RENEWAL EQUATION FOR THE HEAT EQUATION OF AN ARITHMETIC VON KOCH SNOWFLAKE

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Abstract

We investigate the asymptotic behaviour of the heat content as the time $t \to 0^+$ for an arithmetic von Koch snowflake generated by a regular k-gon.

1 Introduction

Let D be an open, bounded and connected set in euclidean space $\mathbb{R}^m$ $(m = 2, 3, \dots)$ with boundary ∂D, and let $u_D : \overline{D} \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \; x \in D, \; t > 0, \tag{1.1}$$

with initial condition

$$u(x; 0) = 0, \; x \in D, \tag{1.2}$$

and with boundary condition

$$u(x; t) = 1, \; x \in \partial D, \; t > 0. \tag{1.3}$$

Let

$$E_D(t) = \int_D u_D(x; t) dx, \tag{1.4}$$

represent the total amount of heat in D at time t.

In this paper we analyse the asymptotic behaviour of the heat content $E_{K_{k,s}}(t)$ as time $t \to 0^+$, where $K_{k,s}$ is a planar region $(m = 2)$ with a piecewise self-similar and fractal boundary. The construction of $K_{k,s}$ is as follows. Fix an integer $k \geqslant 3$ and let $V_j = \frac{1}{2}\left(\operatorname{cosec}\frac{\pi}{k}\right) e^{\frac{2\pi j i}{k}}$, $j = 1, \dots, k$ be the vertices of a regular k-gon with volume $\frac{k}{4} \cot\left(\frac{\pi}{k}\right)$ and boundary length k. Fix $s \in (0, 1)$. We construct $\partial K_{k,s}$ by repeatedly replacing the middle proportion s of each segment, beginning with V_1V_2, $V_2V_3, \dots, V_{k-1}V_k$ and V_kV_1, by the $k-1$ other sides of a regular k-gon. We summarize, without proof, some of the geometric properties of $K_{k,s}$ in the following. See also Chapter 8 in [1] and Chapter 9 in [2].

Proposition 1.1.

(i) *There exists a non-empty relatively closed subset $I_k \subset (0,1)$ such that $K_{k,s}$ is embeddable in $\mathbb{R}^2$ if and only if $s \in I_k$. Moreover, $I_k \subseteq \left(0, \frac{1}{3}\right]$ for $k \geqslant 4$ with equality if and only if $k = 4$. For $s \in I_k$, $K_{k,s}$ is open, bounded and simply connected.*

(ii) *The volume of $K_{k,s}$ is given by*

$$|K_{k,s}| = \frac{k}{4} \frac{(1+s)^2}{1-(2k-1)s^2+2s} \cot\left(\frac{\pi}{k}\right), \quad s \in I_k. \tag{1.5}$$

(iii) *The Hausdorff dimension and the interior Minkowski dimension of $\partial K_{k,s}$, $s \in I_k$ are equal, and are given by the unique positive root $d_{k,s}$ of*

$$(k-1)s^d + 2\left(\frac{1-s}{2}\right)^d = 1. \tag{1.6}$$

(iv) *The interior upper Minkowski content of $\partial K_{k,s}$ is finite and the interior lower Minkowski content of $\partial K_{k,s}$ is strictly positive.*

(v) *$K_{k,s}$ is arithmetic if $s \in A_k$, where*

$$A_k = \left\{ s \in I_k : \frac{\log \frac{1-s}{2}}{\log s} = \frac{p}{q},\ p \in \mathbb{N},\ q \in \mathbb{N},\ (p,q) = 1 \right\}. \tag{1.7}$$

If $K_{k,s}$ is non-arithmetic ($s \in I_k \setminus A_k$), then $\partial K_{k,s}$ is internally Minkowski measurable.

The heat content for $K_{3,\frac{1}{3}}$ has been analysed by Fleckinger, Levitin and Vassiliev in [3, 4]. They proved the existence of two strictly positive, continuous and $(\log 9)$-periodic functions ψ_1 and χ such that for $t \to 0^+$,

$$E_{K_{3,\frac{1}{3}}}(t) = \psi_1(-\log t)t^{1-\frac{\log 5}{\log 9}} - \chi(-\log t)t + O\left(e^{-\frac{1}{1152t}}\right). \tag{1.8}$$

It is not known whether ψ_1 and χ are non-constant functions.

