NONSYMMETRIC ORNSTEIN-UHLENBECK GENERATORS

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Abstract

We review some properties of nonsymmetric Ornstein-Uhlenbeck generators in $L^p(\mu)$, where μ is an invariant measure. We provide a necessary and sufficient condition for the Poincaré Inequality and the Logarithmic Sobolev Inequality, thus extending the result of Rothaus and Simon to the nonsymmetric Gaussian case. Next, we show that the same condition is necessary and sufficient for the compact embedding of the Sobolev space $W_Q^{1,p}(\mu)$ into $L^p(\mu)$. We provide also necessary and sufficient condition for the Ornstein-Uhlenbeck operator to be selfadjoint and in this case its domain in $L^p(\mu)$ is completely characterized.

1 Introduction

Consider a stochastic differential equation

$$\begin{cases} dZ = AZdt + \sqrt{Q}dW, \\ Z(0) = x \in H, \end{cases} \tag{1.1}$$

on a real separable Hilbert space H, where W is a standard cylindrical Wiener process on H, A generates a C_0-semigroup $(S(t))$ on H and the operator $Q \geqslant 0$ is selfadjoint and bounded in H. Under the assumptions listed below equation (1.1) has a solution Z which is a Feller process on H. Our main assumption is that there exists a nondegenerate invariant measure μ for Z.

If $A = -\frac{1}{2}I$ then (1.1) defines the so-called Malliavin process which enjoys many remarkable properties. In particular, if $A = -\frac{1}{2}I$ then the process Z admits a unique invariant measure μ which is zero-mean Gaussian with the covariance operator Q. It was proved by Nelson in [21] that the transition semigroup of this process is hypercontractive. Moreover, the generator L_M of the process Z, when considered in the space $L^p(H,\mu)$, satisfies the Logarithmic Sobolev Inequality and the best constant in this inequality (for $p = 2$) coincides with the size of the spectral gap for the generator. The question when these properties of the operator L_M extend to more general classes of diffusion generators was (and still is) an object of intense study, leading in particular, to the Gross [19] sufficient condition for hypercontractivity given in terms of the Logarithmic Sobolev Inequality and the Simon-Rothaus condition [25], [27] for hypercontractivity of symmetric semigroups.

If we denote by D_M the Malliavin derivative, then $L_M = -\frac{1}{2}D_M^*D_M$ and, in view of the Meyer inequalities, the Sobolev spaces $W_M^{n,p}$ defined by D_M may be identified with the domains of $(I - L_M)^{n/2}$ in $L^p(H,\mu)$. This identification yields, among other things, closed range of the operator D_M in $L^p(H,\mu;H)$ and continuous imbeddings of Sobolev spaces $W_M^{n,p}$.

The generator L_M is related to the Number Operator in Quantum Field Theory and this fact provided an important motivation for the study of the Malliavin process. However, various examples coming from Mathematical Physics, see [13] and [14], and recently from Mathematical Finance, see [20], demand a theory of a more general class of Ornstein-Uhlenbeck processes given by (1.1) which are not necessary symmetric. Let us note that in many problems conditions imposed in this survey on equation (1.1) are too restrictive. For a more general class of Ornstein-Uhlenbeck processes see [3], [22] and [4].

In the present survey we consider a class of Ornstein-Uhlenbeck processes which is, in a sense, a right generalization of the Malliavin process: it enjoys most of its properties described above and covers various applications arising in Mathematical Physics. The paper summarizes recent developments obtained in [6], [7], [8] and [9], but we will be mainly concerned with the consequences of the spectral gap property studied in [10].

We will assume the following condition which is a standing assumption for the rest of the paper:

$$\int_0^\infty \operatorname{tr} S(u)QS^*(u)du < \infty. \tag{1.2}$$

If (1.2) is satisfied then the process

$$Z(t,x) = S(t)x + \int_0^t S(t-s)\sqrt{Q}dW(s)$$

is a solution to (1.1). Moreover, equation (1.1) admits an invariant measure μ which is Gaussian with the mean zero and the covariance operator

$$Q_\infty = \int_0^\infty S(u)QS^*(u)du.$$

Throughout this paper we assume for simplicity of presentation that

$$\ker Q_\infty = \{0\}. \tag{1.3}$$

If (1.2) and (1.3) hold then the transition semigroup

$$R_t\phi(x) = E\phi(Z(t,x))$$

defines a C_0-semigroup of positive contractions in $L^p(H,\mu)$ for all $p \in [1,\infty)$. Its generator L is uniquely determined by the formula

$$L\phi(x) = \frac{1}{2}\operatorname{tr}\left(QD^2\phi(x)\right) + \langle x, A^*D\phi(x)\rangle, \quad x \in H,$$

where D stands for the Fréchet derivative and ϕ belongs to an appropriately defined dense class of cylindrical functions, see [5]. In this way we obtain a large class of Gauss-Markov processes on H which may be thought of as generalizations of the Malliavin process. However, it may be shown that this class contains processes with rather pathological properties and therefore we will restrict to a smaller class of Ornstein-Uhlenbeck generators which is described below.

Let Q_t denote the covariance operator of the Gaussian random variable $Z(t,x)$. It was shown in [6] and [7] that the equality of images

$$\operatorname{im}\left(Q_\infty^{1/2}\right) = \operatorname{im}\left(Q_t^{1/2}\right), \quad t > 0, \tag{1.4}$$

is a necessary condition for many regularizing properties of the transition semigroup (R_t) like hypercontractivity (in this case (1.4) is also sufficient), compactness, smoothing and some others as well. On the other hand (1.4) does not assure sufficiently regular behaviour of the transition semigroup (R_t) for small times. We will show that such a regular behaviour of (R_t) is assured by an inclusion

$$\operatorname{im}\left(Q_\infty^{1/2}\right) \subset \operatorname{im}\left(Q^{1/2}\right), \tag{1.5}$$

which is stronger than (1.4). This assumption is satisfied for the Malliavin process but it holds also in the case of $Q = I$. It was proved in [6], that while (1.4) is equivalent to the hypercontractivity of the Ornstein-Uhlenbeck semigroup the Logarithmic Sobolev Inequality does not need to hold, see [16] and Section 2 below for explicit examples. In Section 3 we use the technique developed in [19] to show that the Logarithmic Sobolev Inequality is satisfied if and only if (1.5) holds. We prove also that (1.5) is equivalent to the existence of spectral gap for L. This result extends the result of Rothaus [25] and Simon [27] to the nonsymmetric Gaussian case.

