong conditions on f'.

Next consider the case of several space dimensions, i.e. Eq. (50) on a bounded domain $\Omega \subset \mathbb{R}^n$. The boundary condition for the homogeneous Dirichlet problem requires that there should be no particles entering the domain along a characteristic direction which leads to

$$u(t,x) = \sqrt{\frac{\tau}{D}}\,\nu^T v(t,x) \quad \text{for} \quad x \in \partial\Omega \tag{65}$$

where ν is the outward normal at $x \in \partial\Omega$. The absorption boundary condition for the telegraph equation (55a) becomes a time dependent Robin condition,

$$\tau u_t = -\sqrt{\tau D}\frac{\partial u}{\partial \nu} - u \quad \text{for} \quad x \in \partial\Omega. \tag{66}$$

Again, one can study the intermediate problem Eq. (55a) with

$$u(t,x) = -\sqrt{\tau D}\frac{\partial u}{\partial \nu} \quad \text{for} \quad x \in \partial\Omega. \tag{67}$$

The homogeneous Neumann condition for Eq. (50) is

$$\nu^T v(t,x) = 0 \quad \text{for} \quad x \in \partial\Omega \tag{68}$$

and

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial\Omega \tag{69}$$

for Eq. (55a).

The u components of stationary solutions of class C^2 of the Dirichlet or Neumann problems satisfy the differential equation

$$-D\Delta u = f(u) \tag{70}$$

and the appropriate boundary conditions Eq. (67) or (69), respectively. Hence known results on the existence of stationary solutions to boundary value problems of reaction diffusion equations (e.g. [63] for Robin boundary conditions) can be used in the hyperbolic case.

We discuss the case $n = 1$ in greater detail. We assume that the nonlinearity $f \in C^1(\mathbb{R})$ has the properties $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f(u) > 0$ for $0 < u < 1$. We are mainly interested in solutions (u^+, u^-) with $u^+ \geq 0$, $u^- \geq 0$ or $0 \leq u \leq 1$. Stationary solutions of the Dirichlet problem (47) and (59) satisfy

$$\gamma u' = -h(u)v, \quad \gamma v' = f(u), \tag{71}$$

$$v(0) = -u(0), \quad v(l) = u(l). \tag{72}$$

For the moment we assume that $h(u) > 0$ for $u \in [0, l]$. Then we can rescale the independent variable (changing the length of the interval) and arrive at

$$\dot{u} = -v, \quad \dot{v} = \tilde{f}(u) \quad \text{where} \quad \tilde{f}(u) = f(u)/h(u). \tag{73}$$

Eq. (73) is a Hamiltonian system with Hamiltonian

$$H(u, v) = \tfrac{1}{2}v^2 + F(u), \quad F(u) = \textstyle\int_0^u \tilde{f}(s)ds. \tag{74}$$

The stationary points are a center at $(0, 0)$ and a saddle point at $(1, 0)$. Solutions of Eqs. (71)–(72) correspond to (pieces of) trajectories that connect the line $v = -u$ to the line $v = u$. There is a one-parameter family of such arcs parametrized by the value $u(l/2)$ ranging from 0 to 1. To each value $\bar{u} \in (0, 1)$ there is a unique solution to the boundary value problem for some l with $u(l/2) = \bar{u}$. For $\bar{u} \to 1$ we have $l \to \infty$, for $\bar{u} \to 0$ the length l converges to some well-defined positive number l^* that can be obtained from the linearization. However, l need not be a monotone function of $\bar{u}$.

The Hamiltonian system (73) describes also the stationary solutions of the Dirichlet problem of the reaction diffusion equation $u_t = u_{xx} + \tilde{f}(u)$, $u(t, 0) = u(t, l) = 0$. Again, there is a branch of nonnegative solutions parametrized by $\bar{u} = u(l/2)$, with $l \to \infty$ for $\bar{u} \to 1$ and $l \to l_0$ for $\bar{u} \to 0$, with some $l_0 > 0$. But there is a marked difference in the two problems. In the parabolic case we look for an orbit that runs from $u = 0$ to $u = 0$, thus l_0 is one half of the period near $(0, 0)$, whereas l^* is usually considerably smaller. Thus the minimal length of an interval that supports a nontrivial stationary solution is considerably shorter in the hyperbolic case. This phenomenon has been discussed in detail in [58]. The difference will be quantitatively explored in terms of the eigenvalues of the linearization. The stationary solutions of the hyperbolic Dirichlet problem look quite different from those of the reaction diffusion equation. Even though particles are absorbed at the boundary, the function u does not vanish at the boundary.

