RANDOM WALK REPRESENTATIONS AND ENTROPIC REPULSION FOR GRADIENT MODELS

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Abstract

We discuss the so-called entropic repulsion for gradient interface models. This phenomenon appears when the random interface is placed on one side of a hard wall. The combined effect of local fluctuations and global stiffness pushes the interface away from the wall. Recently, there has been considerable progress in understanding this effect in a quantitatively precise way for a certain class of models. We describe this progress and related questions on wetting problems.

1 Random surfaces and random walk representations

Our "random surfaces" are special families of real-valued random variables defined on a d-dimensional discrete lattice, in our case \mathbb{Z}^d . Let A be a finite subset of \mathbb{Z}^d . The random variables will be denoted by $\phi_i, i \in A$.

Let

$$\partial A \stackrel{\text{def}}{=} \left\{ i \in \mathbb{Z}^d \setminus A : \exists j \in A \text{ with } |i-j| = 1 \right\},$$

and $\overline{A} = A \cup \partial A$. Furthermore, let $U : \mathbb{R} \to [0, \infty)$ be a symmetric continuous function satisfying some (mild) growth condition for U(x), $|x| \to \infty$. We will not be very precise for the moment, as we will stick to more restrictive assumptions in a moment. We then define the probability measure P_A on \mathbb{R}^A by

$$P_A^U(d\phi) \stackrel{\text{def}}{=} \frac{1}{Z_A} \exp\left[-\frac{1}{2} \sum_{\langle i,j \rangle \in \overline{A}} U(\phi_i - \phi_j)\right] \prod_{i \in A} d\phi_i, \tag{1.1}$$

where the summation is over unordered nearest-neighbor pairs, and $\phi_i \equiv 0$ for $i \notin A$, i.e., we take zero boundary condition. Z_A is of course the norming in order that P_A^U becomes a probability measure. In the case of $A = \Lambda_n = \{-n, -n+1, \ldots, n\}^d$ we just write P_n^U . We will usually drop the U from the notation. Clearly if $d = 1, P_n$ is just the ordinary random walk tied down at both ends of the interval.

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A particular case is the so-called harmonic crystal, where $U(x) = x^2$. In this special case we denote the above measure by P_A^{harm} , which is a centered Gaussian measure. Its covariances have a simple random walk representation: If $i, j \in A$, then

$$\Gamma_A(i,j) \stackrel{\text{def}}{=} \int_{\mathbb{R}^A} \phi_i \phi_j P_A^{\text{harm}}(d\phi) = \mathbb{E}_i^{\text{RW}} \left(\int_0^{\tau_A c} \mathbb{1}_{\{\eta_s = j\}} \, ds \right), \tag{1.2}$$

where $(\eta_s)_{s \ge 0}$ is the continuous-time simple random walk on \mathbb{Z}^d with holding times of expectation 1/2d, starting in *i* under \mathbb{P}_i^{RW} , and τ_{A^c} is the first entrance time in A^c , which is of course finite as we have assumed A to be finite. To prove (1.2), remark that

$$\exp\left[-rac{1}{2}\sum_{\langle i,j
angle \in \overline{A}}(\phi_i-\phi_j)^2
ight] = \exp\left[-rac{1}{2}\left\langle \phi,-Q\phi
ight
angle_A
ight],$$

where $\langle \cdot, \cdot \rangle_A$ is the usual inner product in \mathbb{R}^A , and $Q\phi(i) = \sum_{j \in \overline{A}: |j-i|=1} (\phi_j - \phi_i)$, $i \in A, \phi|_{\partial A} = 0$, i.e., Q is the Q-matrix of the simple random walk with holding times of expectation 1/2d, and with killing at ∂A . Clearly

$$(-Q)^{-1}(i,j) = \mathbb{E}_i^{\mathrm{RW}} \left(\int_0^{\tau_A \epsilon} \mathbf{1}_{\{\eta_s = j\}} \, ds \right),$$

and this is just $E_A^{\text{harm}}\left(\phi_i\phi_j\right)$.

From the random walk representation (1.2) and well-known properties of simple random walk it follows for instance that $E_n^{\text{harm}}(\phi_0^2)$ is of order n for d = 1, $\log n$ for d = 2, and stays bounded for dimensions $d \ge 3$. In the latter case a thermodynamic limit P_{∞}^{harm} exists. One should however remark that this random field has correlations that decay only slowly. In fact, $\Gamma_{\infty}(i,j) = \Gamma_{\infty}(j-i) = E_{\infty}^{\text{harm}}(\phi_i\phi_j)$ behaves like $|j-i|^{-d+2}$ for |j-i| large.

Helffer and Sjöstrand [24] discovered that there is a similar representation in the case where U is uniformly convex. This was put in a probabilistic framework in [20]. In order to avoid technical difficulties, we assume that U is twice continuously differentiable and that there exists a number $K \in (0, 1)$ with

$$K \leqslant U''(x) \leqslant 1/K, \, x \in \mathbb{R}. \tag{1.3}$$

To describe the random walk, we first have to introduce the standard reversible diffusion process on \mathbb{R}^A that has P_A as its invariant measure.

If $H : \mathbb{R}^A \to \mathbb{R}$ is any smooth function, satisfying some appropriate growth condition, then by partial integration we have for any smooth test function $f : \mathbb{R}^A \to \mathbb{R}$

$$\int_{\mathbb{R}^{A}} \exp\left[-H(\phi)\right] \sum_{i \in A} \left(\frac{\partial^{2} f}{\partial \phi_{i}^{2}} - \frac{\partial H}{\partial \phi_{i}} \frac{\partial f}{\partial \phi_{i}}\right) d\phi = 0.$$

Therefore, the diffusion process with generator

$$\sum_{i \in A} \left(\frac{\partial^2}{\partial \phi_i^2} - \frac{\partial H}{\partial \phi_i} \frac{\partial}{\partial \phi_i} \right)$$

has $\exp\left[-H(\phi)\right] d\phi / \int \exp\left[-H(\phi)\right] d\phi$ as its stationary measure. Applying this to

$$H(\phi) \stackrel{\mathrm{def}}{=} rac{1}{2} \sum_{\langle i,j
angle \in \overline{A}} U(\phi_i - \phi_j),$$

one sees that the diffusion process $(X_t)_{t\geq 0}$ on \mathbb{R}^A with generator

$$L \stackrel{\text{def}}{=} \sum_{i \in A} \left(\frac{\partial^2}{\partial \phi_i^2} - \frac{1}{2} \sum_{j \in \overline{A}, |j-i|=1} U'(\phi_i - \phi_j) \frac{\partial}{\partial \phi_i} \right)$$

has P_A as its invariant measure. Based on the paths (X_t) of this diffusion process, one constructs now a continuous-time jump process $(\eta_t)_{t\geq 0}$ on A that is killed upon leaving A, and has jump rates from i to j, |i - j| = 1, given by

$$a^{X}(i,j,t) \stackrel{\text{def}}{=} \frac{1}{2} U''(X_{t,i} - X_{t,j}), \qquad (1.4)$$

where we set $X_{t,j} \equiv 0$ for $j \notin A$. The joint process $((X_t, \eta_t))_{t \ge 0}$, is a Markov process on $\mathbb{R}^A \times A$, with generator

$$\mathcal{L}F(\phi,i) \stackrel{\mathrm{def}}{=} LF(\phi,i) + rac{1}{2} \sum_{j \in \overline{A}, |j-i|=1} U''(\phi_j - \phi_i) \left(F(\phi,j) - F(\phi,i)\right),$$

where we set $F(\phi, j) \equiv 0$ for $j \notin A$. We write $T_t, t \ge 0$, for the semigroup of the diffusion process, and $\mathcal{T}_t, t \ge 0$, for the semigroup of the pair $((X_t, \eta_t))_{t\ge 0}$. We denote by $\mathbb{P}_{(\phi,i)}$ the law of the joint process, starting in (ϕ, i) . If $f : \mathbb{R}^A \to \mathbb{R}$ is smooth, then we denote by $Df : \mathbb{R}^A \times A \to \mathbb{R}$ the function defined by

$$Df(\phi, i) = rac{\partial f}{\partial \phi_i}(\phi).$$

The basic observation that underlies the random walk representation is the following commutation relation:

$$DL = \mathcal{L}D, \tag{1.5}$$

which is easily checked.

We denote by C_A the set of C_{∞} -functions on \mathbb{R}^A that grow not faster than exponentially in any component.

Theorem 1.1. ([24],[20]). Let $f, g \in C_A$. Then

$$\operatorname{cov}_{P_A}(f,g) = \int_0^\infty dt \, \sum_{i \in A} E_A \left(Df(i,\cdot) \mathcal{T}_t Dg(i,\cdot) \right).$$

Proof. We may assume $E_A(g) = 0$. Then

$$E_{A}(fg) = -\int_{0}^{\infty} dt \, E_{A}(fLT_{t}g)$$

=
$$\int_{0}^{\infty} dt \, \sum_{i \in A} E_{A}\left(\frac{\partial f}{\partial \phi_{i}} \frac{\partial T_{t}g}{\partial \phi_{i}}\right)$$

=
$$\int_{0}^{\infty} dt \, \sum_{i \in A} E_{A}\left(Df(i, \cdot)\mathcal{T}_{t}Dg(i, \cdot)\right),$$
 (1.6)

where the first equality uses the well known fact that $D(f,h) \stackrel{\text{def}}{=} \sum_{i \in A} E_A \left(\frac{\partial f}{\partial \phi_i} \frac{\partial h}{\partial \phi_i} \right)$ is the Dirichlet form associated with the diffusion, i.e., $D(f,h) = -E_A(fLh)$ for any nice functions f and h. This follows from partial integration. The second equality in (1.6) uses (1.5).

Applying Theorem 1.1 to the special case $f(\phi) = \phi_i$, $g(\phi) = \phi_j$, we get

$$\int_{\mathbb{R}^A} \phi_i \phi_j P_A(d\phi) = \int_{\mathbb{R}^A} \mathbb{E}_{(i,\phi)} \left(\int_0^{\tau_A^c} ds \, \mathbb{1}_{\{\eta_s = j\}} \right) P_A(d\phi). \tag{1.7}$$

Remark that in the harmonic case where the second derivative of U is constant, the random walk is independent of the diffusion process, and we therefore get back the random walk representation (1.2).

In principle, the random walk representation (1.7) should be nearly as useful as (1.2). Indeed, as the diffusion process X_t is rapidly mixing, the random walk η_t driven by (1.4) should not behave much differently from the standard one. There are however considerable difficulties in implementing this heuristics and many of the fine properties have not been proved. Consequently, many of the qualitatively precise results presented here are known only in the Gaussian case. We will discuss this later on.

