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# Photon Lifetimes in Laser Cavities and Excess Quantum Noise


## Abstract

The concept of photon loss rates from laser cavities is discussed, and the fundamental difference between hot-cavity and cold-cavity loss rates is indicated. This difference gets particularly important in the presence of various loss channels. For a simple cavity model, a quantum mechanical expression is derived for the Hamiltonian that couples the quantized field inside the cavity to the outside world.


## 1 Introduction

Lossy resonators play an essential role in cavity QED as well as in laser physics. In many cases, the cavity modes are described simply by ignoring the losses. The output of the cavity is then calculated by assuming a field distribution corresponding to a perfect cavity. The coupling to the outside world is described by a cavity loss rate.

An important example of present interest is the excess noise factor of a laser. The linewidth of a quantum-limited single-mode laser can be expressed as [1]

$$\Delta\omega = K \Delta\omega_{\mathrm{ST}} , \tag{1}$$

with $\Delta\omega_{\mathrm{ST}}$ the Schawlow-Townes linewidth [2]

$$\Delta\omega_{\mathrm{ST}} = \frac{\Gamma_{\mathrm{h}}^2}{2P_{tot}} . \tag{2}$$

Here

$$\Gamma_{\mathrm{h}} \equiv P_{tot}/W \tag{3}$$

is the loss rate of the lasing cavity (the 'hot-cavity loss rate'), defined as the ratio of the total power loss $P_{tot}$ and the internal energy $W$ of the laser. For simplicity, energies are expressed in units of the photon energy. The enhancement factor $K$ can be expressed in terms of the non-orthogonality of the modes of the lossy cavity [3]. It is often assumed that the hot-cavity loss rate is identical to the inverse lifetime of a photon in the cavity without gain (the 'cold-cavity loss rate' $\Gamma_{\mathrm{c}}$).

In this contribution we discuss the concept of cavity loss rates, both from a semiclassical and a quantum mechanical point of view. First we indicate that the hot-cavity loss rate cannot simply be identified with the inverse of the lifetime of a photon in the non-lasing cavity.

Moreover, unstable laser cavities generally lose power not only through their outcoupling mirror, but also into one or more other loss channels. Even when the total cold- and hot-cavity loss rates are identical, the relative distribution of the losses over the various loss channels may differ in the situation of a lasing system compared to a decaying cavity. This has implications for the experimental determination of the excess noise factor $K$.

In a simple model system of a half-sided open cavity, we discuss the form of the quantum mechanical Hamiltonian that describes the coupling of the internal modes of the cavity, and the external world. In earlier discussions the coupling term in this Hamiltonian was a phenomenological parameter, that is commonly taken to be independent of frequency [4, 5]. We derive an explicit expression for the coupling term. The form of the Hamiltonian is important to describe quantum field effects, such as non-classical output fields from a cavity or spontaneous decay of atoms in cavities.

## 2 Total loss rates

We consider a laser resonator of length $L$, that is lasing in a single transverse mode. As usual, we map the round trip of the light travelling up and down the cavity on the interval $-2L < z < 0$, with z the coordinate in the propagation direction. The gain and loss are represented by the functions $g(z)$ and $\kappa(z)$ respectively. A localized lossy element such as an aperture, positioned at $z_i$ with effective intensity transmissivity $T_i$, is described by the loss function

$$\kappa(z) = -\ln(T_i)\delta(z - z_i) \; . \tag{4}$$

For an outcoupling mirror with intensity reflectivity $R$, a similar expression holds with $T_i$ replaced by $R$. The total loss factor over a round trip can be written as the product of the loss factors of each lossy element, according to

$$T = \Pi_i T_i = \exp[-\int_{-2L}^{0} dz\kappa(z)] \; . \tag{5}$$

The periodicity of the system implies that

$$\int_{-2L}^{0} dz[g(z) - \kappa(z)] = 0, \tag{6}$$

which states that the total gain factor balances the total loss factor.