The elementary analysis presented by van den Berg and Gilkey in [5] for $k = 4$ and $s \in I_4$ extends to all snowflakes and yields the following. For all non-arithmetic snowflakes $K_{k,s}$ there exists a constant $C_{k,s}$ such that for $t \to 0^+$,

$$E_{K_{k,s}}(t) = C_{k,s} t^{1-\frac{d_{k,s}}{2}}(1+o(1)). \tag{1.9}$$

For all arithmetic snowflakes $K_{k,s}$, there exists a $(\frac{2}{q}\log\frac{1}{s})$-periodic, strictly positive and continuous function $\tilde{\psi} : \mathbb{R} \to (0,\infty)$ such that for $t \to 0^+$,

$$E_{K_{k,s}}(t) = \tilde{\psi}(-\log t)t^{1-\frac{d_{k,s}}{2}}(1+o(1)). \tag{1.10}$$

The main result of this paper (Theorem 1.4) is a refinement of (1.10) up to an exponential remainder. The results of Fleckinger, Levitin and Vassiliev in [4] for $k = 3$, $s = \frac{1}{3}$ and of van den Berg in [6] for $k = 4$, $s \in A_4$ are recovered as special cases. The idea in the proof of Theorem 1.4 is to exploit the self-similarity of $\partial K_{k,s}$ in order to obtain an approximate functional equation for the heat content $E_{K_{k,s}}(t)$ (Proposition 1.2). It is convenient to define for $t \geqslant 0$,

$$E(t) = \frac{1}{k} E_{K_{k,s}}(t), \tag{1.11}$$

$$H(t) = E(t) - (k-1)s^2 E\left(\frac{t}{s^2}\right) - 2\left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right). \tag{1.12}$$

Since $0 \leqslant u_{K_{k,s}}(x;t) < 1$ for $x \in K_{k,s}$ and $t > 0$, we see that

$$0 \leqslant E(t) \leqslant \frac{1}{k}|K_{k,s}|, \tag{1.13}$$

$$|H(t)| \leqslant \frac{1}{k}|K_{k,s}|. \tag{1.14}$$

Proposition 1.2. *For each* $k = 3, 4, \dots$ *and* $s \in I_k$ *there exist a function* $F : [0, \infty) \to \mathbb{R}$ *and a constant* $c_k > 0$ *such that*

$$H(t) = F(t) + O\left(e^{-\frac{c_k s^2}{t}}\right), \tag{1.15}$$

where F *is continuous, and satisfies both the linear functional equation*

$$F(t) = \left(\frac{1-s}{2}\right)^2 F\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right), \quad t \geqslant 0, \tag{1.16}$$

and the estimate

$$|F(t)| \leqslant 1000t. \tag{1.17}$$

The main idea in the proof of Proposition 1.2 is to exploit the symmetry of $K_{k,s}$, the self-similarity of $\partial K_{k,s}$ and to use the probabilistic solution of (1.1)-(1.3). The probabilistic tools have the advantage over the analytic tools in that they give simple proofs of (i) the various estimates involving the maximum principle (the principle of not feeling the boundary), and (ii) the scaling properties of the heat equation (scaling of brownian motion).

The proof of Proposition 1.2 is very similar to the proof of the corresponding statement for $k = 4$, $s \in I_4$ in [6], and will be omitted.

The structure of the asymptotic expansion of $E_{K_{k,s}}(t)$, $t \to 0^+$, $s \in A_k$ is governed by the geometry of $\{z \in \mathbb{C} : P_k(z) = 0\}$, where

$$P_k(z) = 1 - (k-1)s^{d_{k,s}} z^q - 2\left(\frac{1-s}{2}\right)^{d_{k,s}} z^p, \tag{1.18}$$

and where p and q are the unique positive integers determined by the choice of $s \in A_k$ in (1.7). We list some geometric properties of $\{z \in \mathbb{C} : P_k(z) = 0\}$ in the following.

Proposition 1.3. *Let $s \in A_k$ and let $z_1, z_2, \dots$ denote the roots of $P_k(z) = 0$, ordered such that*

$$|z_1| \leqslant |z_2| \leqslant \cdots. \tag{1.19}$$

(i) *All roots have multiplicity* 1.