2 Entropic repulsion for dimensions $d \ge 3$.

2.1 Statement of the result for the harmonic case

Entropic repulsion is a general phenomenon in statistical mechanics of random interfaces in the presence of walls. A wall is a fixed (typically nonrandom) layer impenetrable for the random surface. Therefore, the surface has to stay on one side of the wall. We consider here the simplest possible wall, namely the configuration identical to 0, possibly only on some part of the region on which the random surface is defined. In the case of our gradient type interfaces this just means that the surface has to have a definite sign, for convenience a positive one. Thus, we consider the conditional law for the random field $P(\cdot|\Omega_A^+)$ where

$$\Omega^+_A \stackrel{\mathrm{def}}{=} \left\{ \phi: \phi_i \geqslant 0 \ , \ i \in A \right\}.$$

In the physics literature, also more complicated types of wall to surface interactions have been considered. We will give some comments about that in the final section.

What is the effect of the presence of the wall on the surface? The crucial point is that the surface has local fluctuations, which push the interface away from the wall. On the other hand, there are long-range correlations giving the surface a certain global stiffness. In order to understand what is going on, consider first the case where there are no such long-range correlations, in the extreme case, where the ϕ_i are just i.i.d. random variables. In that case, evidently nothing interesting is happening: The variables are individually conditioned to stay positive. In particular, $E(\phi_i|\Omega_A^+)$ stays bounded for $A \uparrow \mathbb{Z}^d$. This picture remains the same for fields with rapidly decaying correlations. However, gradient fields behave entirely differently, and so do interfaces in more realistic statistical physics models. As the surface has some global stiffness, the energetically best way for the surface to leave some room for the local fluctuations is to move away from the wall in some global sense. This effect is called "entropic repulsion" and is well known in the physics literature.

The first mathematically rigorous treatment of entropic repulsion appeared in the paper by Bricmont, Fröhlich and El Mellouki [12], where some qualitative results were proved. In a series of recent papers [4], [17], [18], and [5], sharp quantitative results have been derived, the most accurate ones for the harmonic case.

In most of these and related questions, the two-dimensional case is the most difficult but also the most interesting one. In fact, interfaces in the "real world" are mostly two-dimensional. I give a discussion of the two-dimensional case in the next section. In the present one, I outline what is happening in the easier higher-dimensional case, and mainly stick to the harmonic case. For gradient non-Gaussian models, some results in the same spirit have been obtained in [18], but they are not as precise as the ones obtained in the Gaussian model. This is partly connected with the difficulties to get precise information from the random walk representation (1.7). The case where one starts with the field $P_{\infty}^{\rm harm}$ is somewhat easier than the field on the finite box Λ_n with zero boundary condition. In the latter case, there are some boundary effects complicating the situation without changing it substantially. This is investigated in [17]. I am dropping from now on the superscript "harm", giving special mentioning when the more general situation is considered. Despite the fact that we consider P_{∞} , we consider the wall only on a finite box, i.e., we consider $P_{\infty}\left(\cdot|\Omega_{\Lambda_n}^+\right)$, and we are interested in what happens as $n \to \infty$. We usually write Ω_n^+ for $\Omega_{\Lambda_n}^+$. Our first task is to get information about $P_{\infty}(\Omega_n^+)$.

Theorem 2.1. Let $d \ge 3$, and consider the harmonic case. Then

a)

$$P_{\infty}\left(\Omega_{n}^{+}\right) = \exp\left[-2\Gamma(0)\mathrm{cap}\left(\Lambda\right)n^{d-2}\log n\left(1+o(1)\right)\right],$$

where $\Lambda = [-1,1]^d$, cap(A) denotes the Newtonian capacity of A

$$\operatorname{cap}(A) \stackrel{\mathrm{def}}{=} \inf \left\{ \left\| \nabla f \right\|^2 : f \geqslant 1_A \right\},$$

and $\Gamma(0) = \Gamma_{\infty}(0,0)$ is the variance of ϕ_0 under P_{∞} .

b)

$$E_{\infty}\left(\phi_0|\Omega_n^+
ight) = 2\sqrt{\Gamma(0)\log n}(1+o(1)).$$

c)

$$\mathcal{L}_{P_{\infty}(\cdot|\Omega_{n}^{+})}\left(\left(\phi_{i}-E_{\infty}(\phi_{i}|\Omega_{n}^{+})\right)_{i\in\mathbb{Z}^{d}}\right)\to P_{\infty} \ weakly$$

where $\mathcal{L}_{P_{\infty}(\cdot|\Omega_n^+)}$ denotes the law of the field under the conditioned measure.

Part b) gives the exact rate at which the random surface escapes to infinity, while part c) states that the effect of the entropic repulsion essentially consists of only this shift: after subtraction of the shift, the surface looks as it does without the wall. However, there is some subtlety in this picture. From the Theorem in particular part c), one might conclude that $\lim_{n\to\infty} P_{\infty} \theta_{2\sqrt{\Gamma(0)\log n}}^{-1}(\Omega_n^+) = 1$, where $\theta_a : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is the shift mapping $\theta_a\left((\phi_i)_{i\in\mathbb{Z}^d}\right) = (\phi_i + a)_{i\in\mathbb{Z}^d}$. But this is not the case. In fact $P_{\infty} \theta_{2\sqrt{\Gamma(0)\log n}}^{-1}(\Omega_n^+)$ converges rapidly to 0. As part c) states only the weak convergence, this is no contradiction. Parts a) and b) of Theorem 2.1 had been proved in [4], part c) in [18].

2.2 Sketch of the proof of the lower bound in Theorem 2.1 a).

The proof of part a) partly follows the standard pattern in large deviation theory, but there are some uncommon aspects. I will give some more detailed comments concerning the lower bound which is quite interesting also from the large deviation point of view. In the next section, I will discuss in a bit more detail the proof of the upper bound in the two-dimensional case, which is much more delicate than the proof of the upper bound in higher dimensions.

Before I proceed, I recall one of the basic large deviation "philosophies" in the most simple case of the Cramer Theorem for sums of i.i.d. real-valued random variables X_i , $i \in \mathbb{N}$, having exponential moments, and satisfying $EX_i = 0$. We are interested e.g. in the behavior of $P\left(\frac{1}{n}\sum_{i=1}^{n}X_i \ge a\right)$ for fixed a > 0 and large n. The well-known strategy is to ask what joint law Q_n for the vector (X_1, \ldots, X_n) would achieve that the empirical mean exceeds a with large probability, and Q_n is "entropically closest" to μ^n , where μ denotes the law of the X_i . "Entropically closest" means that the relative entropy $H(Q_n|\mu^n)$ is minimal. The relative entropy is defined by

$$H(P|Q) \stackrel{\text{def}}{=} \int \log \frac{dP}{dQ} dP$$

for two probability measure P and Q, where the expression has to be understood to be infinite if P is not absolutely continuous with respect to Q or if the logarithm of the derivative is not integrable with respect to P. In the Cramér case, it turns out that the optimal Q_n is again a product measure, say ν^n , and that ν has to be chosen inside the exponential family

$$\nu(dx) \stackrel{\text{def}}{=} e^{\lambda x} \mu(dx) \left/ \int e^{\lambda x} \mu(dx) \right.$$
 (2.1)

The parameter λ is uniquely chosen such that ν has exactly mean a. Cramér's Theorem then states that

$$P\left(\frac{1}{n}\sum_{i=1}^{N}X_{i} \ge a\right) = \exp\left[-H\left(Q_{n}|\mu^{n}\right)\left(1+o(1)\right)\right] = \exp\left[-nH(\nu|\mu)(1+o(1))\right].$$
(2.2)

The actual proof of the lower bound can be given by the following well-known (and easy) entropy inequality: If P and Q are two probability measures on an arbitrary measurable space (Ω, \mathcal{F}) and $A \in \mathcal{F}$, then

$$\log \frac{P(A)}{Q(A)} \ge -\frac{H(Q|P) + e^{-1}}{Q(A)}.$$
(2.3)

Typically, this inequality is useful only if $Q(A) \sim 1$, as otherwise one does not get any decent information about P(A). In the Cramér case, this is achieved by choosing $\lambda = \lambda_b$ in (2.1) slightly too large, namely such that $\int x\nu(dx) = b > a$. Then by the law of large numbers one gets $\nu^n \left(\frac{1}{n} \sum_{i=1}^N X_i \ge a\right) \to 1$, and hence the lower bound in (2.2) follows from (2.3) with $P = \mu^n$, $Q = \nu^n$, and $A = \{\sum_{i=1}^n X_i / n \ge b\}$ by letting $b \downarrow a$ in the end.

Let us now look at the modification this argument needs in order to prove the lower bound in part a) or Theorem 2.1.

The first task is to find out what the appropriate (and hopefully optimal) measure transformations are. Not surprisingly, at least in the Gaussian case, the cheapest way to transform the measure P_{∞} is to apply a shift. Furthermore, and this is not completely obvious, the shift should be constant on the box Λ_n , say to a positive level a_n , which still has to be determined. Outside Λ_n , we also apply a shift, otherwise the entropy cost at the boundary would be too large. It is not difficult to see that the optimal thing to do is to apply a shift given by the harmonic extension of $a_n 1_{\Lambda_n}$ outside Λ_n . Therefore, we consider the field $(\phi_i + f_n(i))_{i \in \mathbb{Z}^d}$, where

$$f_n(i) \stackrel{\text{def}}{=} \begin{cases} a_n & \text{for } i \in \Lambda_n \\ a_n \mathbb{P}_i^{\text{RW}}(\tau < \infty) & \text{for } i \notin \Lambda_n \end{cases},$$
(2.4)

where τ is the first entrance time into Λ_n of the random walk. We denote the law of this field by $R_{n,a}$ and we keep in mind that it depends on the choice of the sequence $a = (a_n)$. The relative entropy is obtained by an easy calculation:

$$H(R_{n,a}|P_{\infty}) = \int \log \frac{\exp\left[-\frac{1}{2} \langle \phi - f_n, (-Q)(\phi - f_n) \rangle\right]}{\exp\left[-\frac{1}{2} \langle \phi, (-Q)\phi \rangle\right]} dR_{n,a}$$
(2.5)
$$= \frac{1}{2} \langle f_n, (-Q)f_n \rangle.$$

Remember that Q is the Q-matrix of the random walk, i.e., the discrete Laplacian. It is well known (and easy to see) that this expression behaves for large n in the

following way:

$$H(R_{n,a}|P_{\infty}) = \frac{1}{2}a_n^2 n^{d-2} \operatorname{cap}(\Lambda)(1+o(1)), \qquad (2.6)$$

where $\Lambda = [-1, 1]^d$, and $\operatorname{cap}(\Lambda)$ is the Newtonian capacity of Λ .

The next task is to determine the proper choice of a_n . The discussion of the Cramér case would lead to the conclusion that we should choose this sequence in such a way that $R_{n,a}(\Omega_n^+) \to 1$. But this turns out to be wrong, a fact which leads to some interesting complications as I will explain now.