For a system with a total loss factor $T$ the cold-cavity loss rate is determined by the relation

$$e^{-\Gamma_c \tau} \equiv T \; , \tag{7}$$

with $\tau$ the round-trip time. When $v_{gr}$ is he group velocity in the laser cavity, then its round trip time is

$$\tau = \int_{-2L}^{0} dz \frac{1}{v_{gr}(z)} \; . \tag{8}$$

Above threshold the number of photons per unit length $u(z)$ in the laser is constant in time. Then we find from the continuity equation

$$(g-\kappa)v_{\rm gr}u-\frac{\partial}{\partial z}(v_{\rm gr}u)=\frac{\partial}{\partial t}u=0\,, \tag{9}$$

where $v_{\rm gr}(z)u(z)$ is the photon current. The first term in eq. (9) represents the rate of change of the photon number per unit length due to gain or loss, the second term gives the change due to flow. Formal integration of Eq. (9) gives for the photon current the expression

$$v_{\rm gr}(z)u(z)=v_{\rm gr}(-2L)u(-2L)\exp\Big(\int_{-2L}^{z}dz'[g(z')-\kappa(z')]\Big),\quad -2L<z<0\,. \tag{10}$$

The internal photon number $W$ is found by integrating $u$ over one round trip, which gives

$$W=\int_{-2L}^{0}dz\frac{v_{\rm gr}(-2L)u(-2L)}{v_{\rm gr}(z)}\exp\Big(\int_{-2L}^{z}dz'[g(z')-\kappa(z')]\Big)\,. \tag{11}$$

The total photon loss rate can be found by integrating the loss of the local photon number density $\kappa(z)v_{\rm gr}(z)u(z)$ over a round trip. This gives

$$P_{\rm tot}=\int_{-2L}^{0}dz\kappa(z)v_{\rm gr}(-2L)u(-2L)\exp\Big(\int_{-2L}^{z}dz'[g(z')-\kappa(z')]\Big)\,. \tag{12}$$

After integration of Eq. (12) by parts, and using the periodic boundary condition $\kappa(-2L)=\kappa(0)$, this may alternatively be written as an integration of the power density gain, in the form

$$P_{\rm tot}=\int_{-2L}^{0}dzg(z)v_{\rm gr}(-2L)u(-2L)\exp\Big(\int_{-2L}^{z}dz'[g(z')-\kappa(z')]\Big). \tag{13}$$

The identity of equations (12) and (13) reflects the fact that the power gain exactly compensates the power loss.

This way we arrive at a general expression for the hot-cavity loss rate $\Gamma_{\rm h}$ as the ratio between Eq. (12) (or (13)) and Eq. (11). One notices that the result is by no means always identical to the expression for $\Gamma_{\rm c}$, as determined by Eq. (7). Three different situations can be identified where $\Gamma_{\rm h}$ and $\Gamma_{\rm c}$ are identical. In all cases, the group velocity $v_{gr}$ must be uniform. The first case occurs when also the gain $g$ is uniform, as one recognizes when using eq. (13) for $P_{tot}$. The second case corresponds to the situation of uniform loss $\kappa$, which follows after using eq. (12). Finally, the identity of $\Gamma_{\rm h}$ and $\Gamma_{\rm c}$ also holds when in addition to $v_{\rm gr}$ the photon density $u$ is uniform, which requires that the gain and the loss compensate each other locally. Then the exponential terms in eq. (10) disappears. In these three cases the loss rates are given by

$$\Gamma_{\rm h}=\frac{v_{\rm gr}}{2L}\int_{-2L}^{0}dzg(z)=\frac{v_{\rm gr}}{2L}\int_{-2L}^{0}dz\kappa(z)=\Gamma_{\rm c}\,. \tag{14}$$

We recall that a non-uniform intensity tends to give rise to a longitudinal excess noise factor, due to the combined action of gain and loss [6]. This longitudinal factor must be combined with the transverse enhancement factor [7].

## 3 Partial loss rates

Laser cavities can have various loss channels, both at optical elements or due to absorption in the laser medium. Typical for unstable laser cavities is that intensity is also lost by absorption at an aperture, which is equivalent to spillover at the outcoupling mirror. Accordingly, the total power loss $P_{tot}$ can be written as the sum over the power losses $P_i$ at each lossy element. Then the loss rate (3) can be written as the sum over partial loss rates. According to eqs. (5) and (7), the cold-cavity loss rate can also formally be expressed as a sum over lossy elements. We now consider the case that the group velocity and the gain are uniform, so that $\Gamma_{\rm h}$ and $\Gamma_{\rm c}$ are identical, and equal to $v_{gr}g$. Then we obtain the identity

$$\Gamma_{\rm h} = \Gamma_{\rm c} = \sum_i P_i/W = -\sum_i \frac{v_{gr}}{2L} \ln T_i \, . \tag{15}$$