(ii) $z_1 = 1$, *and*

$$p \leqslant q,\ |z_q| < s^{-\frac{d_{k,s}}{q}}, \quad \textit{for} \quad k \geqslant 5, \tag{1.20}$$

$$p \leqslant q,\ |z_q| < s^{-\frac{d_{4,s}}{q}}, \quad \textit{for} \quad q \textit{ odd},\ k = 4, \tag{1.21}$$

$$p < q,\ z_q = -s^{-\frac{d_{4,s}}{q}}, \quad \textit{for} \quad q \textit{ even},\ k = 4, \tag{1.22}$$

$$|z_{p\vee q}| \leqslant \frac{1+\sqrt{3}}{2} s^{-\frac{d_{3,s}}{q}}, \quad \textit{for} \quad k = 3, \tag{1.23}$$

where $p \vee q = \max\{p, q\}$, and where (1.23) is sharp for $p = 1$, $q = 2$ ($s = 3 - 2\sqrt{2}$) and for $p = 2$, $q = 1$ ($s = \frac{1}{2}$). Moreover, no roots of $P_3(z)$ have modulus $s^{-\frac{d_{3,s}}{q}}$.

(iii) *If z is a root (of $P_k(z) = 0$) with modulus r, then $\overline{z}$ is the only other possible root with modulus r.*

The proof of Proposition 1.3 will be deferred to Section 3.

By Proposition 1.3 (i) we may define

$$\sigma_j = \lim_{z_j \to z} \frac{z_j - z}{P_k(z)}. \tag{1.24}$$

We also put

$$z = -\log t, \tag{1.25}$$

$$\gamma = \frac{2}{q} \log \frac{1}{s}, \tag{1.26}$$

$$\psi(z) = e^{z\left(1 - \frac{d_{k,s}}{2}\right)} H(e^{-z}). \tag{1.27}$$

The main result of this paper reads as follows.

Theorem 1.4.

(i) *ψ is continuous and for $z \in \mathbb{R}$,*

$$\sum_{m \in \mathbb{Z}} w^{-m} \psi(z - m\gamma) \tag{1.28}$$

converges absolutely on the annulus W given by

$$W = \left\{ w \in \mathbb{C} : 1 \leqslant |w| < s^{-\frac{d_{k,s}}{q}} \right\}. \tag{1.29}$$

(ii) *For $w \in W$, define $\psi_w : \mathbb{R} \to \mathbb{C}$ by*

$$\psi_w(z) = \sum_{m \in \mathbb{Z}} w^{-m+\frac{z}{\gamma}} \psi(z - m\gamma). \tag{1.30}$$

Then ψ_w is γ-periodic and uniformly continuous.

(iii) *Let $k = 3, 5, 6, \dots$, $s \in A_k$ or $k = 4$, $s \in A_4$, q odd. Then there exist a $p\gamma$-periodic uniformly continuous function $\chi : \mathbb{R} \to \mathbb{R}$ and γ-periodic uniformly continuous functions $\{\phi_{z_j} : z_j \in V,\ P_k(z_j) = 0,\ j = 1, 2, \dots\}$, where*

$$V = \left\{ w \in \mathbb{C} : |w| > s^{-\frac{d_{k,s}}{q}} \right\}, \tag{1.31}$$

such that for $t \to 0^+$,

$$\begin{aligned} E_{K_{k,s}}(t) \quad = \quad & \sum_{\{j : z_j \in W\}} t^{1-\frac{d_{k,s}}{2}+\frac{1}{\gamma}\log z_j} \frac{\sigma_j}{z_j} \psi_{z_j}(-\log t) \\ & + \sum_{\{j : z_j \in V\}} t^{1-\frac{d_{k,s}}{2}+\frac{1}{\gamma}\log z_j} \phi_{z_j}(-\log t) \\ & - \chi(-\log t)t + O\left(e^{-\frac{c_k s^2}{t}}\right). \end{aligned} \tag{1.32}$$

The proof of Theorem 1.4 will be deferred to Section 2.

We note that, by (1.20) and (1.21) in Proposition 1.3, the second term in the right hand side of (1.32) is absent for $k \geqslant 4$. We also note that the case $k = 4$, $s \in A_4$, q even has been excluded from Theorem 1.4 (iii). In that case there is, by (1.22), a root with modulus $s^{-\frac{d_{k,s}}{q}}$. This delicate case was discussed in detail in [6].