Let us first derive a very crude lower bound for $R_{n,a}(\Omega_n^+)$. As the field of random variables is positively correlated (and Gaussian), we can apply Slepian's inequality (i.e., the FKG inequality for physicists), see [27] Corollary 3.12, and get

$$R_{n,a}(\Omega_n^+) \ge \prod_{i \in \Lambda_n} R_{n,a}(\phi_i \ge 0) = \left(\Phi\left(\frac{a_n}{\sqrt{\Gamma(0)}}\right)\right)^{|\Lambda_n|},$$
(2.7)

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where Φ is the standard normal distribution function. This inequality appears to be of a very doubtful quality, as is revealed by considering the case $a_n = 0$, in which case it only tells that $P_{\infty}(\Omega_n^+) \ge (1/2)^{|\Lambda_n|}$, which is certainly not terribly impressive. None the less, let us look at what happens when $a_n \to \infty$. Using the standard expansion for the standard normal distribution function $1 - \Phi(x) \sim \frac{1}{x}\varphi(x), x \to \infty$, where φ is the standard normal density, we get

$$R_{n,a}(\Omega_n^+) \ge \exp\left[-\left|\Lambda_n\right| \frac{\sqrt{\Gamma(0)}}{a_n} \varphi\left(\frac{a_n}{\sqrt{\Gamma(0)}}\right) \left(1+o(1)\right)\right].$$
(2.8)

The bound goes to 1 as soon as $a_n \ge \sqrt{2d\Gamma(0)\log n}$. On the other hand, if $a_n \le (1-\varepsilon)\sqrt{2d\Gamma(0)\log n}$ with $\varepsilon > 0$ arbitrary, then the bound goes to 0 and therefore does not appear to be very useful. Of course, one may ask whether $R_{n,a}(\Omega_n^+)$ really goes to 0 in the latter case, as the application of the Slepian inequality might be quite crude. However, it can in fact be proved that $R_{n,a}(\Omega_n^+) \to 0$ for $a_n \sim (1-\varepsilon)\sqrt{2d\Gamma(0)\log n}$. Therefore, $a_n = \sqrt{2d\Gamma(0)\log n}$ is the border line case for which we seem to be able to get anything useful out of the entropy inequality (2.3). Applying it together with (2.6) and the above choice of the sequence (a_n) , we get the bound

$$P_{\infty}(\Omega_n^+) \ge \exp\left[-d\Gamma(0)\mathrm{cap}(\Lambda)n^{d-2}\log n(1+o(1))\right].$$
(2.9)

This bound is of the right order, but the constant in the exponent is the wrong one, as is seen from Theorem 2.1 a). We may ask who the culprit of this failure is. It turns out that it is *not* the inequality (2.7) but the entropy inequality (2.3), as I will now explain.

At first sight, it is not clear why the above argument should not give a sharp lower bound. The reader of course knows that this is not the case, at least if he believes in the statement of the Theorem. I present a heuristic argument that gives the asymptotic behavior stated in part a) of Theorem 2.1. However, the reader will realize, that the heuristics is somewhat at odds with the use of the entropic bound propounded above. The device is just to match the right-hand side in the Slepian inequality (2.7) with $\exp\left[-H(R_{n,a}|P_{\infty})\right]$. The prefactor $\sqrt{\Gamma(0)}/a_n$ in (2.8) is of course of no importance here, as the behavior of the right-hand side is mainly determined by $\varphi\left(a_n/\sqrt{\Gamma(0)}\right)$ in the exponent. (As a side-tracking remark, I would like to mention that this prefactor is of crucial importance in the proof of the absence of a wetting transition for dimensions $d \ge 3$ in [5].) Remark also that the right-hand side of (2.8) depends much more sensitively on the sequence (a_n) than does $\exp\left[-H(R_{n,a}|P_{\infty})\right] = \exp\left[-\frac{1}{2}a_n^2n^{d-2}\operatorname{cap}(\Lambda)(1+o(1))\right]$. Therefore, the match essentially is achieved by equating the "capacity rate" n^{d-2} with $|\Lambda_n| \exp\left[-a_n^2/2\Gamma(0)\right]$, leading to

$$a_n \sim 2\sqrt{\Gamma(0)\log n},$$

and with this choice we get

$$\exp\left[-H(R_{n,a}|P_{\infty})\right] = \exp\left[-2\Gamma(0)\operatorname{cap}(\Lambda)n^{d-2}\log n(1+o(1))\right]$$

which is the correct behavior. However, the lower bound cannot be obtained directly via the entropy inequality, as $R_{n,a}(\Omega_n^+) \to 0$.

The way out of this problem is to make a decomposition of the random field into a part which keeps the long-range dependencies but suppresses the local fluctuations, and a second part that keeps the fluctuations but has rapidly decaying correlations. The entropy inequality is then applied only to the first one. In [4], this splitting was done in a rather tricky way. In later applications [18], [5] it was realized that the splitting is a variant of a well-known and simple splitting routinely used in field theory. Remember that the covariances are given by $\Gamma = (-Q)^{-1}$, where Q is the discrete Laplacian. We now introduce a so-called mass, i.e., we consider the operator $-Q + \varepsilon^2$ and a corresponding inverse

$$\Gamma_{\varepsilon} \stackrel{\mathrm{def}}{=} \left(-Q + \varepsilon^2\right)^{-1}.$$

This is the covariance matrix of a Gaussian field with a random walk representation where the random walk has a killing rate ε^2 . Therefore, it is easy to see that these covariances are exponentially decaying. On the other hand, if ε is small, then the above modification has not much effect on the short range correlations. If we consider now

$$\widetilde{\Gamma}_{\varepsilon} \stackrel{\text{def}}{=} \Gamma - \Gamma_{\varepsilon},$$

then this is still positive definite, i.e., the covariance matrix of a Gaussian field. This field essentially keeps the long range correlations of the original field, but suppresses the local fluctuations. According to this splitting, we decompose the field variables ϕ_i into independent parts:

$$\phi_i = \phi_i^\varepsilon + \phi_i^\varepsilon.$$

With this decomposition, we have separated the two problems. We apply the argument with the entropy bound only to the field $\tilde{\phi} \stackrel{\text{def}}{=} \left(\tilde{\phi}_i^{\varepsilon} \right)_{i \in \Lambda_n}$. A simple but crucial observation is that the relative entropy for this field under shifts given by f_n in (2.4) behaves in exactly the same way as for the original field. This is coming from the fact that this relative entropy is a "long range" quantity not affected by the cutting of the local fluctuations. The reader may easily check this. Therefore, we have for any $\varepsilon > 0$:

$$H\left(P_{\infty}\left(f_{n}+\widetilde{\phi}^{\varepsilon}\in\cdot\right)\middle|P_{\infty}\left(\widetilde{\phi}^{\varepsilon}\in\cdot\right)\right)=\frac{a_{n}^{2}}{2}n^{d-2}\mathrm{cap}(\Lambda)(1+o(1)).$$

The o(1)-term is understood to be for $n \to \infty$, and depends on ε . As remarked above, we essentially have killed the local fluctuations in the $\tilde{\phi}^{\varepsilon}$ -field. Applying Slepian's inequality as above, we therefore have $P_{\infty}\left(f_n + \tilde{\phi}^{\varepsilon} \in \Omega_n^+\right) \to 1$ for $a_n = c\sqrt{\log n}$ with any c > 0 provided we choose ε small enough, and from that we would get estimates on $P_{\infty}\left(\tilde{\phi}^{\varepsilon} \in \Omega_n^+\right)$. However, we are not really interested in this probability as we still have to take into consideration the ϕ_i^{ε} variables, which we have neglected for the moment. The idea is not to look at $P_{\infty}\left(\tilde{\phi}^{\varepsilon} \in \Omega_n^+\right)$, but at the probability of some much more restrictive event, namely

$$P_{\infty}\left(\widetilde{\phi}_{i}^{\varepsilon} \ge \rho \sqrt{\log n} \text{ for all } i \in \Lambda_{n}\right),$$

where we choose ρ slightly above the "optimal height", i.e., $\rho = 2\sqrt{\Gamma(0)} + \delta$ with some small δ . If we now choose ε small enough, suppressing the local fluctuations in the $\tilde{\phi}_i^{\varepsilon}$ variables as far as necessary (depending on the choice of δ), we get by the above procedure, using Slepian's inequality and the entropic bound:

$$\begin{split} P_{\infty}\left(\widetilde{\phi}_{i}^{\varepsilon} \geqslant \left(2\sqrt{\Gamma(0)} + \delta\right)\sqrt{\log n} \text{ for all } i \in \Lambda_{n}\right) \\ \geqslant \exp\left[-2(1+c(\delta))\Gamma(0)\mathrm{cap}(\Lambda)n^{d-2}\log n\right], \end{split}$$

with $c(\delta) \to 0$ for $\delta \to 0$. This is just coming from relative entropy considerations, the procedure to apply the entropy inequality being exactly the same as in the Cramér case, this now working as we can suppress the local fluctuations by choosing ε appropriately. Remember that these local fluctuations had brought the problems responsible for not getting a better bound than (2.9). A moment's reflection reveals now that we are done:

$$\begin{split} P_{\infty}\left(\Omega_{n}^{+}\right) &\geq P_{\infty}\left(\Omega_{n}^{+}\left|\widetilde{\phi}_{i}^{\varepsilon}\right| \geq \left(2\sqrt{\Gamma(0)}+\delta\right)\sqrt{\log n} \text{ for all } i \in \Lambda_{n}\right) \\ &\times P_{\infty}\left(\widetilde{\phi}_{i}^{\varepsilon} \geq \left(2\sqrt{\Gamma(0)}+\delta\right)\sqrt{\log n} \text{ for all } i \in \Lambda_{n}\right), \end{split}$$

and

$$P_{\infty}\left(\phi_{i}^{\varepsilon} \geqslant -\left(2\sqrt{\Gamma(0)} + \delta\right)\sqrt{\log n} \text{ for all } i \in \Lambda_{n}\right) \geqslant \exp\left[-n^{d-2-\eta(\delta)}\right],$$

again by Slepian's inequality, with $\eta(\delta) > 0$, $\eta(\delta) \to 0$ as $\delta \to 0$. We therefore obtain the lower bound, by letting $\delta \to 0$ in the end.

2.3 Sketch of the proof of the upper bound in Theorem 2.1 a)

I don't discuss the upper bound in detail, but I give a sketch of an argument giving an upper bound of the right order but not the correct constant.