The other topic of this paper are the imbeddings of Gaussian Sobolev spaces and characterizarion of the domain of L. For the importance of such imbeddings for the study of nonlinear stochastic evolution equations see [14], [11] and [5]. Let $D_Q\phi$ denote the derivative of a function ϕ in the direction of $Q^{1/2}(H)$. Note that for $Q = Q_\infty$ we obtain the Malliavin derivative. If D_Q is closable then we can define the Sobolev space $W_Q^{1,p} \subset L^p(H,\mu)$. We will show that if (1.5) holds then $W_Q^{1,p} \subset W_{Q_\infty}^{1,p}$, the gradient D_Q has closed range and the Helmholtz decomposition of the space $L^p(H,\mu;H)$ holds. Moreover, we show that the embedding of $W_Q^{1,p}$ into $L^p(H,\mu)$ is compact if and only if the imbedding $i : Q_\infty^{1/2}(H) \to Q^{1/2}(H)$ is compact.

We show also that (1.5) yields continuous imbedding of the domain of L in $L^p(H,\mu)$ into some Orlicz spaces, thus extending to the nonsymmetric case the results obtained for the selfadjoint semigroups in [15]. To this end we apply the ideas of [2] developed for the case $A = -\frac{1}{2}I$ to prove that $\operatorname{dom}_p(L)$ is contained in the Orlicz space $L^p\operatorname{Log}^r L$ for $r < p$. Next, using the representation of R_t as a second quantized operator we show that $\operatorname{dom}_2(L)$ is continuously imbedded into the Sobolev space $W_{Q_\infty}^{2,2}$.

Finally, we discuss the symmetric case which is important in numerous physical applications and has been an object of intense study for a long time. First, we provide necessary and sufficient conditions for L to be symmetric which extend the earlier result of [28] obtained for $Q = I$. Next, using the results obtained in [8] we provide a complete characterization of the domain of L in $L^p(H,\mu)$ in terms of Sobolev spaces.

2 General Ornstein-Uhlenbeck semigroup

In this section we present some properties of an arbitrary Ornstein-Uhlenbeck semigroup which satisfies conditions (1.2) and (1.3) without any further assumptions.

Let $H_0 = \operatorname{im}\left(Q_\infty^{1/2}\right)$ and $\|h\|_0 = \left\|Q_\infty^{-1/2}h\right\|$ for $h \in H_0$. For the reader's convenience we recall the basic lemma from [6].

Lemma 2.1. *Assume (1.2) and (1.3). The family of operators $S_0^*(t) = Q_\infty^{1/2}S^*(t)Q_\infty^{-1/2}$ defined on H_0 which is a C_0-semigroup on $(H_0, \|\cdot\|_0)$ has a unique extension to a C_0-semigroup of contractions on H (still denoted by $S_0^*(t)$). The space H_0 is invariant for the semigroup $S(t)$. Moreover, $\|S_0^*(t)\| < 1$ if and only if (1.4) holds.*

In the sequel we denote by K_0 the domain of the operator A_0^* considered as a generator of the C_0-semigroup in H_0. Let $L^2(H,\mu) = \bigoplus_{n\geqslant 0}\mathcal{H}_n$ be the Wiener-Ito decomposition of the space $L^2(H,\mu)$. We will denote by I_n the orthogonal projection of $L^2(H,\mu)$ onto the n-th polynomial chaos $\mathcal{H}_n$. In particular $I_0(\phi) = \langle\phi, 1\rangle$. For any $h \in H$ we denote by ϕ_h the measurable linear function on H such that $\phi_h(x) = \left\langle Q_\infty^{-1/2}x, h\right\rangle$ for $x \in H_0$. The domain of L in $L^p(H,\mu)$ will be denoted by $\operatorname{dom}_p(L)$.

The space

$$\mathcal{P}(K_0) = \operatorname{lin}\left\{I_n\left(\phi_h^n\right) : h \in K_0, n = 0, 1, \dots\right\}$$

is dense in $L^p(H,\mu)$ for all $p \geqslant 1$ and its elements may be identified with polynomials of n variables, $n = 0, 1, \dots$. Note that for any $\phi \in \mathcal{P}(K_0)$

$$R_t\phi(x) = \int_H \phi\left(S(t)x + y\right)\mu_t(dy),$$

where $\mu_t = N\left(0, Q_t\right)$ and

$$Q_t = \int_0^t S(s)QS^*(s)ds.$$

Using this fact one may show, see [6], that

$$\mathcal{P}(K_0) \subset \operatorname{dom}_2(L)$$

and

$$LI_n\left(\phi_{h_1}\dots\phi_{h_n}\right) = \sum_{i=1}^n I_n\left(\phi_{h_1}\dots\phi_{h_{i-1}}\phi_{A_0^*h_i}\phi_{h_{i+1}}\dots\phi_{h_n}\right) \tag{2.1}$$

for $h_1, \dots, h_n \in \operatorname{dom}\left(A_0^*\right)$. Hence, it follows that

$$\mathcal{P}(K_0) \subset \operatorname{dom}_p(L).$$

Lemma 2.2. *Assume (1.2) and (1.3). Then $\mathcal{P}(K_0)$ is a core for L in $L^p(H,\mu)$ for all $p \in [1,\infty)$. Moreover, L is a unique extension of $(L, \mathcal{P}(K_0))$ to a generator of C_0-semigroup on $L^p(H,\mu)$.*