This identity (15) might suggest that the separate terms in the two summations are equal term by term, so that $\Gamma_i \equiv P_i/W = -v_{gr} \ln T_i/(2L)$. However, this suggestion is wrong in general, as we illustrate by a simple example. We consider a laser cavity with length $L$, with one perfect mirror, and an outcoupling mirror with reflectivity $R$. An aperture is positioned in front of the outcoupling mirror, and the effective aperture transmissivity is $T_{\rm a}$. This transmissivity obviously depends on the transverse mode profile incident on the aperture. The gain obeys the relation $v_{gr}g = \Gamma_{\rm c} = \Gamma_{\rm h}$. In order to extract the excess noise factor from measurements, we have to express the Schawlow-Townes linewidth (2) in terms of measurable quantities, as

$$\Delta\omega_{\rm ST} = \frac{\Gamma_{\rm h}\Gamma_{\rm m}}{2P_{\rm m}} \, . \tag{16}$$

The mirror output $P_{\rm m}$ and the hot-cavity loss rate $\Gamma_{\rm h}$ can be measured directly [8]. The partial loss rate through the mirror $\Gamma_{\rm m} = P_{\rm m}/W$ can be calculated for this system, with the result

$$\Gamma_{\rm m} = v_{\rm gr}g\frac{(1-R)T_{\rm a}}{1-RT_{\rm a}} = \Gamma_{\rm h}\frac{(1/R-1)\exp(-\Gamma_{\rm h}\tau)}{1-\exp(-\Gamma_{\rm h}\tau)} \, , \tag{17}$$

where we used that $RT_{\rm a} = \exp(-\Gamma_{\rm h}\tau)$. Note that all quantities appearing in the last term in (17) can be deduced from experiment. The point is now that this expression (17) can deviate appreciably from the expression for the loss rate through the mirror $-v_{gr} \ln R/(2L)$, which one would guess on the basis of the identity (15). This deviation is particularly important when the loss over the aperture is appreciable.

## 4 Quantum mechanical coupling Hamiltonian

We consider a similar model of cavity decay, now from a quantum mechanical point of view. Photon loss from a cavity is a prototype of quantum dissipation. The cavity extends between a perfect mirror at $z = -L$ and a non-absorptive semi-transparent mirror of amplitude reflectivity $r$ and transmissivity $t$ at $z = 0$. The global mode functions of this system can be denoted as

$$F_k(z) = -\sqrt{\frac{2}{\pi}} e^{ikL} \mathcal{L}(k) \sin([z + L]k) \tag{18}$$

for $-L < z < 0$, and

$$F_k(z) = -i\frac{1}{\sqrt{2\pi}}\left(e^{-ikz} + e^{ikz}[r - te^{2ikL}\mathcal{L}(k)]\right) \tag{19}$$

for $z > 0$. The intra-cavity field strength for a global mode is

$$\mathcal{L}(k) = \sum_{l=0}^{\infty} t\left(-re^{i2kL}\right)^l. \tag{20}$$

The operators for the electric and magnetic field are given by their standard expressions

$$\hat{E}(z) = i\int_0^\infty dk\sqrt{\frac{\hbar ck}{2\epsilon_0}}\hat{a}(k)F_k(z) + H.c. \tag{21}$$

and

$$\hat{B}(z) = \int_0^\infty dk\sqrt{\frac{\hbar}{2\epsilon_0 ck}}\hat{a}(k)\frac{dF_k(z)}{dz} + H.c. \tag{22}$$

The global operators $\hat{a}(k)$ satisfy the usual commutation relations for continuous annihilation and creation operators, i.e. $[\hat{a}(k), \hat{a}(k')] = 0$ and $[\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k - k')$. The field Hamiltonian is given by the standard expression

$$\hat{H} = \frac{\hbar c}{2}\int_0^\infty dkk\left\{\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)\right\}. \tag{23}$$

In order to introduce field operators inside and outside the cavity, we start from the normalized mode functions corresponding to the perfect cavity. These functions are given by

$$S_n(z) = -\sqrt{\frac{2}{L}}\sin([z + L]k_n), \tag{24}$$

where $k_n = n\pi/L$ with $n = 1, 2, \cdots$. We know from the theory of Fourier series that these modes form a complete normalized set of functions inside the cavity. However, this statement does not properly account for arbitrary boundary conditions. The field in a perfect cavity always vanishes at the boundary, which is not true in a lossy cavity. This boundary effect of the set $S_n(z)$ is expressed by the closure relation

$$\sum_{n=1}^{\infty} S_n(z)S_n^*(z') = \delta(z - z') - \delta(z + z') - \delta(z + z' + 2L), \tag{25}$$