We distinguish between even and odd points in Λ_n . A point is called even if the sum of its components is even, otherwise it is called odd. Evidently

$$\begin{split} P_{\infty}\left(\Omega_{n}^{+}\right) &\leqslant P_{\infty}\left(\phi_{i} \geqslant 0, \ i \text{ even}\right) \\ &= E_{\infty}\left(\mathbf{1}_{\{\phi_{j} \geqslant 0, \ i \text{ odd}\}}P_{\infty}\left(\phi_{i} \geqslant 0, \ i \text{ even}\left|\left(\phi_{j}\right)_{j \text{ odd}}\right)\right). \end{split}$$

The effect of this conditioning is of course very simple. Let us phrase it for later purposes in a slightly more general situation. Consider $A \subset \Lambda_n$ and the σ -field \mathcal{F}_A generated by $(\phi_i)_{i \in A}$. Then the conditional distribution of the field given \mathcal{F}_A is still Gaussian. Furthermore

$$E\left(\phi_{i}|\mathcal{F}_{A}\right) = \sum_{j \in A} \mathbb{P}_{i}^{\mathrm{RW}}(\eta_{\tau_{A} \wedge \tau_{\Lambda_{n}^{c}}} = j)\phi_{j}, \qquad (2.10)$$

where τ_A is the first entrance time in A, and

$$\operatorname{cov}\left(\phi_{i},\phi_{j}|\mathcal{F}_{A}\right) = \mathbb{E}_{i}^{\operatorname{RW}}\left(\int_{0}^{\tau_{A}\wedge\tau_{\Lambda_{n}^{e}}} 1_{\{\eta_{s}=j\}}ds\right).$$
(2.11)

Applying this to our situation, we see that conditioned on $(\phi_j)_{j \text{ odd}}$ the variables ϕ_i , *i* even, are i.i.d. with a fixed variance $\sigma^2 = 1/2d < \Gamma(0)$. The conditional mean of ϕ_i is just the arithmetic mean of the neighboring ϕ_j , which we denote by $\overline{\phi}_i$. We therefore get

$$P_{\infty}\left(\phi_{i} \ge 0, \ i \text{ even } \left| (\phi_{j})_{j \text{ odd}} \right. \right) = \prod_{i \text{ even}} P\left(\xi \ge -\overline{\phi}_{i}\right), \qquad (2.12)$$

where ξ is a centered normally distributed random variable with variance σ^2 . Remark next that the above expression is very small, unless most of the $\overline{\phi}_i$ are quite large. For instance, if half of the $\frac{1}{2}(2n)^d$ variables $\overline{\phi}_i$ are $\leq \delta \sqrt{\log n}$, then the above expression is roughly

$$\left(1 - \exp\left[-\frac{\delta^2 \log n}{2\sigma^2}\right]\right)^{\frac{1}{4}(2n)^d} \simeq \exp\left[-cn^{d-1}\right] \stackrel{\text{def}}{=} \rho(n), \tag{2.13}$$

if $\delta^2/2\sigma^2 = 1$. This is negligible with respect to the desired bound. Therefore, we get

$$P_{\infty}\left(\Omega_{n}^{+}\right) \leqslant P_{\infty}\left(\overline{\phi}_{i} > \delta\sqrt{\log n} \text{ for half of the even } i, \ \phi_{j} \ge 0 \text{ all odd } j\right)$$
$$+ \rho(n)$$
$$(2.14)$$
$$\leqslant P_{\infty}\left(\frac{1}{(2n)^{d}/2} \sum_{i \text{ even}} \overline{\phi}_{i} \ge \frac{\delta}{2}\sqrt{\log n}\right) + \rho(n).$$

By an elementary computation, we have

$$\operatorname{Var}\left(\frac{1}{(2n)^d/2}\sum_{i \text{ even}}\overline{\phi}_i\right) = n^{-d+2} \iint_{A \times A} (-\Delta)^{-1}(x,y) dx dy \times (1+o(1)),$$

where $(-\Delta)^{-1}(x,y)$ is the standard Green kernel of the *d*-dimensional Laplacian. Therefore

$$P_{\infty}\left(\Omega_{n}^{+}\right)\leqslant\exp\left[-Cn^{d-2}\log n
ight].$$

This is a bound of the correct order, but the constant C is not the optimal one. I will now give a sketch how the above procedure can be trimmed to yield the correct constant.

First I present some straightforward ways to optimize the above procedure. For instance, it is not necessary to have half of the $\overline{\phi}_i$ to be $\leq \delta \sqrt{\log n}$ in order to get (2.13). A bound of the same order is obtained if an arbitrary proportion of the $\overline{\phi}_i$ variables is below this level. Also, on the right hand side of (2.13), one does not really need n^{d-1} in the exponent. It suffices to have something slightly larger than n^{d-2} , and therefore one can nearly take $\delta = 2\sigma$. Using this observation, one obtains easily

$$P_{\infty}\left(\Omega_{n}^{+}\right) \leqslant P_{\infty}\left(\frac{1}{(2n)^{d}/2}\sum_{i \text{ even}}\overline{\phi}_{i} \geqslant \left(2\sigma - \varepsilon\right)\sqrt{\log n}\right)$$

with $\varepsilon > 0$ arbitrary, as soon as n is large enough, and this yields

$$P_{\infty}\left(\Omega_{n}^{+}\right) \leqslant \exp\left[-\frac{2\sigma^{2}(1+o(1))}{\iint_{\Lambda \times \Lambda}(-\Delta)^{-1}(x,y)dxdy}n^{d-2}\log n\right],$$
(2.15)

but the reader, may check that this is still not the correct constant of Theorem 2.1. There are two reasons for this failure. The first one is easy to amend: It is bad to switch to the "global" arithmetic mean of the $\overline{\phi}_i$ variables in the second inequality in (2.14). The correct way to do this step is to chop the box Λ_n first into finitely many subboxes, still of macroscopic scale, say of side length εn , do the arithmetic mean procedure separately for all the subboxes, and letting $\varepsilon \to 0$ in the end (after the $n \to \infty$ limit). The crucial point is that we get estimates of the type (2.13) on the event that a small proportion of the $\overline{\phi}_i$ in any of the εn -boxes is below the critical value δ which is slightly smaller than 2σ . The reader may check that in this way, we can replace $1 / \iint_{\Lambda \times \Lambda} (-\Delta)^{-1}(x, y) dxdy$ on the right hand side of (2.15) by cap(Λ), i.e., we are getting

$$P_{\infty}\left(\Omega_{n}^{+}\right) \leqslant \exp\left[-2\sigma^{2}\operatorname{cap}(\Lambda)(1+o(1))n^{d-2}\log n\right]$$
(2.16)

Now we would be finished, if only we could replace σ^2 by $\Gamma(0)$. Remember however that $\sigma^2 < \Gamma(0)$.

Here is a sketch how to handle the problem properly. The even-odd procedure is just not the proper thing to do. Instead of conditioning on the odd variables, one has to take a larger but still microscopic subgrid. The field on which one is conditioning is $(\phi_j)_{j \in L\mathbb{Z}^d}$, L large. For points *i* away from the grid points, e.g.

$$i \in \Lambda_{n,L} \stackrel{\text{def}}{=} \left(L\mathbb{Z}^d + (L/2, \dots, L/2) \right) \cap \Lambda_n, \tag{2.17}$$

the conditional variance is σ_L^2 , which satisfies

$$\lim_{L \to \infty} \sigma_L^2 = \Gamma(0). \tag{2.18}$$

This follows easily from (2.11) and removes the problem of the conditional variance being too small, but there are evidently some difficulties. A seeming one is that we have to estimate

$$P_{\infty}\left(\Omega_{n}^{+}\right) \leqslant P_{\infty}\left(\phi_{i}, i \in \Lambda_{n,L}\right), \qquad (2.19)$$

which may look bad because $|\Lambda_{n,L}|$ is much smaller than $|\Lambda_n|$. However, it turns out that (2.19) is sharp on the level of precision we are considering. Keep in mind that we are looking at a phenomenon that is essentially of "long range" character, so it should be not surprising that such a thinning is not bad for the estimate we have in mind. We want to do the same conditioning procedure as explained above, but now conditioning on the $(\phi_j)_{j \in L\mathbb{Z}^d}$. An evident problem is that the ϕ_i , $i \in \Lambda_{n,L}$, are no longer conditionally independent. The conditional covariances are expressed by (2.11). From this representation, it is evident that these conditional covariances are rapidly decaying. This is then sufficient to apply hypercontractivity estimates which essentially give an estimate similar to the right-hand side of (2.12). Summarizing: One fixes L and then with some additional work one can get the same estimates as with the even-odd procedure, and one gets an upper bound like (2.16), but with σ^2 replaced by σ_L^2 . Using (2.18), this gives the desired upper bound. For details, see [4]. Anticipating already the discussion for the two-dimensional case in the next section, it should be remarked that the main problem there is that such a "one step conditioning" is no longer appropriate. In two dimensions, any conditioning with a finite L is "killing" too much variance, and only $L = L_n = n^{\alpha}$ with $\alpha < 1$ but arbitrary close to 1 would leave the variance intact. But then, the estimate (2.19) is becoming really too bad. We will give some explanations in the next section how to solve this problem.

2.4 Final remarks

- 1. The analysis of $P_{\infty}(\Omega_n^+)$ as sketched above also makes parts b) and c) of Theorem 2.1 very plausible, but there are some substantial additional technical difficulties to really prove these, especially part c). I will not discuss this here.
- 2. An analysis of the type explained above can also be done for more general non-Gaussian gradient models, where the analysis of the variances and covariances can be performed with the Helffer-Sjöstrand random walk representation introduced in Section 1. This has been done in [18]. However, the estimates available in this case are less precise than in the Gaussian case. For this reason, estimates with precise constants have not (yet) been obtained.

- **3.** I have restricted the discussion to the case where one starts with the infinite-volume Gaussian field P_{∞} . It might be more natural to consider the finite-volume measure P_n with zero boundary condition and condition it on Ω_n^+ . This has been done in [17]. The first task again is to investigate the behavior or $P_n(\Omega_n^+)$. This probability is however dominated by boundary effects: it is substantially smaller than $P_{\infty}(\Omega_n^+)$, because the event that the field is positive near the boundary has small probability. With the zero boundary condition, the variables ϕ_i for *i* near the boundary are essentially nearly independent. It is therefore clear that $P_n(\Omega_n^+)$ is at most of order exp $[-cn^{d-1}]$. This is in fact the correct asymptotic behavior, as has been proved in [17]. The behavior of the variables away from the boundary is however not much affected by the zero boundary condition and b) and c) of our Theorem remain correct (with the same constants) also in this case.
- 4. In the Gaussian case, it is actually possible to get an asymptotic evaluation which goes beyond the leading order term. This has been important in a recent work on so-called wetting transitions [5]. I will make some comments about these problems in the last section.