The leading term in (1.32) corresponds to the root with smallest modulus, i.e. $z_1 = 1$. Comparing (1.32) with (1.10), we have $\tilde{\psi} = \sigma_1 \psi_1$ by (1.30). This also jibes with the special case $k = 3$, $s = \frac{1}{3}$ in (1.8) since $\sigma_1 = 1$ for these values of s and k. Since $K_{k,s}$ is bounded and simply connected, we have by Proposition 1.1 (iv), and by Corollary 1.5 in [7] that $\tilde{\psi}$ is finite and strictly positive. The contributions from the remaining roots in W are $o\left(t^{1-\frac{d_{k,s}}{2}}\right)$ but $\gg t$, while all roots in V give contributions which are $o(t)$. The function χ in (1.32) is directly related to the function F in Proposition 1.2. If we define $\hat{\chi} : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = t\hat{\chi}(-\log t), \tag{1.33}$$

then Proposition 1.2 implies that $\hat{\chi}$ is continuous and $p\gamma$-periodic. The complicated relation between χ and $\hat{\chi}$ can be read off from the various formulae in the proof of Theorem 1.4 in Section 2.

2 Proof of Theorem 1.4

By (1.14) and (1.27),

$$|\psi(z)| \leqslant k^{-1} e^{z\left(1-\frac{d_{k,s}}{2}\right)} |K_{k,s}|. \tag{2.1}$$

For $w \in W$, we have by (1.29) and (2.1),

$$\sum_{m=0}^{\infty} |w^{-m}\psi(z-m\gamma)| \leqslant k^{-1}|K_{k,s}| e^{z\left(1-\frac{d_{k,s}}{2}\right)} \sum_{m=0}^{\infty} e^{-m\gamma\left(1-\frac{d_{k,s}}{2}\right)} < \infty. \tag{2.2}$$

Moreover, by (1.14), (1.15) and (1.17) there exists a constant C_1 such that $|H(t)| \leqslant C_1 t$ for all $t \geqslant 0$. By (1.27),

$$|\psi(z)| \leqslant C_1 e^{-z\frac{d_{k,s}}{2}}, \ z \in \mathbb{R}. \tag{2.3}$$

For $w \in W$, we have by (2.3),

$$\sum_{m=1}^{\infty} |w^{m}\psi(z+m\gamma)| \leqslant C_1 e^{-z\frac{d_{k,s}}{2}} \sum_{m=1}^{\infty} \left(|w| e^{-\gamma\frac{d_{k,s}}{2}}\right)^m < \infty. \tag{2.4}$$

The absolute convergence of the series in (1.30) follows from (2.2) and (2.4). The continuity of ψ follows directly from the continuity properties of $E_{K_{k,s}}$. The continuity of ψ_w then follows directly from the exponential decay of $|w^{-m}\psi(z-m\gamma)|$ for $m \to \infty$ and for $m \to -\infty$ by (2.3) and (2.1) respectively. The uniform continuity follows from the γ-periodicity of ψ_w.

To prove part (iii) of Theorem 1.4, we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(z) = e^{z\left(1-\frac{d_{k,s}}{2}\right)} E(e^{-z}). \tag{2.5}$$

Substitution of (2.5) into (1.12) gives, by (1.27),

$$f(z) = (k-1)s^{d_{k,s}} f(z-q\gamma) + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} f(z-p\gamma) + \psi(z). \tag{2.6}$$

Equation (2.6) is an inhomogeneous renewal equation of arithmetic type. From (1.13) and (2.5) we obtain

$$\lim_{z \to -\infty} f(z) = 0. \tag{2.7}$$

The inhomogeneous term ψ in (2.6) is continuous, and satisfies, by (2.1) and (2.3),

$$|\psi(z)| \leqslant (C_1 + k^{-1}|K_{k,s}|) e^{-|z| \min\left\{\frac{d_{k,s}}{2}, 1-\frac{d_{k,s}}{2}\right\}}. \tag{2.8}$$

It follows by the renewal theorem for the arithmetic case (p.198 in [8]) that (2.6) - (2.7) has a unique continuous solution given by (see also (3.10) - (3.16) in [6])

$$f(z) = \sum_{m=0}^{\infty} \sum_{j} \sigma_j z_j^{-1-m} \psi(z - m\gamma). \tag{2.9}$$