For the stationary solutions of the homogeneous Neumann problem we have again the system (71). In the stationary situation the boundary conditions $u_x = 0$ and $v = 0$ are equivalent. Thus the situation is the same as in the parabolic case. In particular, $(0, 0)$ and $(1, 0)$ are the only nonnegative stationary solutions.

In the homogeneous Dirichlet problem of the scalar reaction diffusion equation (2) with $f'(0) > 0$ the stability of the zero solution is lost when the length l of the interval exceeds a certain threshold. This threshold can be obtained either by studying the period of the Hamiltonian system near the origin or by a discussion of the corresponding eigenvalue problem. In the hyperbolic case it is difficult to determine the critical length from the Hamiltonian system. The systematic approach to the spectral problem is simpler.

One can show ([55], [40]) that in standard function spaces $(L_p(0, l))^2$ the generators of the solution semigroups of the linearized problems (47), (59) or (63) have pure point spectrum and that the eigenvalues are the zeros of an analytic characteristic function. Here we derive this function explicitly and we extract some important quantities. Later we return to the case of several space dimensions.

We start from the Dirichlet problem (47) and (59). We linearize at the zero solution and we put $a = f'(0)$, $b = h(0)$. The linearized system is

$$\begin{aligned} u_t + \gamma v_x &= au, \\ v_t + \gamma u_x &= -bv \end{aligned} \tag{75}$$

with the same boundary condition. If $(u, v)\exp\{\lambda t\}$ is an exponential solution then u, v, λ satisfy

$$\gamma v' = (a - \lambda)u, \quad \gamma u' = -(b + \lambda)v, \tag{76}$$

$$u(0) + v(0) = 0, \quad u(l) - v(l) = 0. \tag{77}$$

Put

$$\delta = (b + \lambda)/\gamma, \quad \kappa^2 = (b + \lambda)(\lambda - a)/\gamma^2. \tag{78}$$

Then $u'' = \kappa^2 u$. We are looking for solutions of the form $u(x) = c_1 e^{\kappa x} + c_2 e^{-\kappa x}$. Then $u'(x) = c_1\kappa e^{\kappa x} - c_2\kappa e^{-\kappa x}$, and the boundary conditions yield a linear system $\kappa(c_1 - c_2) = \delta(c_1 + c_2)$, $\kappa(e^{\kappa l}c_1 - e^{-\kappa l}c_2) = -\delta(e^{\kappa l}c_1 + e^{-\kappa l}c_2)$. The determinant vanishes if and only if $(\kappa - \delta)^2/(\kappa + \delta)^2 = e^{2\kappa l}$. Replacing κ and δ we find one form of the characteristic equation

$$\left(\frac{b + \lambda - \sqrt{(b + \lambda)(\lambda - a)}}{b + \lambda + \sqrt{(b + \lambda)(\lambda - a)}}\right)^2 = e^{2\sqrt{(b+\lambda)(\lambda-a)}\,l/\gamma}. \tag{79}$$

This equation can also be written

$$\frac{b - a + 2\lambda - 2\sqrt{(b + \lambda)(\lambda - a)}}{b - a + 2\lambda + 2\sqrt{(b + \lambda)(\lambda - a)}} = e^{2\sqrt{(b+\lambda)(\lambda-a)}\,l/\gamma}, \tag{80}$$

or, showing the analyticity,

$$-\frac{2(b - a + 2\lambda)}{(b + a)^2} = \frac{\sinh(2\sqrt{(b + \lambda)(\lambda - a)}\,l/\gamma)}{2\sqrt{(b + \lambda)(\lambda - a)}\,l/\gamma}\,\frac{l}{\gamma}. \tag{81}$$