3 Entropic repulsion in two dimensions

3.1 Outline of the two-dimensional situation

We again consider only the harmonic case. If the lattice is two-dimensional, a thermodynamic limit of the measures P_n does not exist as the variance blows up. $P_n(\Omega_n^+)$ is of order exp[-cn], as has been shown in [17]. As remarked in the last section, this is mainly a boundary effect and is not so relevant for the phenomenon of the entropic repulsion. To copy somehow the procedure of the last section, consider a subset $D \subset V = [-1, 1]^2$ which has a nice boundary and a positive distance from the boundary of V. To be specific, just think of taking $D \stackrel{\text{def}}{=} \lambda V$ for some $\lambda < 1$. Then let $D_n \stackrel{\text{def}}{=} nD \cap \mathbb{Z}^2$ and $\Omega_{D_n}^+ \stackrel{\text{def}}{=} \{\phi_i \ge 0, i \in D_n\}$. In contrast to $P_n(\Omega_n^+)$, $P_n(\Omega_{D_n}^+)$ decays much slower, but still faster than any polynomial rate. In [6] we proved the following result:

Theorem 3.1. Let $g \stackrel{\text{def}}{=} 1/2\pi$.

a)

$$\lim_{n \to \infty} \frac{1}{(\log n)^2} \log P_n(\Omega_{D_n}^+) = -2g \operatorname{cap}_V(D),$$

where $\operatorname{cap}_V(D)$ is the relative capacity of D with respect to V:

$$\operatorname{cap}_{V}(D) \stackrel{\text{def}}{=} \inf \left\{ \|\nabla f\|_{2}^{2} : f \in H_{0}^{1}(V), \, f \ge 1 \text{ on } D \right\}.$$

Here, $H_0^1(V)$ is the Sobolev space of (weakly) differentiable functions f with square integrable gradient and $f|_{\partial V} = 0$.

b) For any $\varepsilon > 0$

$$\lim_{n \to \infty} \sup_{x \in D_n} P_n \left(\left| \phi_x - 2\sqrt{g} \log n \right| \ge \varepsilon \log n \right| \Omega_{D_n}^+ \right) = 0.$$

This corresponds to parts a) and b) of Theorem 2.1. Part c) does not make sense here as P_{∞} does not exist.

I will not give the proof of Theorem 3.1, it being quite complicated. I will however explain the main technique for an easier hierarchical model. First, one should point out that the lower bound in a) is quite easy, in fact easier than in the case $d \ge 3$ discussed in the last section. In order to understand why this is so, we step back to the heuristic explanation of the last section, which works here too, and gives the correct answer. We need precise information on the variances and covariances $\Gamma_n(i, j)$ given by (1.2). For $i \in \Lambda_n$ we write $\sigma_n^2(i) \stackrel{\text{def}}{=} \Gamma_n(i, i)$.

Lemma 3.2.

$$\sup_{i\in D_n} \left|\sigma_n^2(i) - g\log n\right| \leqslant c(\delta),$$

where $c(\delta)$ depends only on $\delta \stackrel{\text{def}}{=} \operatorname{dist}(D, \partial \Lambda)$.

The proof is an easy consequence of Theorem 1.3.3 of [26].

According to the heuristics given in Subsection 2.2, we should find a sequence $a = (a_n)$ such that there is an approximate match between $\exp\left[-H(R_{n,a}|P_n)\right]$ and $R_{n,a}(\Omega_{D_n}^+)$, where

$$R_{n,a} \stackrel{\text{def}}{=} P_n \left(\phi + f_n \in \cdot \right),$$

 f_n being given in our case given by

$$f_n(i) \stackrel{\mathrm{def}}{=} \left\{ egin{array}{cc} a_n & \mathrm{for}\, i \in D_n \ \mathbb{P}^{\mathrm{RW}}_i(au_{D_n} < au_{\partial \Lambda_n}) & \mathrm{for}\, i \in \Lambda_n \setminus D_n \end{array}
ight.$$

The relative entropy in the two-dimensional case is given by

$$H(R_{n,a}|P_n) = \frac{a_n^2}{2} \operatorname{cap}_V(D)(1+o(1)).$$

(See Lemma 2.4 of [1].) Estimating $R_{n,a}(\Omega_{D_n}^+)$ again by Slepian's inequality, we get

$$R_{n,a}(\Omega_{D_n}^+) \ge \prod_{i \in D_n} P_n(\phi_i + a_n \ge 0) \simeq \prod_{i \in D_n} \left(1 - \exp\left[-\frac{a_n^2}{2\sigma_n^2(i)} \right] \right)$$
(3.1)
$$\simeq \left(1 - \exp\left[-\frac{a_n^2}{2g \log n} \right] \right)^{|D_n|},$$

the last approximation by Lemma 3.2. An approximate matching of the right-hand side of (3.1) with $\exp\left[-H\left(R_{n,a}|P_n\right)\right]$ is achieved by taking $a_n \sim 2\sqrt{g}\log n$ and

then $\exp\left[-H\left(R_{n,a}|P_n\right)\right] = \exp\left[-2g\left(\log n\right)^2\left(1+o(1)\right)\right]$. Remark however, that if we choose a_n a tiny bit larger, then the right-hand side of (3.1) is tending to 1, without $\exp\left[-H\left(R_{n,a}|P_n\right)\right]$ being much changed. This contrasts sharply with the case $d \ge 3$, and makes things easier for the lower bound. Indeed, we can just apply Slepian's inequality together with the entropy bound.

On the other hand, proving the upper bound is *much* more delicate. To see the reason for this, we try to repeat the procedure outlined in Section 2.3. Consider again the field variables ϕ_i for $i \in \Lambda_{n,L}$ which was defined by (2.17) conditioned on $L\mathbb{Z}^2$. It is not difficult to see from (2.11) that for a fixed number L, σ_L^2 stays bounded in n. This variance is in fact up to some negligible correction the same as if we would just consider the box Λ_L with zero boundary conditions and calculate the variance of the field in the midpoint. Therefore, σ_L^2 is of order $g \log L$. This means that we have to choose $L = L_n$ essentially of order n, or maybe slightly smaller, in order that $\sigma_L^2 \sim \operatorname{var}(\phi_i)$. But this is of course disastrous for the proof outlined in Section 2.3, because $|\Lambda_{n,L}|$ is then way too small.

The way out of this problem is by applying a multiscale analysis, where one does successive conditionings on many intermediate scales. The details are quite delicate, and it is best to explain the procedure for a simplified hierarchical model which is however catching all the essentials that are necessary for a treatment of the harmonic case.

3.2 Entropic repulsion for a hierarchical model

We call a sequence $\alpha = \alpha_1 \alpha_2 \dots \alpha_m$, $\alpha_i \in \{0, 1\}$, a binary string; $\ell(\alpha) = m$ is the length. \emptyset is the empty string of length 0. We write T for the set of all such strings of finite length, and $T_m \subset T$ for the set of strings of length m. If $\alpha \in T_m$, $0 \leq k \leq m$, we write $[\alpha]_k$ for the truncation at level k:

$$[\alpha_1\alpha_2\ldots\alpha_m]_k \stackrel{\mathrm{def}}{=} \alpha_1\alpha_2\ldots\alpha_k.$$

Also, if $\alpha \in T$, $\beta \in T$, then we write $\alpha\beta \in T$ for the natural concatenation. If α , $\beta \in T_m$ we define the hierarchical distance

$$d_H(\alpha,\beta) \stackrel{\text{def}}{=} m - \max \left\{ k \leqslant m : [\alpha]_k = [\beta]_k \right\}$$

We consider the following family $(X_{\alpha})_{\alpha \in T}$ of centered Gaussian random variables: $X_{\varnothing} \stackrel{\text{def}}{=} 0$, and for $\alpha \in T$, $\ell(\alpha) = m \ge 1$

$$X_{\alpha} \stackrel{\text{def}}{=} \xi_{\alpha_1}^1 + \xi_{\alpha_1\alpha_2}^2 + \ldots + \xi_{\alpha_1\alpha_2\ldots\alpha_m}^m$$

where $(\xi_{\alpha_1\alpha_2...\alpha_i}^i)_{i\in\mathbb{N}, \alpha_1...\alpha_i\in T_i}$ is a family of independent centered normally distributed random variables with variance γ . Evidently, the 2^m random variables X_{α} , $\alpha \in T_m$ all have variance γm . More generally

$$\operatorname{cov}\left(X_{\alpha}, X_{\beta}\right) = \gamma\left(m - d_{H}(\alpha, \beta)\right).$$
(3.2)

It is also easily proved by induction on m that

$$\operatorname{var}\left(2^{-m}\sum_{\alpha\in T_m}X_{\alpha}\right) = \gamma\left(1-2^{-m}\right).$$

We argue now that there is much similarity between the two-dimension harmonic field $(\phi_i)_{i\in D_n}$ and the field $(X_{\alpha})_{\alpha\in T_m}$. To see this, we first match the number of variables, i.e., put $2^m = |D_n|$. As $|D_n|$ is of order n^2 , this means that $m \sim 2\log n/\log 2$. Then we should also match the variances, i.e., take $\gamma = g/2\log 2$. For the free field (ϕ_i) , it is known that $\operatorname{cov}(\phi_i, \phi_j)$ behaves like $g(\log n)/\log |i-j|$, if i, j are not too close to the boundary. This follows from the random representation and standard results on two-dimensional random walks (see [26]). Comparing this with (3.2), we see that for any number $s \in (0, g)$

$$\# \{ j \in D_n : \operatorname{cov}(\phi_i, \phi_j) \leqslant s \log n \} \sim \# \{ \beta \in T_m : \operatorname{cov}(X_\alpha, X_\beta) \leqslant s \log n \}$$
(3.3)

to leading order, for any $i \in D_n$, $\alpha \in T_m$. Therefore, the two fields have roughly the same covariance structure. The above criterion (3.3) looks a bit formal, but in fact the two fields are qualitatively (and even in many quantitative aspects) very close. The hierarchical field clearly has a much simpler structure. For instance, there is no real geometry involved. The zero boundary condition for the ϕ -field corresponds for the hierarchical field to the setting $X_{\emptyset} = 0$.

From now on, we set $\gamma = 1$. This is just a scaling which is of no importance.

Theorem 3.3. Let $\Omega_m^+ = \{X_\alpha \ge 0, \text{ for all } \alpha \in T_m\}$

a)

$$P\left(\Omega_{m}^{+}
ight)=\exp\left[-m^{2}\log2(1+o(1))
ight]$$

b) For any $\varepsilon > 0$

$$\lim_{m \to \infty} P\left(\left| X_{\alpha} - \sqrt{2\log 2m} \right| \ge \varepsilon \right| \Omega_m^+ \right) = 0,$$

where α is understood to belong to T_m .

We give only a closer discussion of part a). As already remarked, the lower bound is easy, and is a direct consequence of Slepian's inequality. The upper bound follows from properties of the distribution of the maximum of the variables, as will be explained now.

Proposition 3.4. Given $\eta > 0$, there exists $b(\eta) > 0$ such that

$$P\left(\max_{lpha\in T_m}X_lpha\leqslant\sqrt{2\log 2}(1-\eta)m
ight)\leqslant\exp\left[-b(\eta)m^2
ight]$$

for large enough m.

Before proceeding further with the discussion of this result, we show that it implies Theorem 3.3 a).