In order to analyse the behaviour of the double sum in (2.9) for $z \to \infty$, we first consider the contribution from the j^{th} term in (2.9) such that $z_j \in W$. We write

$$\sum_{m=0}^{\infty} \sigma_j z_j^{-1-m} \psi(z - m\gamma) = \sigma_j z_j^{-1-\frac{z}{\gamma}} \psi_{z_j}(z) - \sum_{m=1}^{\infty} \sigma_j z_j^{-1+m} \psi(z + m\gamma). \tag{2.10}$$

By (1.15), (1.33) and the continuity of $\hat{\chi}$ and H, there exists a constant C_2 such that for $z \geqslant 0$, $m \geqslant 0$,

$$|H(e^{-z-m\gamma}) - e^{-z-m\gamma} \hat{\chi}(z + m\gamma)| \leqslant C_2 e^{-c_k s^2 e^{z+m\gamma}}. \tag{2.11}$$

By (1.27), (2.11) and the $p\gamma$-periodicity of $\hat{\chi}$ we obtain that (see also (3.23) in [6])

$$\begin{aligned} \sum_{m=1}^{\infty} \sigma_j z_j^{-1+m} \psi(z + m\gamma) &= \sum_{m=1}^{p} \sigma_j z_j^{-1+m} \left(1 - z_j^p s^{p\frac{d_{k,s}}{q}}\right)^{-1} e^{-d_{k,s}\frac{z+m\gamma}{2}} \hat{\chi}(z + m\gamma) \\ &\quad + O\left(e^{-c_k s^2 e^z}\right). \end{aligned} \tag{2.12}$$

In order to estimate the contribution from a term in (2.9) with j such that $z_j \in V$, we define $K : \mathbb{R} \to \mathbb{R}$ by

$$\psi(z) = e^{z\left(1 - \frac{d_{k,s}}{2}\right)} \{e^{-z} \hat{\chi}(z) + K(z)\}. \tag{2.13}$$

It follows by (1.27) and (2.11) that K satisfies

$$|K(z)| \leqslant C_2 e^{-c_k s^2 e^z}, \; z \geqslant 0. \tag{2.14}$$

The contribution from the j^{th} term in (2.9) with $z_j \in V$ can be written as

$$\begin{aligned} &\sigma_j z_j^{-1} \sum_{m=0}^{\infty} \left\{ e^{-(z-m\gamma)\frac{d_{k,s}}{2}} z_j^{-m} \hat{\chi}(z - m\gamma) + e^{(z-m\gamma)\left(1-\frac{d_{k,s}}{2}\right)} z_j^{-m} K(z - m\gamma) \right\} \\ &= \sigma_j z_j^{-1} e^{-z\frac{d_{k,s}}{2}} \left(1 - s^{p\frac{d_{k,s}}{q}} z_j^{-p}\right)^{-1} \sum_{m=0}^{p-1} \left(s^{\frac{d_{k,s}}{q}} z_j^{-1}\right)^m \hat{\chi}(z - m\gamma) \\ &\quad + z_j^{-\frac{z}{\gamma}} \hat{\phi}_{z_j}(z), \end{aligned} \tag{2.15}$$

where

$$\hat{\phi}_{z_j}(z) = \sigma_j z_j^{\frac{z}{\gamma}-1} \sum_{m=0}^{\infty} e^{(z-m\gamma)\left(1-\frac{d_{k,s}}{2}\right)} z_j^{-m} K(z - m\gamma). \tag{2.16}$$

The first series in the right hand side of (2.15) converges since $\hat{\chi}$ is bounded, and $z_j \in V$. The infinite sum reduces, by the $p\gamma$-periodicity of $\hat{\chi}$, to the finite sum in the first term in the right hand side of (2.15). It remains to investigate the asymptotic behaviour of $\hat{\phi}_{z_j}(z)$, $z \to \infty$ in (2.16). It is straightforward to check that $\hat{\phi}_{z_j}$ satisfies

$$\hat{\phi}_{z_j}(z+\gamma) = \hat{\phi}_{z_j}(z) + \sigma_j z_j^{\frac{z}{\gamma}} e^{(z+\gamma)\left(1-\frac{d_{k,s}}{2}\right)} K(z+\gamma). \tag{2.17}$$

The second term in the right hand side is, by (2.14), exponentially small for $z \to \infty$. By Lemma 2.5 and (2.32) in [4], we conclude that there exists a γ-periodic continuous function ϕ_{z_j} such that

$$\hat{\phi}_{z_j}(z) = \phi_{z_j}(z) + O\left(e^{-c_k s^2 e^z}\right), \quad z \to \infty. \tag{2.18}$$

Theorem 1.4 follows from (2.9), (2.10), (2.12), (2.15), (2.18), where χ can be read off from the first terms in the right hand sides of (2.12) and (2.15) respectively.