Let $D\phi$ denote the Fréchet derivative of $\phi \in \mathcal{P}(K_0)$. We define the first Sobolev norm of $\phi \in \mathcal{P}(K_0)$ by

$$\|\phi\|_{1,p}^2 = \|\phi\|_p^2 + \left\|Q^{1/2}D\phi\right\|_p^2$$

and for $n \geqslant 2$

$$\|\phi\|_{n,p}^2 = \|\phi\|_{n-1,p}^2 + \left\|\left(Q^{1/2}\right)^{\otimes n} D^n\phi\right\|_p^2,$$

where the norm of the operator $\left(Q^{1/2}\right)^{\otimes n} D^n\phi(x)$ is the Hilbert-Schmidt norm in the space $H^{\otimes n}$. The completion of $\mathcal{P}(K_0)$ in the norm $\|\cdot\|_{n,p}$ is denoted by $W_Q^{n,p}$. The space $W_Q^{1,p}$ may be identified with a subspace of $L^p(H,\mu)$ if and only if the operator $D_Q = Q^{1/2}D$ with the domain $\mathcal{P}(K_0)$ is closable. In particular, for the Malliavin derivative D_M we have $D_M = Q_\infty^{1/2} \circ D_{Q_\infty}$. A necessary and sufficient condition for the closability of D_Q in $L^2(H,\mu)$ was given in [18].

Let N be an operator in $L^2(H,\mu)$ with the domain $\mathcal{P}(K_0)$. We say that N is closable on $\mathrm{dom}_2(L)$ endowed with the graph norm if $N\phi_n \to 0$ for every sequence $(\phi_n) \subset \mathcal{P}(K_0)$ such that

$$\phi_n \to 0 \quad \text{and} \quad L\phi_n \to 0.$$

The closure of N in this norm will be denoted by $\overline{N}$. The next result proved in [10] gives the precise formulation of the well known property of diffusion semigroups.

Theorem 2.3. *([10]) Assume (1.2) and (1.3). Then the operator D_Q with the domain $\mathcal{P}(K_0)$ is closable on* $\mathrm{dom}_2(L)$ *endowed with the graph norm and*

$$\langle\phi, L\phi\rangle_2 = -\frac{1}{2}\left\|\overline{D_Q}\phi\right\|_2^2, \quad \phi \in \mathrm{dom}_2(L).$$

Let $V = Q^{1/2}Q_\infty^{-1/2}$ be an operator in H with the domain H_0. Putting $\phi = \phi_h$ in the above theorem and taking (2.1) into account we obtain

$$\langle -A_0^*h, h\rangle = \frac{1}{2}\|Vh\|^2, \quad h \in K_0. \tag{2.2}$$

Proposition 2.4. *([10]) Assume that (1.2) and (1.3) hold. Then the space $W_Q^{1,p}$ is continuously imbedded into $L^p(H,\mu)$ if and only if V is closable.*

Let us recall the result from [6] which provides necessary and sufficient conditions for the hypercontractivity of the semigroup (R_t).

Theorem 2.5. *For every $t \geqslant 0$ and $p \in [1, \infty)$*

$$\|R_t\|_{L^p \to L^{q(t)}} = 1,$$

where

$$q(t) = 1 + \frac{p-1}{\|S_0(t)\|^2}.$$

In particular, R_t is hypercontractive if and only if $\|S_0(t)\| < 1$, or equivalently if and only if (1.4) holds.

2.1 The symmetric case

In this section we present some properties of symmeteric Ornstein-Uhlenbeck generators obtained in [9]. We start with the necessary and sufficient conditions for the semigroup (R_t) to be selfadjoint in $L^2(H, \mu)$. This problem has been solved in [28] for the case $Q = I$.

Theorem 2.6. *Assume (1.2) and (1.3). Then the semigroup (R_t) is selfadjoint in $L^2(H, \mu)$ if and only if $Q(\mathrm{dom}(A^*)) \subset \mathrm{dom}(A)$ and*

$$AQx = QA^*x, \quad x \in \mathrm{dom}(A^*).$$

Remark 2.7. If the semigroup (R_t) is symmetric, then V is closable and $A_0 = -\frac{1}{2}V^*\overline{V}$. Moreover, $\ker(Q) = \{0\}$.

It is shown in [9] that the operator $(AQ, \mathcal{P}(K_0))$ has Friedrichs extension to a selfadjoint operator in H and $\mathcal{P}(K_0)$ is a core for $(-AQ)^{1/2}$. We denote by $W^{1,p}_{AQ}$ the closure of $\mathcal{P}(K_0)$ in the norm

$$\|\phi\|_{1,p,AQ} = \left(\|\phi\|_p^p + \left\|(-AQ)^{1/2} D\phi\right\|_p^p\right)^{1/p}, \quad \phi \in \mathcal{P}(K_0).$$

Similarly, $W^{2,p}_Q$ denotes the closure of $\mathcal{P}(K_0)$ with respect to the norm

$$\|\phi\|_{2,p,Q} = \left(\|\phi\|_p^p + \int_H \left\|Q^{1/2}D\phi(x)\right\|^p \mu(dx) + \int_H \left\|Q^{1/2}D^2\phi(x)Q^{1/2}\right\|_{HS}^p \mu(dx)\right)^{1/p},$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm of an operator. The next theorem characterizes the domain of L in $L^p(H, \mu)$ for all $p \in (1, \infty)$. More general results of this type may be found in [8] and [9]. The domain of $(-L)^{1/2}$ in $L^p(H, \mu)$ was characterized in [26].

Theorem 2.8. *Assume that L is selfadjoint in $L^2(H, \mu)$. Then for every $p \in (1, \infty)$*

$$\mathrm{dom}_p\,(-L)^{1/2} = W^{1,p}_Q,$$

and

$$\mathrm{dom}_p\,(L) = W^{2,p}_Q \cap W^{1,p}_{AQ},$$

where AQ denotes the Friedrichs extension of the operator AQ defined on K_0.