Proof that Proposition 3.4 implies Theorem 3.3 a). We fix $k \in \mathbb{N}$ and assume m > k. We write the elements $\alpha \in T_m$ as $\alpha = \beta \gamma, \beta \in T_k, \gamma \in T_{m-k}$ and

$$X_{\alpha} = X_{\beta} + Y_{\beta\gamma}^{(m-k)},$$

where $(Y_{\beta\gamma}^{(m-k)})_{\gamma\in T_{m-k}}$ for varying β are independent copies of $(X_{\gamma})_{\gamma\in T_{m-k}}$. Set $\mathcal{F}_k = \sigma(X_{\beta}: \beta \in T_k)$ and fix an arbitrary $\lambda > 0$. For $l \in \mathbb{N}$, let

$$A_l^{(k)} = \{ \#\{ \beta \in T_k : X_\beta \leqslant \sqrt{2\log 2}(1-\lambda)m\} \ge l \} \in \mathcal{F}_k.$$

According to Proposition 3.4, there exists $b(\lambda) > 0$ such that on $A_l^{(k)}$

$$P(X_{\alpha} \ge 0 \forall \alpha \in T_{m} | \mathcal{F}_{k})$$

$$\leqslant P\left(\inf_{\gamma \in T_{m-k}} Y_{\beta\gamma}^{(m-k)} \ge -\sqrt{2\log 2}(1-\lambda)m, \text{ for } l \text{ fixed } \beta l \text{s})\right)$$

$$\leqslant \exp(-b(\lambda)lm^{2}).$$

(for m large enough). We therefore get

$$P(X_{\alpha} \ge 0, \alpha \in T_m) \le P(A_l^{(k)c}) + \exp\left[-b(\lambda)lm^2\right],$$
(3.4)

for any l, k, λ , if m is large enough. Depending on λ , we choose l so large that $b(\lambda)l \ge 10$ ($l \ge l_0(\lambda)$, say). Let $\Lambda = \{\beta \in T_k : X_\beta \le (\sqrt{2\log 2} - \lambda)n\}$. On $A_l^{(k)c}$ we have $|\Lambda| \le l-1$. Putting $B = \{2^{-k} \sum_{\beta \in \Lambda} X_\beta^{(k)} \le -\lambda n\}$, we get

$$P\left(B \cap A_l^{(k)c}\right) \leqslant \binom{2^k}{l} \max_{\substack{S \subset T_k \\ |S| \leqslant l-1}} P\left(2^{-k} \sum_{\beta \in S} X_{\beta}^{(k)} \leqslant -\lambda m\right).$$

As $\operatorname{Var}\left(2^{-k}\sum_{\beta\in S}X_{\beta}^{(k)}\right) \leq 2^{-2k}k(l-1)^2$, we can estimate the right-hand side by

$$\leq 2^{lk} \exp(-\frac{\lambda^2 m^2}{22^{-2k}k(l-1)^2}) \leq \exp(-10m^2),$$

provided $k \ge k_0(\lambda, l)$ (*m* large enough). Combining this with (3.4), we get

$$P(X_{\alpha} \ge 0, \alpha \in T_m) \le P(A_l^{(k)c} \cap B^c) + 2\exp(-10m^2)$$
(3.5)

for $l \ge l_0(\lambda), \ k \ge k_0(l,\lambda).$ On $A_l^{(k)c} \cap B^c$ we have

$$2^{-k} \sum_{\beta \in T_k} X_{\beta} = 2^{-k} \sum_{\beta \in \Lambda} X_{\beta} + 2^{-k} \sum_{\beta \notin \Lambda} X_{\beta}$$
$$\geqslant -\lambda m + \frac{2^k - l}{2^k} (\sqrt{2\log 2} - \lambda) m \geqslant \sqrt{2\log 2}m - 3\lambda m$$

by still increasing k (if necessary). Implementing this in (3.5), we get

$$\limsup_{m \to \infty} \frac{1}{m^2} \log P(X_{\alpha} \ge 0, \alpha \in T_m) \leqslant -\frac{1}{2} \frac{(\sqrt{2\log 2} - 3\lambda)^2}{1 - 2^{-k}}.$$

Letting $k \to \infty$, and then $\lambda \to 0$, we get the upper bound in Theorem 3.3 a). As remarked before, the lower bound is an easy application of Slepian's inequality.

3.3 Discussion of Proposition 3.4

The result was actually known since long, except perhaps the Gaussian tail estimate. For instance Biggins in [3] proved that

$$\lim_{m \to \infty} \frac{\max_{\alpha \in T_m} X_\alpha}{m} = \sqrt{2\log 2}$$
(3.6)

in probability, and this has been reproved many times. The constant is somewhat surprising, since it is the same as if the variables would be independent: If Y_{α} are 2^m i.i.d. normally distributed random variables with variance m, then it is easy to see that

$$\lim_{m \to \infty} \frac{\max_{\alpha \in T_m} Y_\alpha}{m} = \sqrt{2 \log 2}.$$
(3.7)

Therefore, the Biggins result states that, for the maximum, the hierarchical model behaves to leading order in the same way as if the field variables would be independent.

Perhaps the easiest way to understand (3.6) (but perhaps not to prove it) is to replace the binary tree of depth m by one with a fixed (large) number K of branching levels. So we consider variables

$$X_{\alpha}^{(K)} \stackrel{\text{def}}{=} \xi_{\alpha_1}^1 + \xi_{\alpha_1 \alpha_2}^2 + \ldots + \xi_{\alpha_1 \ldots \alpha_K}^K, \tag{3.8}$$

where $\alpha_i \in \{1, \ldots, 2^{m/K}\}$, and the $\xi^i_{\alpha_1, \ldots, \alpha_i}$ are normally distributed with expectation 0 and variance m/K. We again get 2^m random variables with variance m, but a slightly different covariance structure. By (3.7), as $m \to \infty$ (K fixed), we have

$$\frac{1}{m} \max_{\alpha_1} \xi_{\alpha_1}^1 \to \frac{\sqrt{2\log 2}}{K}$$

in probability, and for any $\alpha_1, \ldots, \alpha_i$

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$$\frac{1}{m} \max_{\alpha_{i+1}} \xi_{\alpha_1, \dots, \alpha_{i+1}}^{i+1} \to \frac{\sqrt{2\log 2}}{K}.$$

From this one gets

$$\lim_{m \to \infty} \frac{1}{m} \max_{\alpha} X_{\alpha}^{(K)} \ge \sqrt{2\log 2}$$
(3.9)

for any fixed K. The upper bound follows directly from Slepian's inequality (also for the binary tree case). This of course does not prove (3.6) in the binary tree case, but as K is arbitrary, it makes it very plausible. Our proof of Proposition 3.4 (and actually also in the ϕ -field case) is based on a refinement and extension of the above "finite K" argument.

Remark 3.5. Much more than just (3.6) or the statement of Proposition 3.4 is known in the binary tree case, namely also the next order corrections which are of order log m, and even the correct tail estimates: For m and u large enough, one has

$$P\left(\max_{\alpha\in T_m} X_{\alpha} \geqslant \sqrt{2\log 2} \, m - \frac{3}{2\sqrt{2\log 2}} \log m + u\right) \leqslant \exp\left[-cu\right] \tag{3.10}$$

$$P\left(\max_{\alpha\in T_m} X_{\alpha} \leqslant \sqrt{2\log 2} \, m - \frac{3}{2\sqrt{2\log 2}} \log m - u\right) \leqslant \exp\left[-cu^2\right] \tag{3.11}$$

There is no published proof for this, not even for the fact that $-\frac{3}{2\sqrt{2\log^2}}\log m$ is the proper correction. The result is however close to a result of Bramson [9] on branching Brownian motions, and can be proved by an adaptation of his approach. Bramson had announced results of this type in his paper, but they have never been published. The Gaussian lower tail estimate in (3.11) is not proved in [9], also not for the branching Brownian case, but it is actually not very difficult. It should also be remarked that for the supremum of 2^m i.i.d. Gaussian random variables with variance m, the correction to the leading order $\sqrt{2\log 2m}$ is only $-\frac{1}{2\sqrt{2\log^2}}\log m$, which is easily checked. Also the lower tail in this case is doubly exponential, while the Gaussian tail in (3.11) is the correct behavior in the tree case.

Remark 3.6. Sidetracking a bit further, let us observe that the binary tree case is the border line case where the above triviality of the maximum (to leading order) is correct, i.e., where the maximum of the field of random variables is to leading order at the same level as if they were independent. To give this statement a precise meaning, consider again the above binary tree, but where the variances of the variables $\xi_{\alpha_1...\alpha_k}^k$, $k \leq m$, may vary with k, but still remain independent. For instance, consider a continuous function $f: [0,1] \to (0,\infty)$, satisfying $\int f(x)dx = 1$, and set $\operatorname{var}(\xi_{\alpha_1...\alpha_k}^k) = f(k/m)$. Then the variances of the variables X_{α} for $\alpha \in T_m$ is still m (approximately). One may ask under what conditions on f (3.6) remains true. One can prove that this is the case if and only if f is nondecreasing. The binary tree case discussed before is the case with $f \equiv 1$. For a discussion of various aspects of this and related models and its connections with spin glass theory, see [16].

I give a proof of Proposition 3.4 including the Gaussian tail estimate. The proof given here has the advantage to be quite robust, and a modification of it, using a cascade of conditionings, has been used in the free field case in [6] to prove entropic repulsion.

Before starting with the proof of Proposition 3.4, it has to be observed that the most straightforward approach, namely an application of the second moment method

fails. Define for $\lambda > 0$

$$N_m(\lambda) \stackrel{\text{def}}{=} \# \left\{ \alpha \in T_m : X_\alpha \geqslant \alpha m \right\}.$$

Then, neglecting factors which are only polynomial in m, we have

$$E(N_m(\lambda)) \sim 2^m \exp\left[-\frac{\lambda^2 m}{2}\right],$$

which goes rapidly to ∞ if $\lambda < \sqrt{2 \log 2}$. If we would know that

$$\sqrt{\operatorname{var}\left(N_m(\lambda)\right)} \ll E\left(N_m(\lambda)\right),\tag{3.12}$$

then we could apply the Tchebychev inequality, or sharper versions of it, and conclude that $N_m(\lambda)$ is large with large probability, and therefore $\max_{\alpha \in T_m} X_\alpha \ge \lambda m$ with large probability. The reader is invited to check that (3.12) is correct if λ is small enough, but that the opposite is true for λ sufficiently close to $\sqrt{2 \log 2}$, an effect which is due to the correlations of the field. In fact, if the X_α would be i.i.d., then we could argue in this way.

One way to proceed is to consider intermediate levels like in (3.8), and prove, by induction along the K levels, that the maximum is surpassing the appropriate heights corresponding to the different levels.