3 Proof of Proposition 1.3

The proof of Proposition 1.3 (i), (iii) for $k = 4$ can be found, after a suitable transformation, in Lemma 6.4 of [9].

(i) The statement is trivial if $p = q = 1$ (i.e. $s = \frac{1}{3}$). Since $(p, q) = 1$, it remains to consider the case $p \neq q$. We argue by contradiction. Suppose z is a root with multiplicity larger than 1. Then both $P_k(z) = 0$ and $P_k'(z) = 0$. Hence

$$(k-1)s^{d_{k,s}} z^q + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} z^p = 1, \tag{3.1}$$

$$(k-1)s^{d_{k,s}} q z^q + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} p z^p = 0. \tag{3.2}$$

Since $p \neq q$, we can solve (3.1) - (3.2) for z^p and z^q. This gives

$$z^q = \frac{p}{(k-1)s^{d_{k,s}}(p-q)}, \tag{3.3}$$

$$z^p = \frac{q}{2\left(\frac{1-s}{2}\right)^{d_{k,s}}(q-p)}. \tag{3.4}$$

From (3.3) - (3.4) and (1.7) we obtain that p and q have to satisfy

$$2^q p^p (q-p)^{q-p} = (k-1)^p q^q (-1)^p. \tag{3.5}$$

First suppose $q > p$. Then the left hand side of (3.5) is positive. Hence $(-1)^p > 0$, so that p is even. Since $(p, q) = 1$, we have that q is odd. Since the left hand side of (3.5) is even, $k - 1$ is even. Let l, m, n and r be the unique positive integers such that $k = 2^l m + 1$, $p = 2^n r$, where m and r are odd. Counting the factors of 2 in both sides of (3.5), we obtain that $q + rn2^n = rl2^n$. Since n is a positive integer, we conclude that q is even. This is a contradiction.

Next suppose that $q < p$. Rewriting (3.5), we have

$$2^q p^p = (k-1)^p (p-q)^{p-q} q^q (-1)^q. \tag{3.6}$$

Since the left hand side of (3.6) is positive, we have $(-1)^q > 0$. Hence q is even. Since $(p, q) = 1$, we have that p is odd. Hence (3.6) can be rewritten as

$$p^p = (k-1)^p (p-q)^{p-q} \left(\frac{q}{2}\right)^q (-1)^q. \tag{3.7}$$

Since p is odd, the left hand side of (3.7) is odd. Hence $q = 2r$, where r is odd. Since $(p, q) = 1$, we also have $(p, r) = 1$. Since $q = 2r$, the right hand side of (3.7) contains a factor r^{2r}. Hence $r^{2r} | p^p$. Since $(p, r) = 1$, we have to conclude that $r = 1$ and $q = 2$. Hence (3.7) reduces to

$$p^p = (k-1)^p (p-2)^{p-2}, \ p \text{ odd}. \tag{3.8}$$

Since $q = 2$ and $p > q$, we have $p \geqslant 3$, p odd. It follows from (3.8) that $\left(\frac{p}{k-1}\right)^p \in \mathbb{N}$. Hence there exists an odd integer n such that $p = (k-1)n$. From (3.8) we conclude that

$$\left(k - 1 - \frac{2}{n}\right)^{(k-1)n} = ((k-1)n - 2)^2. \tag{3.9}$$

Since the right hand side of (3.9) is a positive integer, $k - 1 - \frac{2}{n}$ is an integer. Since n is odd, we conclude that $n = 1$, $p = k - 1$. Hence $(p-2)^{p-2} = 1$, and so $p = 3$, $k = 4$ and $q = 2$. This contradicts the fact that $q \geqslant p$ for $s \in A_4 \subset (0, \frac{1}{3}]$.