3 The spectral gap inequality

In this section we study the Ornstein-Uhlenbeck generator L under the assumption (1.5). The main result says that (1.5) is equivalent to the existence of a gap in the spectrum of L, that is for the spectrum $\sigma_2(L)$ of L in $L^2(H,\nu)$ we have

$$\sigma_2(L)\backslash\{0\} \subset \{\lambda \in C : \operatorname{Re}\lambda \leqslant -r\}, \tag{3.1}$$

for a certain $r > 0$. We present the results obtained in [10]. It is well known (see Proposition B.1. in [13]) that (1.5) is equivalent to the following condition.

There exists $a > 0$ such that for all $x \in H$

$$\left\|Q^{1/2}x\right\| \geqslant a\left\|Q_\infty^{1/2}x\right\|. \tag{3.2}$$

Obviously, (3.2) holds if and only if

$$\|Vx\| \geqslant a\|x\| \tag{3.3}$$

for all $x \in H_0$. Let us recall that by (11.22) in [13]

$$H_0 = \operatorname{im}(T), \tag{3.4}$$

where $T : L^2(0,\infty;H) \to H$ is given by the formula

$$Tu = \int_0^\infty S(s)Q^{1/2}u(s)ds.$$

Hence, applying Proposition B.1 from [13] and taking into account that $T^*h(t) = Q^{1/2}S^*(t)h$ we obtain the following result. Let

$$a_2(L) = \sup\left\{a > 0 : \left\|Q^{1/2}x\right\| \geqslant a\left\|Q_\infty^{1/2}x\right\|, x \in H\right\}.$$

Proposition 3.1. *Condition (1.5) holds if and only if for a certain $c > 0$*

$$\int_0^\infty \left\|Q^{1/2}S^*(t)h\right\|^2 dt \leqslant c\left\|Q^{1/2}h\right\|^2, \quad h \in H. \tag{3.5}$$

or equivalently

$$\int_0^\infty \left\|Q^{1/2}S^*(t)Q^{-1/2}h\right\|^2 dt \leqslant c\|h\|^2, \quad h \in Q^{1/2}(H). \tag{3.6}$$

Moreover, if (3.6) holds then

$$\frac{1}{a_2^2(L)} = \sup\left\{\int_0^\infty \left\|Q^{1/2}S^*(t)Q^{-1/2}h\right\|^2 dt : h \in Q^{1/2}(H), \|h\| = 1\right\}.$$

Proof. In view of the previous considerations, it is enough to noitce that if (3.6) holds then the operator

$$Q^{1/2}(H) \ni h \to Ch = Q^{1/2}S^*(\cdot)Q^{-1/2}h \in L^2(0,\infty;H), \tag{3.7}$$

extends to a bounded operator on H and

$$\left\|V^{-1}h\right\|_H = \|Ch\|_{L^2(0,\infty;H)}\,.$$

∎

Remark 3.2. Note that, $V^{-1}: H \to H$ is compact if and only if $C: H \to L^2(0,\infty;H)$ is compact.

Let

$$L_0^2(H,\mu) = \left\{\phi \in L^2(H,\mu) : \int_H \phi(x)\mu(dx) = 0\right\}.$$

Clearly, $R_t\left(L_0^2(H,\mu)\right) \subset L_0^2(H,\mu)$. The restriction of R_t to $L_0^2(H,\mu)$ will be denoted by R_t^0. The theorem below provides necessary and sufficient conditions for the spectral gap inequality (3.8) for L.

Theorem 3.3. *Assume (1.2) and (1.3), and let $a > 0$ be fixed. Then the following conditions are equivalent.*

(i) *Condition (3.2) holds.*

(ii) $\|S_0(t)\| \leqslant e^{-ta_2^2(L)/2}$ *for all* $t \geqslant 0$.

(iii) $\left\|R_t^0\right\|_2 \leqslant e^{-ta_2^2(L)/2}$ *for all* $t \geqslant 0$.

(iv) *The generator L of the semigroup (R_t) in $L^2(H,\mu)$ enjoys the property:*

$$\langle -L\phi, \phi\rangle \geqslant \frac{1}{2}a_2^2(L)\|\phi\|_2^2, \quad \phi \in \mathrm{dom}_2(L) \cap L_0^2(H,\mu), \tag{3.8}$$

(and hence hence (3.1) holds).

Proof. If $\|S_0^*(t)\| \leqslant e^{-a^2t/2}$ then by properties of contraction semigroups

$$\langle -A_0^*h, h\rangle \geqslant \frac{1}{2}a^2\|h\|^2 \tag{3.9}$$

for all $h \in \mathrm{dom}\,(A_0^*)$. Hence, by (2.2) $\|Vh\|^2 \geqslant a^2\|h\|^2$ for all $h \in K_0$ and by the limiting argument (3.2) follows for all $h \in H_0$. Conversely, if (3.2) holds then (2.2) yields (3.9) for all $h \in K_0$ and since K_0 is a core for A_0^* inequality (3.9) is satisfied for all $h \in \mathrm{dom}\,(A_0^*)$. Thereby $\|S_0^*(t)\| \leqslant e^{-a^2t/2}$ which concludes the proof that (i) and (ii) are equivalent. The equivalence of (iii) and (iv) follows by the well known properties of contraction semigroups. Next, (ii) and (iii) are equivalent by the result in [6]. ∎

Corollary 3.4. *(i) If D_Q is closable then (3.8) is equivalent to the Poincaré inequality,*

$$\|\phi - \langle \phi, 1\rangle\|_2^2 \leqslant \frac{2}{a_2^2(L)} \|D_Q\phi\|_2^2, \quad \phi \in W_Q^{1,2}.$$

(ii) If any of the conditions of Theorem 3.3 is satisfied then (1.4) holds.

Proof. Part (i) is obvious. To prove (ii) note that (ii) of Theorem 3.3 yields $\|S_0^*(t)\| < 1$ for all $t > 0$ and by Proposition 2 in [7] this property is equivalent to (1.4). ∎

Below we provide some necessary and sufficient conditions for (1.5) to hold.