We will repeatedly use some of the standard exponential inequalities for independent random variables. A convenient one is the following:

Lemma 3.7. Let ξ_1, \ldots, ξ_n be *i.i.d.* real-valued random variables satisfying $E\xi_i = 0$, $\sigma^2 = E\xi_i^2$, $\|\xi_i\|_{\infty} \leq 1$. Then

$$P\left(\left|\sum_{i=1}^{n} \xi_{i}\right| \ge t\right) \le 2 \exp\left[-\frac{t^{2}}{2n\sigma^{2} + 2t/3}\right].$$

For a proof, see [2]

We first prove the following preliminary result:

Lemma 3.8. There exist $\delta > 0$, b > 0 such that

$$P(\#\{\alpha \in T_m : X_\alpha \ge 0\} \le 2^{\delta m}) \le e^{-bm^2}.$$

Proof. We use c as a generic positive constant, not necessarily the same at different occurrences. Also, all inequalities are required to hold only for large enough m. Without loss generality, we assume that m is even. If $\alpha \in T_{m/2}$, let $Y_{\alpha}^1 \stackrel{\text{def}}{=} X_{\alpha}$, $Y_{\alpha}^2 \stackrel{\text{def}}{=} \xi_{\alpha 0}^{m/2+1} + \xi_{\alpha 00}^{m/2+2} + \cdots + \xi_{\alpha 0\dots 0}^m$. Then

$$P(\#\{\alpha \in T_m : X_\alpha \ge 0\} \le 2^{m\delta}) \le P(\#\{\alpha \in T_{m/2} : Y_\alpha^1 + Y_\alpha^2 \ge 0\} \le 2^{m\delta})$$

Let $\Lambda \stackrel{\text{def}}{=} \{ \alpha \in T_{m/2} : Y^1_{\alpha} \ge -m/10 \}$ and define the event $A \stackrel{\text{def}}{=} \{ |\Lambda| \ge 2^{m/4} \}$. Then

$$P(A^c) \leqslant P(A^c, \max_{\alpha \in T_{m/2}} Y^1_{\alpha} \leqslant m^2) + P(\max_{\alpha \in T_{m/2}} Y^1_{\alpha} > m^2).$$
 (3.13)

Remark that

$$P(\max_{\alpha \in T_{m/2}} Y_{\alpha}^{1} > m^{2}) \leqslant e^{-cm^{3}}.$$
(3.14)

On $A^c \cap \{ \max_{\alpha} Y^1_{\alpha} \leq m^2 \}$ one has

$$\begin{split} 2^{-m/2} \sum_{\alpha \in T_{m/2}} Y_{\alpha}^1 &= 2^{-m/2} (\sum_{\alpha \in \Lambda} Y_{\alpha}^1 + \sum_{\alpha \notin \Lambda} Y_{\alpha}^1) \leqslant 2^{-m/2} (|\Lambda| m^2 - |\Lambda^c| \frac{m}{10}) \\ &\leqslant -\frac{m}{10} + 2^{-m/2} |\Lambda| (m^2 + \frac{m}{10}) \leqslant -\frac{m}{20}, \end{split}$$

as $|\Lambda| < 2^{m/4}$ on A^c . Using this in (3.13) and using (3.14), we get

$$P(A^c) \leqslant P(2^{-m/2} \sum_{\alpha \in T_{m/2}} Y_{\alpha}^1 \leqslant -\frac{m}{20}) + e^{-cm^3} \leqslant \exp(-cm^2),$$

by Lemma 3.7.

Let now $\mathcal{F}_1 \stackrel{\text{def}}{=} \sigma(Y^1_{\alpha} : \alpha \in T_{m/2})$. We then have

$$\begin{split} &P\left(\#\left\{\alpha\in T_{m/2}:Y_{\alpha}^{1}+Y_{\alpha}^{2}\geqslant 0\right\}\leqslant 2^{\delta m},A\right)\\ &=E\left(1_{A}P\left(\sum_{\alpha\in\Lambda}1_{\left\{Y_{\alpha}^{2}\geqslant-Y_{\alpha}^{1}\right\}}\leqslant 2^{m\delta}\middle|\,\mathcal{F}_{1}\right)\right)\\ &\leqslant E\left(1_{A}P\left(\sum_{\alpha\in\Lambda}1_{\left\{Y_{\alpha}^{2}\geqslant m/10\right\}}\leqslant 2^{m\delta}\middle|\,\mathcal{F}_{1}\right)\right)\\ &\leqslant P\left(\sum_{j=1}^{2^{m/4}}1_{\left\{Y_{j}\geqslant m/10\right\}}\leqslant 2^{m\delta}\right), \end{split}$$

where $Y_1, \ldots, Y_{2^{m/4}}$ are i.i.d. centered Gaussian random variables with variance m/2. Let $\eta_j \stackrel{\text{def}}{=} 1_{\{Y_j \ge m/10\}} - P(Y_j \ge m/10)$. Remark that $P(Y_j \ge m/10) \ge \exp[-m/90]$. Choosing δ small enough, we have

$$P\left(\sum_{j=1}^{2^{m/4}} \mathbb{1}_{\{Y_j \ge m/10\}} \le 2^{m\delta}\right) \le P\left(\left|\sum_{j=1}^{2^{m/4}} \eta_j\right| \ge \frac{2^{m/4}}{2} \exp\left[-m/90\right]\right)$$
$$\le \exp\left[-2^{cm}\right],$$

again by Lemma 3.7.

We will apply Lemma 3.8 not directly to m, but to a small proportion: Corollary 3.9. Let $\eta > 0$. Then there exist $\delta(\eta) > 0$ and $b(\eta) > 0$ such that

$$P\left(\#\left\{\alpha\in T_{\eta m}: X_{\alpha} \ge 0\right\} \le 2^{m\delta(\eta)}\right) \le \exp\left[-b(\eta)m^2\right].$$

Proof of Proposition 3.4 We fix $\eta > 0, K \in \mathbb{N}$, to be chosen later on (depending on β). We put $\eta_i \stackrel{\text{def}}{=} \eta + \frac{i}{K}(1-\eta), 0 \leq i \leq K$. Without further notice, we will assume that $\eta_i m \in \mathbb{N}$. Let

$$\Gamma_0 \stackrel{\mathrm{def}}{=} \{ \alpha \in T_{\eta m} : X_\alpha \ge 0 \}$$

$$\Gamma_{i} \stackrel{\text{def}}{=} \left\{ \alpha \in T_{\eta_{i}m} : [\alpha]_{\eta_{i-1}m} \in \Gamma_{i-1} , X_{\alpha} \geqslant \sqrt{2\log 2} \left(\frac{i}{K} - \frac{i}{K^{2}} \right) (1-\eta)m \right\}.$$

We fix some $\delta > 0$, which is at most equal to the $\delta(\eta)$ of Corollary 3.9, and put

$$A_i \stackrel{\text{def}}{=} \{ |\Gamma_i| \ge 2^{m\delta} \}.$$

We will now show that A_i has large probability. Before proceeding, a comment is in order why we define the events A_i in this way. Evidently, we are shooting for having $|\Gamma_K| \ge 1$ with large probability, and therefore it appears that we should rather and more simply consider the events $\{|\Gamma_i| \ge 1\}$. Trying to copy somehow the proof of (3.9) given above, we would like to prove that $P(|\Gamma_i| \ge 1|\mathcal{F}_{i-1})$ has large probability on $\{|\Gamma_{i-1}| \ge 1\}$, where $\mathcal{F}_i = \sigma(\xi_{\alpha_1...\alpha_j}^j : j \le \eta_i m)$. This would then do the job. However, it seems impossible to proceed in this way, because of the still present dependencies between the levels, which is exactly the reason why the direct second moment failed, as we have explained above. The trick to define the A_i as we do is to have on each level $i, 0 \le i \le K$, sufficiently many variables overshooting a certain bound, in order that the second moment method works to perform the induction from one level to the next.

Let us now proceed with the induction from level to level. We already know from Corollary 3.9 that

$$P(A_0) \ge 1 - \exp(-b(\eta)m^2).$$
 (3.15)

We claim that for arbitrary $\eta > 0$, and corresponding $\delta(\eta) > 0$, we may choose $K_0(\eta) \in \mathbb{N}$ such that for $K \ge K_0$, there exists $d_K > 0$ such that

$$P(A_i|\mathcal{F}_{i-1}) \ge 1 - \exp(-e^{d_K m}) \tag{3.16}$$

on A_{i-1} , $i \ge 1$. Clearly, (3.15) and (3.16) prove Proposition 3.4. In fact, depending on $\beta > 0$, we take $\eta = \beta/2$ and $K \ge K_0(\eta)$ such that $1/K < \beta/2$. Then

$$P(A_K) \ge 1 - \exp\left[-b(\eta)m^2\right] - K \exp\left[-e^{d_K m}\right] \ge 1 - \exp\left[-\frac{b(\eta)}{2}m^2\right],$$

and evidently

$$A_K \subset \left\{ \max_{\alpha \in T_m} X_\alpha \geqslant \sqrt{2\log 2} (1-\beta)m \right\}.$$

It therefore only remains to prove (3.16).

On A_{i-1} , we have at least $\mu \stackrel{\text{def}}{=} 2^{m\delta}$ parameters $\alpha \in T_{\eta_{i-1}m}$ such that

$$X_{\alpha} \geqslant \sqrt{2\log 2} \left(\frac{i-1}{K} - \frac{i-1}{K^2} \right) (1-\eta)m$$

We pick μ such α 's, call them $\alpha_1, \ldots, \alpha_{\mu}$. To each α_i there correspond 2^M , $M = \frac{1}{K}(1-\eta)m$, Gaussian variables $Z_{\alpha_i\alpha'_j}$, $1 \leq j \leq 2^M$, of variance M such that the variables X_{α} with $[\alpha]_{\eta_{i-1}m} \in \{\alpha_1, \ldots, \alpha_{\mu}\}$ are given by $X_{\alpha_i} + Z_{\alpha_i\alpha'_j}$, $1 \leq i \leq \mu$, $1 \leq j \leq 2^M$. It therefore suffices to estimate

$$P\left(\sum_{i=1}^{\mu}\sum_{j=1}^{2^{M}}1_{Z_{\alpha_{i}\alpha_{j}^{\prime}}\geqslant\sqrt{2\log 2}(1-\eta)\frac{K-1}{K^{2}}m}\leqslant 2^{m\delta}\right)$$
(3.17)

from above. Let

$$\eta_{ij} \stackrel{\text{def}}{=} 1_{Z_{\alpha_i \alpha'_j} \geqslant \sqrt{2 \log 2}(1-\eta) \frac{K-1}{K^2} m},$$

and remark, that for different *i*, the η_{ij} are independent. Put $\eta_i \stackrel{\text{def}}{=} \sum_{j=1}^{2^M} \eta_{ij}$. Then

$$E\left(\sum_{j=1}^{2^m} \eta_{ij}\right) = 2^{\frac{1-\eta}{K}m} P\left(Z_{ij} \ge \sqrt{2\log 2}(1-\eta)\frac{K-1}{K^2}m\right) \ge 2$$

if m is large enough. Therefore,

$$P\left(\sum_{i=1}^{\mu}\eta_{i}\leqslant 2^{m\delta}\right)\leqslant P\left(\sum_{i=1}^{\mu}\left(\eta_{i}-E\eta_{i}\right)\leqslant-2^{m\delta}\right)$$
$$\leqslant P\left(\left|\sum_{i=1}^{\mu}\left(\eta_{i}-E\eta_{i}\right)\right|\geqslant 2^{m\delta}\right).$$

Evidently, $\|\eta_i\|_{\infty} \leq 2^M = 2^{\frac{(1-\eta)}{K}m}$. Our requirement now is that $K \geq K_0(\beta)$, where $(1-\eta)/K_0 \leq \delta/3$. (Remember that η, δ are already chosen, depending on β .) Putting $\overline{\eta}_i \stackrel{\text{def}}{=} (\eta_i - E\eta_i)/\|\eta_i - E_{\eta_i}\|_{\infty}$, we get

$$P\left(\sum_{i=1}^{\mu}\eta_i \leqslant 2^{m\delta}\right) \leqslant P\left(\left|\sum_{i=1}^{\mu}\overline{\eta}_i\right| \geqslant 2^{2\delta m/3}\right) \leqslant \exp\left[-\frac{2^{m\delta/3}}{3}\right],$$

by Lemma 3.7. We therefore see that the expression in (3.17) is bounded above by $\exp\left[-2^{m\delta/3}/3\right]$, which proves (3.16). Therefore, Proposition 3.4 is proved.