Consider Eq. (47) and (49a). Then $b = 2\mu$. From (80) one finds that the critical length l of the interval and the parameter $a = f'(0)$ are connected by the equation

$$\tan\frac{\sqrt{2\mu a}\,l}{\gamma} = -\frac{2\sqrt{2\mu a}}{2\mu - a}. \tag{82}$$

For a qualitative discussion of Eq. (82) we keep μ fixed. The quotient l/γ is a decreasing function of the parameter a. For $a \to 2\mu$ the right hand side goes to infinity, thus the argument of the tangent is $\pi/2$, and $a = 2\mu$ corresponds to $l/\gamma = \pi/(4\mu)$. For $a \to \infty$ the right hand side goes to zero like $2\sqrt{2\mu}/\sqrt{a}$. Since $\tan x \approx x$ near $x = 0$, we find

$$\frac{l}{\gamma} \sim \frac{2}{a} \quad \text{for} \quad a \to \infty. \tag{83}$$

For $a \to 0$ the argument of the tangent goes to π, hence

$$\frac{l}{\gamma} \sim \frac{\pi}{\sqrt{2\mu a}} \quad \text{for} \quad a \to 0. \tag{84}$$

In the parabolic case we have, using Eq. (36) for a comparison,

$$\frac{l}{\gamma} = \frac{\pi}{\sqrt{2\mu a}}. \tag{85}$$

Thus the behavior for small l is markedly different.

The linear system (6) and (58) preserves positivity and the spectral bound (the eigenvalue with maximal real part) is real. The spectral bound can be obtained from the characteristic equation with $a = 0$. We put $\lambda = \mu\nu$ and we write the characteristic equation in the form

$$\frac{1 + \nu - \sqrt{(2+\nu)\nu}}{1 + \nu + \sqrt{(2+\nu)\nu}} - e^{2\sqrt{(2+\nu)\nu}(\mu l/\gamma)} = 0. \tag{86}$$

Let λ_0 be the spectral bound. Then λ_0/μ depends only on the parameter $\mu l/\gamma$. Notice that this number is different from the parameter $\mu l/\gamma^2$ that appears in the parabolic case. λ_0 is always negative. With some effort one can show [65] that the quotient λ_0/μ is an increasing function of $\mu l/\gamma$ and assumes the following special values,

$$\begin{array}{llll} \mu l/\gamma \to 0 & \Longleftrightarrow & \lambda_0/\mu \to -\infty, & \mu l/\gamma = 1 \quad \Longleftrightarrow \quad \lambda_0/\mu = -2, \\ \mu l/\gamma = \pi/2 & \Longleftrightarrow & \lambda_0/\mu = -1, & \mu l/\gamma \to -\infty \quad \Longleftrightarrow \quad \lambda_0/\mu \to 0. \end{array}$$

The Neumann problem is somewhat simpler. The eigenvalues are $\lambda_0 = a$, with eigenvector $(1, 0)$, and

$$\lambda_k^{\pm} = \tfrac{1}{2}[a - b \pm \{(a+b)^2 - 4k^2\gamma^2/l^2\}^{1/2}, \quad k = 1, 2, \ldots. \tag{87}$$

Now we return to the case of several space dimensions. With $f(0) = 0$, $f'(0) = a > 0$, the linearized system to (54) is

$$\begin{aligned} u_t + \operatorname{div} v &= au, \\ \tau v_t + D\operatorname{grad} u + v &= 0, \end{aligned} \tag{88}$$

with boundary condition (65) or (68), respectively. The corresponding telegraph equation is

$$\tau u_{tt} + (1 - \tau a)u_t = D\Delta u + au. \tag{89}$$

The eigenvalue problem to Eq. (88) is

$$\begin{aligned} \lambda u + \operatorname{div} v &= au, \\ \tau\lambda v + D\operatorname{grad} u + v &= 0, \end{aligned} \tag{90}$$

with boundary condition (65) or (68), respectively. In Eq. (90) one can eliminate v and find

$$\Delta u = \Lambda u \tag{91}$$

with

$$(\lambda - a)(1 + \tau\lambda) = D\Lambda \tag{92}$$

and boundary conditions

$$(1 + \tau\lambda)u(x) = -\sqrt{\tau D}\,\frac{\partial u}{\partial \nu}(x), \quad x \in \partial\Omega, \tag{93}$$

or

$$\frac{\partial u}{\partial \nu}(x) = 0, \quad x \in \partial\Omega, \tag{94}$$

respectively. Eqs. (91), (92) and (94) could also be obtained from Eqs. (89)and (69).