Corollary 3.5. *Assume that $S(t)Q = QS(t)$. Then (1.5) holds if and only there exist $M \geqslant 1$ and $\beta > 0$ such that*

$$\|S(t)\| \leqslant Me^{-\beta t}, \quad t \geqslant 0. \tag{3.10}$$

If (3.10) holds then

$$\frac{1}{a_2^2(L)} = \sup_{\|x\|\leqslant 1} \int_0^\infty \|S^*(t)x\|^2 \, dt.$$

Proof. Note first that Q commutes with $S^*(t)$ and so does $Q^{1/2}$. It follows from (i) that $Q^{1/2}(H)$ is dense in H. Indeed, assume that $Qx = 0$ for a certain $x \neq 0$. Then $QS^*(t)x = S(t)^*Qx = 0$ and thereby $Q_\infty x = 0$ which contradicts (1.3). Assume first that (3.10) holds. Then the operator

$$T_1 u = \int_0^\infty S(t)u(t)dt, \quad u \in L^2(0,\infty,H),$$

is bounded from $L^2(0,\infty;H)$ to H. Since $T = Q^{1/2}T_1$ it follows that (1.5) holds. Conversely, assume that (1.5) holds. Hence, by (3.5), for all $x \in Q^{1/2}(H)$

$$\int_0^\infty \|S^*(t)x\|^2 \, dt \leqslant c\,\|x\|^2 \tag{3.11}$$

and by the Fatou Lemma (3.11) follows for all $x \in H$. In view of the Datko-Pazy theorem [23], (3.11) yields (3.10). Finally, if (3.10) holds then

$$\left(V^{-1}\right)^* V^{-1} = Q^{-1/2}Q_\infty Q^{-1/2} = \int_0^\infty S(t)S^*(t)dt.$$

Hence,

$$\frac{1}{a_2^2(L)} = \sup_{\|x\|\leqslant 1} \left\|V^{-1}x\right\|^2 = \sup_{\|x\|\leqslant 1} \int_0^\infty \|S^*(t)x\|^2 \, dt$$

which concludes the proof. ∎

It follows from Theorem 3.3 that if (1.5) holds then $S_0^*(t)$ is a strict contraction for $t > 0$ and therefore the semigroup (R_t) is hypercontractive by the result in [6]. It was shown by Gross (see [19] and references therein) that hypercontraction property is closely related to the following Logarithmic Sobolev Inequality

$$\int_H |\phi(x)|^p \log |\phi(x)| \, \mu(dx) \leqslant c(p) \, \langle (\gamma(p) - L)\phi, \phi_p \rangle + \|\phi\|_p^p \log \|\phi\|_p \tag{3.12}$$

for $\phi \in \mathrm{dom}_p(L)$, $\phi_p = \operatorname{sgn}\phi|\phi|^{p-1}$ and $p > 1$. It is known that (3.12) implies the Spectral Gap inequality. However, it was shown in [16] that (3.12) does not hold for arbitrary hypercontractive Ornstein-Uhlenbeck semigroup. Below we give a necessary and sufficient condition for (3.12) to hold. This result may be viewed as a generalization of the result of Rothaus [25] and Simon [27] to the case of nonsymmetric semigroups.

Theorem 3.6. *The generator L of the Ornstein-Uhlenbeck semigroup satisfies the Logarithmic Sobolev Inequality (3.12) if and only if (1.5) holds and in that case for* $\phi \in \mathrm{dom}_p(L)$

$$\int_H |\phi(x)|^p \log |\phi(x)| \, \mu(dx) \leqslant \frac{p}{p-1} \frac{1}{a_2^2(L)} \langle -L\phi, \phi_p \rangle + \|\phi\|_p^p \log \|\phi\|_p, \tag{3.13}$$

where the constant a is given in (3.2).

Proof. Assume (1.5). By the result in [6] the semigroup (R_t) is hypercontractive from $L^p(H, \mu)$ to $L^q(H, \mu)$ for all

$$q \leqslant q(t,p) = 1 + \frac{p-1}{\|S_0(t)\|^2} \tag{3.14}$$

and $\|R_t\|_{p,q} = \infty$ for $q > q(t,p)$. If (1.5) holds, then, in particular, (3.14) is satisfied for

$$q \leqslant 1 + (p-1)e^{a_2^2(L)t}.$$

Hence by Theorem 3.12 in [19] (3.12) holds with $\gamma = 0$ and $c(p) = \frac{p}{(p-1)a_2^2(L)}$. Conversely, assume that the Logarithmic Sobolev Inequality (3.12) holds for every $p > 1$ and certain $\gamma(p), c(p) > 0$. Then by Theorem 3.7 in [19]

$$\|R_t\|_{p,\rho(t,p)} \leqslant e^{M(t,p)} \tag{3.15}$$

for a function ρ which is a solution to the initial value problem

$$c(\rho)\frac{d\rho}{dt} = \rho, \qquad \rho(0,p) = p, \quad t \geqslant 0$$

and with

$$M(t,p) = \int_0^t \gamma(\rho(s,p)) \, ds.$$

But, by the above mentioned result in [6], (3.15) implies that $\rho(t,p) \leqslant q(t,p)$ and in this case $\|R_t\|_{\rho(t,p)} \leqslant 1$. Therefore we can assume that $M = 0$ which together with (3.15) and Theorem 3.12 from [19] yields (3.13). Take $h \in \operatorname{dom}(A_0^*)$ such that $\|h\| = 1$ and define on H the measurable linear function $\phi_h(x) = \left\langle Q_\infty^{-1/2} x, h \right\rangle$. Then (5.2) yields

$$\int_H |\phi_h(x)|^2 \log |\phi_h(x)| \; \mu(dx) \leqslant c(2) \langle -A_0^* h, h \rangle$$

with the left-hand side being an absolute constant. As a consequence we find that

$$\langle -A_0^* h, h \rangle \geqslant c\|h\|^2$$

for a certain positive c and all $h \in \operatorname{dom}(A_0^*)$. Finally, (2.2) yields $\|Vh\|^2 \geqslant 2c\|h\|^2$ and this inequality concludes the proof. ∎

Corollary 3.7. *Assume (1.2), (1.3) and* $\dim H < \infty$. *Then the Logarithmic Sobolev Inequality (3.12) holds if and only if* $\det Q \neq 0$.