(ii) It follows from (1.6) that $z = 1$ is a root of $P_k(z) = 0$. Suppose z is any other root. Then

$$\begin{aligned} 1 &= (k-1)s^{d_{k,s}} z^q + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} z^p \\ &= \left|(k-1)s^{d_{k,s}} z^q + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} z^p\right| \\ &\leqslant (k-1)s^{d_{k,s}} |z|^q + 2\left(\frac{1-s}{2}\right)^{d_{k,s}} |z|^p. \end{aligned} \tag{3.10}$$

Since the right hand side of (3.10) is strictly increasing in $|z|$ and equal to 1 for $|z| = 1$, we obtain $|z| \geqslant 1$. Proposition 1.3(iii) (the proof of which is independent of parts (i) and (ii)) implies that $z = 1$ is the only root with modulus 1. Hence $z_1 = 1$. Let

$$w = s^{d_{k,s}} z. \tag{3.11}$$

Then $P_k(z) = 0$ if and only if

$$(k-1)w^q + 2w^p = 1. \tag{3.12}$$

Suppose $k \geqslant 5$, and w is a root of (3.12) with $|w| \geqslant 1$. By Proposition 1.1, $A_k \subset (0, \frac{1}{3}]$ for $k \geqslant 5$. Hence $q \geqslant p$ and $(k-1)|w|^q = |1-2w^p| \leqslant 1+2|w|^p \leqslant 3|w|^p \leqslant 3|w|^q$. This is a contradiction.

Suppose $k = 4$ and w is a root of (3.12) with $|w| > 1$. By Proposition 1.1, $A_4 \subset (0, \frac{1}{3}]$. Hence $q \geqslant p$ and $(k-1)|w|^q = |1-2w^p| \leqslant 1+2|w|^p < 3|w|^p \leqslant 3|w|^q$. This is a contradiction. To complete the proof for $k = 4$, we suppose that $w = e^{i\theta}$ is a root of (3.12) with $0 < \theta < \pi$. Then $\overline{w}$ is another root of (3.12). Hence

$$3e^{iq\theta} + 2e^{ip\theta} = 1, \tag{3.13}$$
$$3e^{-iq\theta} + 2e^{-ip\theta} = 1. \tag{3.14}$$

It follows that

$$e^{ip\theta} + e^{-ip\theta} = -2, \tag{3.15}$$
$$e^{iq\theta} + e^{-iq\theta} = 2. \tag{3.16}$$

Hence $p\theta = (2l+1)\pi$, $q\theta = 2m\pi$, $l, m \in \mathbb{Z}$. Since $0 < \theta < \pi$, we have $l < \frac{p-1}{2}$ and $q < \frac{m}{2}$. Since

$$\frac{p}{q} = \frac{2l+1}{2m}, \tag{3.17}$$

we conclude that $(p, q) > 1$. This is a contradiction. Finally, $w = -1$ is a root of (3.12) if and only if q is even and p is odd.

Suppose $k = 3$. First we show that there are no roots of (3.12) with $|w| = 1$. Suppose to the contrary. Then $w = e^{i\theta}$ and $\overline{w} = e^{-i\theta}$ are roots of (3.12). Since $w = -1$ is not a root, we may assume $0 < \theta < \pi$. Hence

$$2e^{iq\theta} + 2e^{ip\theta} = 1, \tag{3.18}$$
$$2e^{-iq\theta} + 2e^{-ip\theta} = 1. \tag{3.19}$$

It follows that

$$\cos(p\theta) = \cos(q\theta) = \frac{1}{4}, \tag{3.20}$$
$$\sin(p\theta) = \sin(q\theta) = 0. \tag{3.21}$$

Hence there exist $l \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ such that

$$p\theta = \arccos\frac{1}{4} + 2l\pi,\ q\theta = -\arccos\frac{1}{4} + 2m\pi, \tag{3.22}$$

or there exist $l \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ such that

$$p\theta = -\arccos\frac{1}{4} + 2l\pi,\ q\theta = \arccos\frac{1}{4} + 2m\pi, \tag{3.23}$$

In either case, we conclude that $\frac{1}{\pi}\arccos\frac{1}{4}$ is rational. This contradicts Theorem 6.16 in [10]. To prove (1.23), we note that if $q > p$ and w is a root of (3.12) with $|w| > 1$, then

$$2|w|^q = |1 - 2w^p| \leqslant 1 + 2|w|^p \leqslant 1 + 2|w|^{q-1}. \tag{3.24}$$

Hence

$$2|w|^q\left(1 - \frac{1}{|w|}\right) \leqslant 1. \tag{3.25}$$

Since $q \geqslant 2$ we have $2|w|^2 - 2|w| \leqslant 1$. This implies (1.23) in the case $q > p$. The case $p > q$ follows by the symmetry of (3.12). This proof also shows that we have equality in (1.23) if and only if $p = 1$, $q = 2$ ($s = 3 - 2\sqrt{2}$) or $p = 2$, $q = 1$ ($s = \frac{1}{2}$).