Let Λ_k, U_k, $k = 0, 1, 2, \ldots$ be the sequence of eigenvalues and eigenvectors of Eqs. (91) and (94), normalized in $L_2(\Omega)$, with $\Lambda_0 = 0$. Then the eigenvalues of problem (90) and (94) are $\lambda_0 = a$ and

$$\lambda_k^{\pm} = -\frac{1 - \tau a}{2\tau} \pm \frac{1}{2\tau}\sqrt{(1 + \tau a)^2 + 4\tau D\Lambda_k}, \quad k = 1, 2, \ldots. \tag{95}$$

The eigenvalues λ with large absolute values are complex and their asymptotic behavior is $\Re\lambda = -(1 - a\tau)/(2\tau)$, $\Im\lambda \sim \sqrt{D|\Lambda|/\tau}$. Thus, for small τ, the eigenvalues approach a line parallel to the imaginary axis at about $\Re\lambda \approx -1/(2\tau)$.

As in the one-dimensional case [67], [6] one can find solutions to the linear initial boundary value problem (88) and (69) by separation of variables. The ansatz

$$u(t, x) = \sum_k T_k(t)U_k(x) \tag{96}$$

for (89) leads to the ordinary differential equation

$$\tau\ddot{T}_k + (1 - \tau a)\dot{T}_k = (D\Lambda + a)T_k. \tag{97}$$

The characteristic exponents are $\lambda_k^{\pm}$ as given by Eq. (95). If $u(0, x) = \sum_k c_k U_k(x)$, $u_t(0, x) = \sum_k c_k' U_k(x)$, then $T_k(0) = c_k$, $\dot{T}_k(0) = c_k'$. Hence we find the representation

$$u(t, x) = c_0 e^{at}U_0(x) + \sum_{k=1}^{\infty} \frac{1}{\lambda_k^+ - \lambda_k^-}\left[(e^{\lambda_k^+ t} - e^{\lambda_k^- t})c_k' + (e^{\lambda_k^- t}\lambda_k^+ - e^{\lambda_k^+ t}\lambda_k^-)c_k\right]U_k(x). \tag{98}$$

As in the reaction diffusion case the ultimate goal is the description of the qualitative behavior of the solutions of the semilinear equations with appropriate boundary conditions. A useful tool are invariants and Lyapunov functions which have been found by Brayton and Miranker [4] for $n = 1$ and can be generalized to several space dimensions and also to some vector-valued problems (see also [40]). Let

$$V(u, v) = \frac{1}{2}\int_\Omega (u_t^2 + \frac{\tau}{D}v_t^2)dx - M\int_\Omega (\frac{1}{2D}v^2 + F(u) - u\nabla v)dx. \tag{99}$$

Then along trajectories

$$\frac{d}{dt}V(u,v) = -\int_\Omega [(M - f'(u))u_t^2 + \frac{1}{D}(1 - M\tau)v_t^2]dx - \int_{\partial\Omega}(u_t + Mu)\nu^T v_t dS. \quad (100)$$

If the damping condition $\sup_u f'(u) < 1/\tau$ is satisfied then one can find M such that $\sup_u f'(u) < M < 1/\tau$. Then the term containing the space integral becomes nonpositive. The boundary integral vanishes in the case of the Neumann condition.

Several authors have shown that specific damped wave equations have compact attractors (Webb [71], Lopes [50], Fereisl [16], Hale [36]).