We will apply the results obtained above to the symmetric case. In this case it follows from (2.2) that $\|Vx\| = \sqrt{2} \left\| (-A_0)^{1/2} x \right\|$ on K_0 and therefore

$$H_0 \subset \operatorname{dom} \overline{V|\,K_0} = \operatorname{dom}(-A_0)^{1/2}.$$

This implies that V with the domain H_0 is closable and

$$\operatorname{dom} \overline{V} = \operatorname{dom}(-A_0)^{1/2}.$$

It follows from Theorem 2.3 that

$$\operatorname{dom}(L^*) = \operatorname{dom}(L) \subset W_Q^{1,2}.$$

Note that in general $\operatorname{dom}(L)$ is not a subset of $W_{Q_\infty}^{\alpha,2} = \operatorname{dom}\left((I - L_M)^{\alpha/2}\right)$ for any $\alpha > 0$.

As a consequence of Theorem 3.6 and Proposition 3.5 we obtain a complete characterization of the hypercontractivity properties in the case when $S(t)Q = QS(t)$. Equivalence of (iii) and (iv) below is well known [19].

Corollary 3.8. *Assume that* $S(t)Q = QS(t)$. *Then the following conditions are equivalent.*

(i) *The semigroup* $(S(t))$ *is exponentially stable.*

(ii) $\operatorname{im}\left(Q_t^{1/2}\right) = \operatorname{im}\left(Q_\infty^{1/2}\right)$ *for* $t > 0$ *(i.e. (1.4) holds).*

(iii) *The generator* L *of* (R_t) *satisfies the Logarithmic Sobolev Inequality (3.12).*

(iv) (R_t) *is hypercontractive from* $L^p(H,\mu)$ *to* $L^q(H,\mu)$ *for all* $1 < p < q \leqslant 1 + \frac{p-1}{\|S_0(t)\|^2}$.

Example 3.9. We provide here an example of the transition semigroup for which (1.4) holds but (1.5) is not satisfied. The first example of this type, related to the finite dimensional Ornstein-Uhlenbeck process, was given in [16]. We shall present another one which is of some importance in Mathematical Finance (see [20] and [29]) for details). In the space $L^2(0,1)$ we consider the equation

$$\begin{cases} dZ = AZdt + bdW, \\ Z(0) = x, \end{cases}$$

where the operator

$$A = \frac{\partial}{\partial \zeta}, \quad \operatorname{dom}(A) = \left\{x \in H^1(0,1) : x(1) = 0\right\},$$

generates the semigroup

$$S(t)x(\zeta) = \begin{cases} x(t+\zeta) & \text{if} \quad t+\zeta \leqslant 1, \\ 0 & \text{if} \quad t+\zeta > 1. \end{cases}$$

We assume that W is a one dimensional Wiener process and $b \in H$ is such that

$$\operatorname{lin}\{S(t)b : t \geqslant 0\}$$

is dense in H. Obviously R_t is hypercontractive for $t > 1$ and $H_0 \not\subset \operatorname{im}\left(Q^{1/2}\right)$. If we take for example $b \equiv 1$ then it is easy to see that H_0 is dense in H but for $0 < t_1 < t_2 < 1$

$$\overline{\operatorname{im}\left(Q_{t_1}^{1/2}\right)} \not\subseteq \overline{\operatorname{im}\left(Q_{t_2}^{1/2}\right)} \not\subseteq H.$$

Therefore $\operatorname{im}\left(Q_t^{1/2}\right) \neq \operatorname{im}\left(Q_\infty^{1/2}\right)$ and the semigroup (R_t) is not hypercontractive for $t < 1$.

Example 3.10. Let $\mathcal{O} \subset R^d$ be a bounded domain with a smooth boundary and let $H = L^2(\mathcal{O})$. We consider a differential operator

$$Af(\zeta) = \sum_{|\alpha|=2m} a_\alpha(\zeta)\frac{\partial^\alpha f}{\partial \zeta^\alpha}(\zeta) + \sum_{|\alpha|<2m} a_\alpha(\zeta)\frac{\partial^\alpha f}{\partial \zeta^\alpha}(\zeta)$$

with the zero Dirichlet boundary conditions. We assume that the operator

$$A_1 f(\zeta) = \sum_{|\alpha|=2m} a_\alpha(\zeta)\frac{\partial^\alpha f}{\partial \zeta^\alpha}(\zeta)$$

is strongly elliptic, all the coefficients a_α are bounded and measurable and for $|\alpha| = 2m$ the coefficients a_α are continuous in $\overline{\mathcal{O}}$. Then by the result in [1] the operator A with the domain $H^{2m}(\mathcal{O}) \cap H_0^m(\mathcal{O})$ generates an analytic semigroup $(S(t))$ in H. We will also assume that the semigorup $(S(t))$ is exponentially stable. Consider a linear stochastic differential equation in $H = L^2(\mathcal{O})$:

$$\begin{cases} dZ = AZdt + BdW, \\ Z(0) = x. \end{cases}$$

We assume that B is an isomorphism of H onto $H_\theta = \operatorname{dom}(-A)^\theta$, where

$$\frac{2m}{d}(1+2\theta) > 1.$$

Then by [1] condition (1.2) holds and the process Z is well defined. To be more explicit one may consider the operator $Bx(\zeta) = (-A)^{-\theta} b(\zeta) x(\zeta)$, where b is a measurable function on $\mathcal{O}$ such that

$$0 < m \leqslant b(\zeta) \leqslant M, \quad \zeta \in \mathcal{O}.$$

For $Q = BB^*$ we have

$$\operatorname{im}\left(Q^{1/2}\right) = H_\theta$$

and the norm in H_θ is equivalent with the graph norm in $Q^{1/2}(H)$. Hence, the space $Q^{1/2}(H)$ is invariant for the semigroup $(S(t))$ which is strongly continuous and exponentially stable in $Q^{1/2}(H)$ endowed with the graph norm. Equivalently, the operators $S_Q(t) = Q^{-1/2}S(t)Q^{1/2}$ are well defined on H and $\{S_Q(t) : t \geqslant 0\}$ is an exponentially stable C_0-semigroup on H. Since (3.5) may be rewritten in the form

$$\int_0^\infty \left\| Q^{1/2} S^*(t) Q^{-1/2} h \right\|^2 dt \leqslant c \|h\|^2, \quad h \in Q^{1/2}(H), \tag{3.16}$$

the above discussion shows that (3.16) holds for every $h \in H$ which yields (1.5).