4 Concluding remarks: Pinning and the wetting transition

The problem of entropic repulsion is closely related to the so-called wetting transition. Wetting appears for interfaces having also an attracting wall-to-surface interaction. This attraction is usually assumed to be very local: Only if the surface is very close to the wall, then it feels the attraction. There are several ways to define such a local attraction. The standard way in the physics literature is to change the Hamiltonian in (1.1), i.e., $\frac{1}{2} \sum_{\langle i,j \rangle \in \overline{A}} U(\phi_i - \phi_j)$, by adding some pinning potential, e.g. replace it by

$$H(\phi) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\langle i,j \rangle \in \overline{A}} U(\phi_i - \phi_j) + \sum_{i \in A} \psi(\phi_i), \tag{4.1}$$

where ψ is some function with values in \mathbb{R}^- , being nonzero only in a neighborhood of 0. If a hard wall is present as in Section 2, the function ψ has only to be defined on \mathbb{R}^+ . The wetting problem is arising when both types of influences are present, which lead to a competition between the entropic repulsion and the pinning. A transition occurs when at specific parameters there is a transition from a pinning dominated situation to a repulsion dominated one. In physics jargon, this is a transition from partial wetting to complete wetting.

It is however appropriate to discuss separately the pinning problem and leave out for the moment the hard wall condition. I therefore first describe what is known about the pinning phenomenon, in the absence of entropic repulsion. It is reasonable to assume that $\psi : \mathbb{R} \to \mathbb{R}^-$ is symmetric. A slightly different model which has been introduced in [7] is called δ -pinning. Here one modifies the definition (1.1) by defining

$$\hat{P}_{n,\varepsilon}(d\phi) \stackrel{\text{def}}{=} rac{1}{\hat{Z}_{n,\varepsilon}} \exp\left[-rac{1}{2}\sum_{\langle i,j
angle \in \overline{\Lambda}_n} U(\phi_i - \phi_j)
ight] \prod_{i \in \Lambda_n} (d\phi_i + \varepsilon \delta_0(d\phi_i)),$$

where ε is a positive parameter. This model is technically slightly easier than the pinning with the ψ -function, which we call the "bump pinning" case. However, all the results which have been proved for the δ -pinning case can also be derived (with some technical complications) for the bump pinning one. A remarkable fact is that, in any dimension, and for arbitrarily weak pinning, the field gets localized in a strong sense, namely

a)

$$\sup_{n} \int |\phi_0| \, \hat{P}_{n,\varepsilon}(d\phi) < 0$$

b)

$$\sup_{n} \int \phi_{i} \phi_{j} \hat{P}_{n,\varepsilon}(d\phi) \leqslant C \exp\left[-m \left|i-j\right|\right],$$

for some positive C, m (which depend on ε).

This has been proved in [13] for the harmonic case in dimension $d \ge 3$ (for bump pinning) using reflection positivity. The drawback of this method is that it works

only with periodic boundary conditions and the formulation actually needs some modification. In the two-dimensional case, a) has been proved in [22] (also for the Gaussian case). The proof however could not be used for b). Both a) and b) have been proved in [8] by a modification of the arguments of [13], again with the usual restrictions when applying reflection positivity. A full and satisfactory treatment has been given in [21] and [25] covering also the non-Gaussian case (with U uniformly convex). The dependence for instance of $\sup_n \int |\phi_0| \hat{P}_{n,\varepsilon}(d\phi)$ on ε for small ε , or of the mass m in b) is quite an interesting problem. In [22] a bound of order $\sqrt{\log(1/\varepsilon)}$ (for d = 2) has been obtained. The correct dependence however is of order $\log(1/\varepsilon)$, as comes out of the analysis in [8] and [25].

The wetting phenomenon is appearing when one considers both pinning and a hard wall condition. Therefore we consider

$$\hat{P}_{n,\varepsilon}^+(d\phi) \stackrel{\text{def}}{=} \hat{P}_{n,\varepsilon}(d\phi|\Omega_n^+).$$

A natural question is which of the interaction effects is dominating the overall behavior of the random field. It is not difficult to see that, for large ε , the model is "pinning dominated". This is quite easy and has been proved in [7], but actually only in a somewhat weaker formulation, namely by a pressure estimate. To be precise, we consider the pressure

$$p^{+}(\varepsilon) \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{1}{|\Lambda_{n}|} \log \frac{\hat{Z}_{n,\varepsilon}\hat{P}_{n,\varepsilon}(\Omega_{n}^{+})}{Z_{n}P_{n}(\Omega_{n}^{+})}.$$

The existence of the limit is easily established (see [7]), and also that $p^+(\varepsilon) \ge 0$. Naturally, $p^+(\varepsilon) > 0$ would mean that the model is pinning dominated, whereas $p^+(\varepsilon) = 0$ essentially would mean that the pinning has no effect (at least not to leading order). It is natural to expect that in the former case one would have strong localization properties like a) and b) above, but this has not been proved and is probably quite difficult. On the other hand, one would expect that if $p^+(\varepsilon) = 0$, then the model would behave essentially in the same way as without pinning, but again, this has not been proved. Clearly, $p^+(\varepsilon)$ is monotone in ε , and $p^+(0)$ is 0. The question of a wetting transition (in a weak sense) is whether or not there is a positive interval of ε -values where $p^+ = 0$. The following facts are known:

- a) In any dimension, $p^+(\varepsilon) > 0$ for large enough ε . This has been proved in [7] for the Gaussian case and d = 2, but the argument works easily also in the other cases.
- b) The existence of a wetting transition is easy for d = 1. This has been proved for a discrete random walk case by M. Fisher [22], but it is easy also for the gradient models considered here (see [5]). The existence of a wetting transition is also known for the Ising model in d = 2, i.e., where the interface is one-dimensional.
- c) Recently, it has been proved that the Gaussian model in $d \ge 3$ does not have a wetting transition, i.e., that $p^+(\varepsilon) > 0$ for all ε . This is true both for δ -pinning and for bump pinning (see [5]).

d) It has recently been shown by Caputo and Velenik [14] that the Gaussian model in d = 2 has a wetting transition. Even more surprisingly in the light of c) above, other slightly modified models have a wetting transition in all dimensions. The case discussed in [14] is the continuous SOS-model, where U(x) = |x|. It is quite remarkable that the physics of the model is depending so strongly on the special details of the interaction. The proof given by Caputo and Velenik uses a simple but very clever argument of Chalker [15].

In the physics literature, slightly different models have been considered. The one having attracted most attention is a Gaussian gradient model of the type considered here, but not with a hard wall condition. Instead, the measure has a potential ψ and a Hamiltonian as in (4.1), where $\psi : \mathbb{R} \to \mathbb{R}$ is a mixture of two exponentials:

$$\psi(x) = -\varepsilon \exp\left[-x/\xi\right] + \exp\left[-2x/\xi\right],\tag{4.2}$$

where $\varepsilon, \xi > 0$ are parameters. The most interesting questions are connected with the behavior for ε close to 0. Remark that $\psi(x) \to \infty$ rapidly as $x \to -\infty$, so this is acting like a somewhat softer wall. On the other hand, there is a negative part of the ψ -function near 0, whose depth and width depends on the parameter ε . In fact, the width is of order $\log(1/\varepsilon)$ and the depth is of order ε for $\varepsilon \sim 0$. For $x \to \infty$, $\psi(x)$ is evidently going to 0. There is a paper by Lemberger [28] on this model, but the most interesting problems, especially the ones concerning the small ε behavior, are completely open. From non-rigorous renormalization group considerations, one expects a rich behavior of this model for $\varepsilon \sim 0$ for different ranges of the ξ parameter (see e.g. [10], [11]).

Let me sketch the background and motivation for the consideration of this "double exponential" model. I am indepted to François Dunlop who introduced me to this topic. In physically more realistic models, like the Ising model, there is still considerable uncertainty about the nature of the wetting transition, even with non-rigorous methods, and a solution is completely beyond reach of a mathematically rigorous treatment. I give a short discussion of the situation for the Ising model. Consider the three-dimensional Ising model, where interfaces are two-dimensional. Let us consider the Ising model, below the critical temperature T_c , and defined on a cubic box, say $B_n \stackrel{\text{def}}{=} \{-n, -n+1, \dots, n\}^3$. Taking minus boundary conditions on the lower half $x \in \partial B_n$, $x_1 \leq 0$, and plus boundary conditions on the upper half, introduces a two-dimensional random interface, which however may have "overhangs". It is known (by works of Dobrushin) that this interface remains stiff if the temperature is small enough, i.e., the fluctuations stay of order one. However, it is believed, but not proved rigorously, that there is a so-called roughening transition at some temperature $T_r < T_c$, meaning that for temperatures between these two critical values, the interface has fluctuations which grow logarithmically in n, whereas only below T_r they stay of order one. This is certainly one of the most prominent and challenging open problems in rigorous statistical mechanics, but physicists appear to be confident. Let us now modify the boundary condition slightly, taking the minus boundary condition only at the bottom of the box, while the rest of the boundary has plus boundary condition. The interface is then repelled by the bottom through the entropic repulsion

effect. Let us now still modify the model by making it energetically advantageous for the interface to stick to the bottom layer. This can be done by introducing an external field on the boundary layer at the bottom just inside the box. If we have a minus boundary condition on the bottom layer then this external field should favor plus spins. It is believed that such a local attraction of the interface to the bottom layer leads still to another transition temperature $T_w \in (T_r, T_c)$, the so-called wetting transition temperature. For temperatures below, the interface is believed to stick to the wall in the sense that the deviation from the bottom layer is of order one, despite the fact that the temperature is above the roughening transition. This is the region of "partial wetting". On the other hand, for temperatures above T_w , the entropic repulsion should win, and the interface moves away from the boundary, i.e., one has "complete wetting". Many of the problems about the precise nature of this transition are however completely unclear. A purely phenomenological theory has led for temperature $T < T_w$, but very close to, to a Gaussian approximation leading to the above Gaussian gradient model with ψ given by (4.2). However, the derivation of this approximation is very far from rigorous (saying nothing about the existence of T_r and T_w for the Ising model which of course has to be supposed in order to make sense of this approximation). The questions around this wetting transition are quite fascinating and to a large extent completely unsolved.

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