(iii) It is sufficient to prove that if w_1 is a root of (3.12) with modulus r, then $\overline{w}_1$ is the only other possible root with the same modulus. Suppose to the contrary, and let

$$w_1 = re^{i\psi_1},\ w_2 = re^{i\psi_2} \tag{3.26}$$

be roots of (3.12) with

$$0 \leqslant \psi_1 \leqslant \pi,\ 0 \leqslant \psi_2 \leqslant \pi,\ \psi_1 \neq \psi_2. \tag{3.27}$$

Then

$$|(k-1)w_1^{p-q} + 2| = |(k-1)w_2^{p-q} + 2|, \tag{3.28}$$

and

$$|(k-1)w_1^{p-q} + 2||(k-1)\overline{w}_1^{p-q} + 2| = |(k-1)w_2^{p-q} + 2||(k-1)\overline{w}_2^{p-q} + 2|. \tag{3.29}$$

It follows that

$$w_1^{p-q} + \overline{w}_1^{p-q} = w_2^{p-q} + \overline{w}_2^{p-q}. \tag{3.30}$$

From (3.26) and (3.30), we obtain

$$\cos((p-q)\psi_1) = \cos((p-q)\psi_2). \tag{3.31}$$

Without loss of generality, we may assume that $p \neq q$. By (3.31) we have

$$\psi_1 = \frac{2\pi m}{p-q} \pm \psi_2 \tag{3.32}$$

for some $m \in \mathbb{Z}$. Suppose we have the $+$ sign in (3.32). Since $re^{i\psi_1}$ is a root of (3.12), we have

$$1 = (k-1)r^q e^{iq\psi_2 + \frac{2\pi mqi}{p-q}} + 2r^p e^{ip\psi_2 + \frac{2\pi mpi}{p-q}}. \tag{3.33}$$

Dividing both sides of (3.33) by $e^{\frac{2\pi mpi}{p-q}}$ gives

$$e^{-\frac{2\pi mpi}{p-q}} = (k-1)r^q e^{iq\psi_2} + 2r^p e^{ip\psi_2}. \tag{3.34}$$

Since $re^{i\psi_2}$ is also a root of (3.12), we conclude that for some $l \in \mathbb{Z}$,

$$\frac{mp}{p-q} = l. \tag{3.35}$$

By (3.27), $|\psi_1 - \psi_2| \leqslant \pi$. Since we assumed the + sign in (3.32), we conclude that

$$\left|\frac{2m}{p-q}\right| \leqslant 1. \tag{3.36}$$

From (3.35) and (3.36), we obtain that both $|l| \leqslant \frac{p}{2}$ and $\frac{p}{q} = \frac{l}{l-m}$. This implies $(p,q) > 1$, contradicting the choice of p and q. Suppose we have the $-$ sign in (3.32). Since $re^{i\psi_1}$ is a root of (3.12), we have

$$1 = (k-1)r^q e^{-iq\psi_2 + \frac{2\pi mqi}{p-q}} + 2r^p e^{-ip\psi_2 + \frac{2\pi mpi}{p-q}}. \tag{3.37}$$

Dividing both sides of (3.37) by $e^{\frac{2\pi mpi}{p-q}}$ and noting that $re^{-i\psi_2}$ is a root of (3.12) yields (3.35) for some $l \in \mathbb{Z}$. By (3.27), $|\psi_1 + \psi_2| < 2\pi$. Hence (3.32) (with the $-$ sign) implies

$$\left|\frac{m}{p-q}\right| < 1. \tag{3.38}$$

From (3.35) and (3.38), we obtain that both $|l| < p$ and $\frac{p}{q} = \frac{l}{l-m}$. This implies $(p,q) > 1$, contradicting the choice of p and q.

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