Example 3.11. Let $H = L^2_\rho(R)$ be a weighted L^2-space with the weight $\rho(\zeta) = \left(1 + k\zeta^2\right)^{-1}$, $k > 0$. For different values of $k > 0$ we obtain equivalent norms so dependence on k is omitted. Let $A = \Delta - mI$ with $m > 0$, where Δ denotes the Laplace operator in $L^2(R)$ with the domain $H^2(R)$. The operator A generates a symmetric semigroup $S(t) = e^{tA}$ on $L^2(R)$ which extends to an analytic C_0-semigroup on $L^2_\rho(R)$. Moreover, by Proposition 9.4.5 in [14]

$$\|S(t)\|_{L^2_\rho} \leqslant e^{-\beta t},$$

where

$$\beta = m - \frac{3k}{4} > 0$$

for k sufficiently small. We define $Q = JJ^*$, where J is the imbedding of $L^2(R)$ into $L^2_\rho(R)$. Then L is a well defined generator of the Ornstein-Uhlenbeck transition semigroup which has an invariant measure (see [14] for details). It is easy to check that $(S(t))$ is an exponentially stable C_0-semigroup in $L^2(R)$ and therefore by Proposition 3.1 condition (1.5) holds. Since $Q^{1/2}(H) = L^2(R)$ this also follows directly from the inclusion

$$H_0 = \left\{ \int_0^\infty S(t)u(t)dt : u \in L^2\left(0,\infty; L^2(R)\right)\right\} \subset L^2(R) = Q^{1/2}(H).$$

4 Sobolev spaces $W_Q^{1,p}$ and characterizations of L and its domain

In this section we study properties of the imbeddings of the Sobolev space $W_Q^{1,p}$ which hold under condition (1.5) and give some better characterizations of $\mathrm{dom}_p(L)$ which follow from (1.5). We will start with the result which shows that if (1.5) holds then similarly as in the case of Malliavin operator L_M the generator L is a composition, loosely speaking of gradient and divergence operator. Let $\mathcal{D}$ denote the maximal domain of the operator $D^*_{Q_\infty}A_0^*D_{Q_\infty}$ in $L^p(H,\mu)$, that is $\phi \in \mathcal{D}$ if and only if $\phi \in W^{1,p}_{Q_\infty}$, $D_{Q_\infty}\phi \in L^p\left(H,\mu; \mathrm{dom}\left(A_0^*\right)\right)$ and $A_0^*D_{Q_\infty}\phi \in \mathrm{dom}_p\left(D^*_{Q_\infty}\right)$.

Corollary 4.1. *([10]) If (1.5) holds then*

$$L\phi = D^*_{Q_\infty}A_0^*D_{Q_\infty}\phi, \quad \phi \in \mathcal{D} \cap \mathrm{dom}_2(L).$$

In the theorems below we characterize imbedding of the Sobolev space $W_Q^{1,p}$ into $L^p(H,\mu)$ and give a sufficient condition for D_Q to have a closed range in $L^p(H,\mu)$. We show also that (1.5) yields the Helmholtz decomposition of p-integrable vector fields on H. Theorem 4.4 extends the results of [12], [24], and [14] to the case when $p \neq 2$ and the operators Q and Q_∞ do not commute. In the theorem below we use the notation

$$W^{1,p}_{0,Q} = W_Q^{1,p} \cap L_0^p(H,\mu).$$

Theorem 4.2. *([10]) Assume (1.2) and (1.3). Then for every $p \in (1,\infty)$ the following holds.*

(i) *The space $W_Q^{1,p}$ is continuously imbedded into $W^{1,p}_{Q_\infty}$ if and only if V is closable and (1.5) holds.*

(ii) *Assume that V is closable. If (1.5) holds then $\mathrm{im}\left(\overline{D_Q}\right)$ is closed in $L^p(H,\mu)$ and for every $F \in L^p(H,\mu;H)$ we have a unique decomposition*

$$F = D_QU + F_0,$$

with $U \in W^{1,p}_{0,Q}$ and $D_Q^*F_0 = 0$.

Theorem 4.3. *([10]) Assume (1.2) and (1.3). If V is closable and the operator $C : H \to L^2(0,\infty;H)$ defined in Section 3 is compact then the imbedding of $W_Q^{1,p}$ into $L^p(H,\mu)$ is compact for all $p \in (1,\infty)$. Conversely, if for a certain $p > 1$ the imbedding of $W_Q^{1,p}$ into $L^p(H,\mu)$ is compact then V is closable and C is compact.*