for $-L \leq z \leq 0$. The last two terms in (25) contribute only when both $z$ and $z'$ located exactly at a boundary. For arbitrary continuous functions $f(z)$, the well-known expansion

$$f(z) = \sum_{n=1}^{\infty} S_n(z) \int_{-L}^{0} dz' S_n^*(z') f(z') \tag{26}$$

is valid for $z$ in the open interval $(-L, 0)$. However, exactly at the boundaries $z = -L$ or $z = 0$ the expansion (26) is valid only for functions $f(z)$ that vanish at the boundary. In fact, the r.h.s. of eq. (26) vanishes for $z = 0$ and for $z = -L$, regardless the value of $f$ in those points. This failure to reproduce the electric field on the boundary with the outside is the reason why a quantum description in terms of perfect cavity modes breaks down as the cavity quality factor Q decreases.

We choose to expand the quantized electric field operator inside the cavity (i.e. for $-L \leq z < 0$) as

$$\hat{E}_{in}(z) = \sum_{n=1}^{\infty} \sqrt{\frac{\hbar c k_n}{2\epsilon_0}} \hat{\mathcal{E}}_{in,n} S_n(z) \ . \tag{27}$$

This operator must be identical to the expression (21) for $-L \leq z < 0$, which defines the operators $\hat{\mathcal{E}}_{in,n}$ in terms of the global field operators $\hat{a}(k)$.

Another complete normalized set is formed by

$$C_n(z) = -\sqrt{\frac{2}{L}} \cos([z + L]k_n) \ , \tag{28}$$

with $n = 0, 1, 2, \cdots$. These functions obey the natural boundary conditions for the magnetic field in a perfect cavity. The quantized magnetic field operator inside the cavity is now expanded as

$$\hat{B}_{in}(z) = \sum_{n=1}^{\infty} \sqrt{\frac{\hbar k_n}{2\epsilon_0 c}} \hat{\mathcal{B}}_{in,n} C_n(z), \tag{29}$$

which basically defines the operators $\hat{\mathcal{B}}_{in,n}$. Because the mode functions are real, $\hat{\mathcal{E}}_{in,n}$ and $\hat{\mathcal{B}}_{in,n}$ are Hermitian operators. We want to introduce local field operators $\hat{a}_n$, in such a way that the expansions for the electric and the magnetic field operator inside the cavity have the same form as in a perfect cavity. This is accomplished by the expressions

$$\begin{aligned} i(\hat{a}_n - \hat{a}_n^\dagger) &= \hat{\mathcal{E}}_{in,n}, \\ \hat{a}_n + \hat{a}_n^\dagger &= \hat{\mathcal{B}}_{in,n}. \end{aligned} \tag{30}$$

If we equate the expressions (27) and (29) to the global expressions (21) and (22) for $-L \leq z < 0$, we can express the local operaors $\hat{a}_n$ in terms of the global operators $\hat{a}(k)$. Explicit expressions are obtained after multiplying eqs. (21) and (22) with $S_n(z)$ and $C_n(z)$ respectively, and integrating over the cavity length. The resulting expressions are found as

$$\hat{a}_n = \int_0^{\infty} dk \left\{ \alpha_{n1}^*(k) \hat{a}(k) - \alpha_{n2}(k) \hat{a}^\dagger(k) \right\}, \tag{31}$$

where

$$\alpha_{n1}(k) = \frac{1}{\sqrt{\pi L}}\sqrt{\frac{k}{k_n}}\frac{\sin([k-k_n]L)}{k-k_n}e^{-ikL}\mathcal{L}^*(k), \tag{32}$$

$$\alpha_{n2}(k) = -\frac{1}{\sqrt{\pi L}}\sqrt{\frac{k}{k_n}}\frac{\sin([k+k_n]L)}{k+k_n}e^{-ikL}\mathcal{L}^*(k). \tag{33}$$

In a similar way, we can introduce field operators $\hat{b}(k)$ corresponding to the outside, by starting from the expressions for the electric and magnetic field in the form

$$\hat{E}_{out}(z) = \int_0^\infty dk\sqrt{\frac{\hbar ck}{2\epsilon_0}}\hat{\mathcal{E}}_{out}(k)S_k(z), \tag{34}$$

$$\hat{B}_{out}(z) = \int_0^\infty dk\sqrt{\frac{\hbar k}{2\epsilon_0 c}}\hat{\mathcal{B}}_{out}(k)C_k(z), \tag{35}$$

with the continuum of mode functions for the outside part with a perfect mirror at $z = 0$ defined by

$$\begin{aligned} S_k(z) &= -\sqrt{\tfrac{2}{\pi}}\sin(kz), \\ C_k(z) &= -\sqrt{\tfrac{2}{\pi}}\cos(kz). \end{aligned} \tag{36}$$