The arguments for the scalar case can be easily carried over to the case of several dependent variables. Then one arrives at hyperbolic systems that mimics reaction diffusion systems. For one space dimension, let $u = (u_1, \ldots, u_m)$ be a vector of species, $M = (\mu_i\delta_{ij})$, $\Gamma = (\gamma_i\delta_{ij})$ and let $f : \mathbb{R}^m \to \mathbb{R}^m$ be the vector field that describes the interaction of species. Then the generalization of (44) is

$$\begin{aligned} u_t^+ + \Gamma u_x^+ &= M(u^- - u^+) + \tfrac{1}{2}f(u), \\ u_t^- - \Gamma u_x^- &= M(u^+ - u^-) + \tfrac{1}{2}f(u). \end{aligned} \quad (101)$$

This system is equivalent to

$$\begin{aligned} u_t + \Gamma v_x &= f(u), \\ v_t + \Gamma u_x &= -2Mv, \end{aligned} \quad (102)$$

and the function u is a solution to the vector telegraph equation

$$u_{tt} + (I - 2Mf'(u))u_t = \Gamma^2 u_{xx} + 2Mf(u). \quad (103)$$

5 Branching processes, random walks and travelling fronts

In the standard interpretation of the reaction diffusion equation (2), with $f(u) = u(1-u)$, say, either the function $u(t,\cdot)$ represents a particle density at time t or the number $u(t,x)$ is a probability describing an event at time t at the point x. McKean [51] has given a totally different interpretation of the same equation in terms of branching processes. He designs a stochastic process of the following form. At any time $t \geq 0$ there are $\nu(t)$ particles. These have positions on the real axis $X_1(t), \ldots, X_{\nu(t)}(t)$. Thus there are $\nu(t) + 1$ random variables. The process is constructed in such a way that these are independent. Each of the particles performs a Brownian motion independent of all the other particles. At the same time the particles are subject to a branching process. Any existing particle has exponential holding time. When it splits it gives rise to finitely many daughters (two in the special case). The daughters start their motion at the position of the mother. Brownian motion and branching act independently of each other.

Thus the process is characterized by the following parameters: The diffusion rate D, the Poisson parameter b and the distribution of the number of daughters given by its generating function

$$g(z) = \sum_{k=2}^{\infty} g_k z^k, \quad g_k \geq 0, \quad k = 2, 3, \ldots; \quad \sum_{k=2}^{\infty} g_k = 1.$$

In the simplest case, branching into two daughters, we have $g(z) = z^2$.

McKean defines a function $u(t, x)$ as a probability for the position of the most advanced particle,

$$u(t, x) = Prob\{X_i(t) < x, \, i = 1, \ldots, \nu(t)\}. \tag{104}$$

Clearly, $u(t, x)$ is a nondecreasing function of x with $u(t, x) \to 0$ for $x \to -\infty$, and $u(t, x) \to 1$ for $x \to +\infty$. He proves that the function u satisfies a reaction diffusion equation

$$u_t = \tfrac{1}{2} u_{xx} + f(u), \quad f(u) = b(g(u) - u). \tag{105}$$

Consider the process that starts with a single particle at $x = 0$. Then the initial datum is

$$u(0, x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 1. \end{cases} \tag{106}$$

Then the solution to the initial value problem will develop into a travelling front solution. Thus we have the same situation as in Fisher's model.

McKean's idea of connecting a branching process to a process for spatial spread has been carried over to correlated random walks by Dunbar and Othmer [14], [15]. They consider a stochastic process with the following properties. At time t there are $\nu(t)$ particles at positions $X_1(t), \ldots, X_{\nu(t)}(t)$. These particles multiply according to a branching process and move according to a correlated random walk. It is assumed that the branching of particles and the motion of particles act independently. Again they consider the function (104) and they show that it satisfies a reaction telegraph equation

$$u_{tt} + (2\mu + 2f'(u))u_t = \gamma^2 u_{xx} - (2\mu + b) f(u) \tag{107}$$

Here μ and γ are the parameters of the correlated random walk, and b defines the holding time of the branching process. For $\mu \to \infty$, $\gamma \to \infty$, $\gamma^2/\mu \to 1$, Eq. (107) becomes Eq. (105).