We will sketch the proof of sufficiency only. We consider first the case $p = 2$. If $\overline{V}^{-1}$ is compact then $V^*\overline{V}$ has a complete orthonormal system of eigenvectors $\{e_j : j \geqslant 1\}$ with the corresponding sequence of eigenvalues $\{\gamma_j : j \geqslant 1\}$ such that $0 < \gamma_1 \leqslant \gamma_2 \leqslant \dots$ and $\lim \gamma_j = \infty$. Let $\alpha = (\alpha_1, \dots)$ denote an arbitrary sequence of nonnegative integers such that $|\alpha| = \alpha_1 + \alpha_2 + \dots < \infty$. If V is closable then the strongly contiuous semigroup on $L^2(H,\mu)$ generated by $L^V = -D_Q^*\bar{D}_Q$ will be denoted by $\left(R_t^V\right)$. The functions

$$f_\alpha = \prod_{j=1}^{\infty} \frac{1}{\sqrt{\alpha_j!}} I_{\alpha_j}\left(\phi_{e_j}^{\alpha_j}\right)$$

form a complete orthonormal system in $L^2(H,\mu)$ and (2.1) yields

$$L^V f_\alpha = \langle \alpha, \gamma \rangle f_\alpha,$$

where $\langle \alpha, \gamma \rangle = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \dots$. Since $\lim_{j\to\infty} \gamma_j = \infty$ and $\gamma_j > 0$ the only accumulation point of the sequence $\langle \alpha, \gamma \rangle$ is infinity and all of them are of finite multiplicities since γ_j are. Since

$$\|\phi\|_{1,2}^2 = \left\| \left(I - L^V\right)^{1/2} \phi \right\|^2 \tag{4.1}$$

the compactness of the operator $\left(I - L^V\right)^{-1/2}$ concludes the proof of sufficiency for $p = 2$. Since the semigroup $\left(R_t^V\right)$ is symmetric in $L^2(H,\mu)$, we find that $\left(R_t^V\right)$ is compact in $L^2(H,\mu)$. Since $\left(R_t^V\right)$ is positive and contractive on $L^p(H,\mu)$ for $p \in [1,\infty]$ the interpolation arguments yield compactness of the semigroup $\left(R_t^V\right)$ in $L^p(H,\mu)$ for all $p \in (1,\infty)$. By the result in [8] we have

$$\operatorname{dom}_p \left(-L^V\right)^{1/2} = W_Q^{1,p},$$

hence the imbedding of $\operatorname{dom}_p \left(-L^V\right)^{1/2}$ into $L^p(H,\mu)$ is compact and the theorem follows.

Remark 4.4. If $\dim H < \infty$ then the imbedding of $W_Q^{1,p}$ into $L^p(H,\mu)$ is compact if and only if $\det Q \neq 0$.

Remark 4.5. The imbedding of $W_Q^{1,2}$ into $L^2(H,\mu)$ is never of Hilbert-Schmidt type. It is enough to consider the case $\dim H = 1$ and $Q = I = Q_\infty$. Let f_n denote the n-th normalized Hermite polynomial. Then $Df_n = \sqrt{n} f_{n-1}$ and for sufficiently smooth $\phi \in L^2(R,\mu)$ we have

$$\left(I - L^I\right)\phi = \sum_{n=0}^{\infty} (n+1) \langle \phi, f_n \rangle f_n,$$

hence $\left(I - L^I\right)^{-1/2}$ is not Hilbert-Schmidt.

Theorem 4.6. *([10]) If (1.2), (1.3) and (1.5) hold then* $\mathrm{dom}_2(L)$ *is continuously imbedded into* $W^{2,2}_{Q_\infty}$. *Moreover,*

$$\|\phi\|_{W^{2,2}_{Q_\infty}} \leqslant \max\left(1, \frac{2}{a_2^2(L)}\right) \|(I - L)\,\phi\|_2\,, \tag{4.2}$$

Proof. Let

$$J = \int_0^\infty e^{-t} R_t \, dt$$

be the resolvent of R_t. It is enough to show that the operator $(I - L_M)\,J$ is bounded on $L^2(H,\mu)$. To this end note that by the results of [6]

$$J\phi = \sum_{n\geqslant 0} \int_0^\infty e^{-t} R_t I_n(\phi)\, dt = \sum_{n\geqslant 0} J_n\phi,$$

and J_n is the resolvent of the semigroup R_t restricted to $\mathcal{H}_n$. Since the operator L_M acts invariantly in $\mathcal{H}_n$ it remains to show that

$$\sup_{n\geqslant 0} \|(I - L_M)\,J_n\| < \infty.$$

By Theorem 4 in [6] and (1.5)

$$\|R_t I_n\| \leqslant e^{-na^2 t/2}, \tag{4.3}$$

where $a = a_2(L)$. Thereby

$$\|(I - L_M)\,J_n\phi\| \leqslant \int_0^\infty e^{-t}\,\|(I - L_M)\,R_t I_n(\phi)\|\,dt$$

$$\leqslant (n+1)\int_0^\infty e^{-t\left(1+na^2/2\right)}\,dt\,\|I_n(\phi)\| = \frac{2(n+1)}{na^2+2}\,\|I_n(\phi)\|\,.$$

This yields the estimate (4.2). ∎

Theorem 4.7. *([10]) Assume (1.2) and (1.3), (1.5) and* $p > 1$. *Then*

(i) $\mathrm{dom}_p(L) \subset L^p\mathrm{Log}^r L$, *for* $0 \leqslant r < p$,

(ii) *If moreover the semigroup* (R_t) *is analytic then*

$$\mathrm{dom}_p(L^\epsilon) \subset L^p\mathrm{Log}^r L.$$

for $\epsilon > 0$ *and* $0 \leqslant r < p\epsilon$.

Proof. This theorem has been proved in [2] for $L = L_M$ but the close inspection of the proof shows that it can be repeated for a general Ornstein-Uhlenbeck semigroup if the Logarithmic Sobolev Inequality (3.13) holds. ∎

Remark 4.8. Conditions of analyticity of (R_t) are given in [17]. In particular, if $\dim H < \infty$ then (R_t) is analytic if and only if (1.5) hold.

Finally, as an immediate consequence of Theorems 4.3 and (4.6) we obtain

Corollary 4.9. *Assume that $S(t)Q = QS(t)$. Then the conclusions of Theorem 4.6 hold if and only if the semigroup $(S(t))$ is exponentially stable. Moreover, if the semigroup $(S(t))$ is exponentially stable then the conclusions of Theorem 4.3 hold.*

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