Then the outside field operators are defined by the expressions

$$\begin{aligned} i\left(\hat{b}(k) - \hat{b}^\dagger(k)\right) &= \hat{\mathcal{E}}_{out}(k), \\ \hat{b}(k) + \hat{b}^\dagger(k) &= \hat{\mathcal{B}}_{out}(k), \end{aligned} \tag{37}$$

which makes the external field operators identical in form as in the case of a perfect mirror. Again, we can derive expressions for the outside operators $\hat{b}(k)$ in terms of the global operators $\hat{a}(k)$. The expressions (21) and (22) obey the well-known canonical commutation relations, from which one can argue that also the inside and outside operators obey the standard commutation relations

$$\begin{aligned} [\hat{a}_n, \hat{a}_{n'}] &= 0, \\ [\hat{a}_n, \hat{a}^\dagger_{n'}] &= \delta_{nn'}, \end{aligned} \tag{38}$$

and

$$\begin{aligned} [\hat{b}(k), \hat{b}(k')] &= 0, \\ [\hat{b}(k), \hat{b}^\dagger(k')] &= \delta(k-k'). \end{aligned} \tag{39}$$

In order to derive an explicit form of the Hamiltonian in terms of the inside and outside operators $\hat{a}_n$ and $\hat{b}(k)$, we have to express the global operators $\hat{a}(k)$ in terms of these. However, the set of inside and outside operators is not complete in general, since they cannot properly describe the field at the mirror. For most models of the mirror, the reflection $r$ deviates from its perfect-cavity value already to first order in the transmittivity, and non-orthogonal modes (with non-vanishing field values at the mirror) are needed already in this order. Therefore, we consider a simple case where $r \approx -1$ up to first order for non-vanishing $t$. We take $r = -\sqrt{1-\epsilon^2}$ and $t = i\epsilon$ for real, positive $\epsilon < 1$. In this case, the global operators can be expressed as expansion in the the inside and outside operators $\hat{a}_n$ and $\hat{b}(k)$. Substituting this expansion in Eq. (23), and retaining only first order terms in $\epsilon$ leads to the expression for the Hamiltonian

$$\begin{aligned} \hat{H} &= \sum_{n=1}^{\infty} \frac{\hbar c k_n}{2} \left(\hat{a}_n^\dagger \hat{a}_n + \hat{a}_n \hat{a}_n^\dagger\right) \\ &+ \int_0^\infty dk \frac{\hbar c k}{2} \left\{\hat{b}^\dagger(k)\hat{b}(k) + \hat{b}(k)\hat{b}^\dagger(k)\right\} \\ &+ \sum_{n=1}^{\infty} \int_0^\infty dk \left\{V_n(k)\hat{b}^\dagger(k)\hat{a}_n + V_n^*(k)\hat{a}_n^\dagger \hat{b}(k)\right\}, \end{aligned} \tag{40}$$

with the coupling term given by

$$V_n(k) = -\frac{\hbar\epsilon}{2\sqrt{\pi L}} e^{-ikL} \frac{\sin([k-k_n]L)}{(k-k_n)L}. \tag{41}$$

The Hamiltonian (40) has the same form as the phenomenological Hamiltonian introduced by Gardiner and Collett (1985), which has also been used by Barnett and Radmore (1988). However, we have obtained an explicit expression for the coupling term $V$, which deviates from the common assumption of a constant strength. Moreover, we did not make the rotating-wave approximation. The counter-rotating terms, consisting of products of two creation or two annihiltaion operators, vanish exactly in first order in the transmissivity.

The Hamiltonian (40) can be used to study the decay of the cavity. One readily obtains the result

$$\frac{d}{dt} < \hat{a}_n^\dagger \hat{a}_n > = -\Gamma < \hat{a}_n^\dagger \hat{a}_n > , \tag{42}$$

with $\Gamma = |\epsilon|^2/(2cL)$. However, it is more interesting to study the properties of the outside field. The explicit Hamiltonian allows us to study the quantum properties of the radiation field leaking out of a cavity containing a non-classical field initially. Moreover, the derivation indicates that for low-Q cavities, the operators for the cavity field and the outside can no longer be expected to commute. The modes needed to describe all possible fields inside, including the correct boundary conditions, must be expected to be non-orthogonal.

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