As indicated in the introduction, reaction diffusion equations started with a travelling front problem. In the parabolic case consider the scalar equation (2) with $f \in C^1[0, 1]$, $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f(u) > 0$ for $0 < u < 1$. A travelling front is a solution $u(t, x) = \phi(x - ct)$ where the shape function ϕ satisfies $0 < \phi(x) < 1$ and $\lim_{x \to -\infty} = 1$ and the speed c is positive. There is a minimal speed c_P (depending on f) and for every $c \geq c_P$ there is, up to translation, exactly one travelling front (see [48], [2], [35], the literature in [31], and [21] for more recent developments in a stochastic setting).

The wave speeds for the scalar reaction diffusion equations have two different interpretations. In the Fisher-KPP interpretation the speed is the velocity of propagation of the wave front, i.e. the point $m(t)$ with $u(t, m(t)) = 1/2$, in McKean's interpretation it describes how the position of the most advanced particle moves.

For the hyperbolic problem (44), with the same hypothesis on f, a travelling front is a solution $u^+(x - ct)$, $u^-(x - ct)$ with $u^+(x) > 0$, $u^-(x) > 0$, $u^+(x) + u^-(x) < 1$, $u^s(-\infty) = 1/2$, $u^s(+\infty) = 0$ for $s = \pm$. Under suitable conditions on f and μ there is a positive minimal speed $c_H \in (0, \gamma)$, and for every $c \in [c_H, \gamma)$ there is, up to translation, a unique travelling front. A sufficient condition for this statement is the inequality $f'(u) < 2\mu$ for all $u \in [0, 1]$. For the proof and extensions see [30], [31].

Travelling front solutions can be found in other reaction random walk systems, where such solutions exist in the corresponding reaction diffusion equations. As an example one can consider a set of equations that describe epidemic spread by migrating infectives

$$\begin{aligned} u_t &= -\beta u(v^+ + v^-)\,, \\ v_t^+ + \gamma v_x^+ &= \mu(v^- - v^+) - \alpha v^+ + \beta u(\tau v^+ + (1-\tau)v^-)\,, \\ v_t^- - \gamma v_x^- &= \mu(v^+ - v^-) - \alpha v^- + \beta u((1-\tau)v^+ + \tau v^-)\,. \end{aligned} \tag{108}$$

Here one can ask for travelling front solutions that connect two stationary points of the underlying reaction system [32].

6 Pattern formation and other problems from biology

From the view point of Biology one of the main applications of reaction diffusion equations are models for pattern formation. Although there are several schools which have developed such models (mostly with rather specific nonlinearities, not so much supported by experimental evidence but rather by tradition, to name some of the earliest, [22], [24]), the underlying idea is that of Turing [70]: A stable spatially constant equilibrium of the underlying reaction scheme can be destabilized by diffusion, if different species have rather distinct diffusion rates. The simplest case occurs for two variables. Consider

$$u_t = Du_{xx} + Au \tag{109}$$

with

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{110}$$

Assume that the zero solution is stable with respect to the reaction, $a_{11} + a_{22} < 0$, $a_{11}a_{22} - a_{12}a_{21} > 0$, and that u_1 acts as an "activator", u_2 as an "inhibitor", i.e. $a_{11} > 0$, $a_{22} < 0$, $a_{12} < 0$, $a_{21} > 0$. Let $[0, l]$ be a bounded interval. Consider the system (109) with homogeneous Neumann conditions. If d_1 is small and d_2 is large (a short range activator with a long range inhibitor) then there are modes k and lengths l such that these modes grow exponentially. Hillen [39] has studied systems of the

form (109) that show the same behavior and he has determined the exact stability conditions.

Several authors, e.g. Hadeler [30], Sánchez-Garduño and Maini [69] have considered density-dependent diffusion, i.e. diffusion equations where the diffusion coefficient itself depends on density,

$$u_t = (D(u)u_x)_x + f(u). \tag{111}$$

Systems of this form have been extensively studied ("cross diffusion") by Mimura, e.g. in [54]. In the case where D is uniformly positive, many results carry over from the case $D \equiv const.$ Particular attention has been paid to the porous medium version [69]. A similar dependence on density can be incorporated in Eq. (44),

$$\begin{aligned} u_t^+ + (\gamma(u)u^+)_x &= \mu(u)(u^- - u^+) + \tfrac{1}{2}f(u), \\ u_t^- - (\gamma(u)u^-)_x &= \mu(u)(u^+ - u^-) + \tfrac{1}{2}f(u). \end{aligned} \tag{112}$$

or Eq. (46)

$$\begin{aligned} u_t + (\gamma(u)v)_x &= f(u), \\ v_t + (\gamma(u)u)_x &= -2\mu v. \end{aligned} \tag{113}$$

The boundary conditions remain (58-63). The travelling front problem can be solved if γ'/γ is not too large [33].

Greenberg [27] has studied the classical Stefan problem for a generalized heat equation. As we have observed earlier, the boundary value problem for the random walk system does not approximate to that of the diffusion equation if the appropriate limit in the coefficients is taken. Using an energy balance argument, Greenberg specifies a nonlinear Stefan condition for the hyperbolic problem that in the limit yields the Stefan boundary. Here we propose to stay in the class of linear problems (i.e. the only nonlinearity is the dependence on the boundary itself). Thus we consider the standard problem (44) in a domain

$$\Omega = \{(t,x) : 0 \le t \le T,\ 0 \le x \le s(t)\}.$$

The boundary condition at $x = 0$ is of Dirichlet type $u^+(t,0) = \varphi(t)$. The boundary condition at $x = s(t)$ is of Stefan type. We assume that a proportion $\kappa \in (0,1)$ of particles is reflected, $u^-(t,s(t)) = \kappa(u^+(t,s(t))$ and the boundary is pushed forward by the unreflected particles, $\dot{s}(t) = \tau\big(u^+(t,s(t)) - u^-(t,s(t))\big)$. Equivalently, we can assume $\dot{s}(t) = \tau(1-\kappa)u^+(t,s(t))$. Initial conditions are $u^+(0,x) = u_0(x)$, $u^-(0,x) = u_0^-(x)$ for $0 < x < s(0)$.

Reaction diffusion equations play a prominent role in neurobiology (see [49],[11]. The Hodgkin-Huxley model and its simplified version, the Fitzhugh-Nagumo model, are actually three models each. The basic model is an ordinary differential equations system that describes the excitation of a (short) piece of nerve membrane (the so-called space clamp situation). The same equations with an additional parameter modeling the dendritic input describe, via a Hopf bifurcation, the onset of oscillatory

behavior at the axon hillock ([34], [29]). Finally, one can model the axon as a one dimensional domain, the ordinary differential equation is acting at each space point, and the transmission between adjacent areas is modeled by the cable equation (which corresponds to nearest neighbor coupling in a discrete setting). Then a (degenerate) reaction diffusion system results. There is a vast literature on these problems [11]. The first proof of existence for a travelling pulse solution was given by Carpenter [7]. Fitzgibbon and Parrot [19] reconsidered the original papers and found, that originally the system was designed as a hyperbolic system and later has been "simplified" to a parabolic system by putting a presumably small parameter equal to zero. Already Lieberstein [49] pointed out that this parameter (self-induction) should not be neglected. In the Fitzhugh-Nagumo case the hyperbolic system has the form

$$\begin{aligned} \epsilon u_{tt} + (1 + \epsilon g(u))u_t &= u_{xx} + f(u) - \delta v, \\ v_t &= u - \nu v \end{aligned} \tag{114}$$

with $f(u) = u(1-u)(u-\alpha)$, $\alpha \in (0,1)$. One of the most interesting features of these equations is the existence of travelling pulses and travelling wave trains. A travelling wave ansatz $u(x-ct)$, $v(x-ct)$ leads to

$$\begin{aligned} \epsilon c^2 \ddot{u} - c(1 + \epsilon g(u))\dot{u} &= \ddot{u} + f(u) - \delta v, \\ \dot{v} &= (\nu v - u)/c. \end{aligned} \tag{115}$$

We put $w = \dot{u}$ to obtain a three-dimensional first order system,

$$\begin{aligned} (1 - \epsilon c^2)\dot{w} &= -c(1 + \epsilon g(u))w - f(u) + \delta v, \\ \dot{v} &= (\nu v - u)/c, \\ \dot{u} &= w. \end{aligned} \tag{116}$$

For this system, for small ϵ, one can again use the arguments of [7].

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