PATH DISTRIBUTIONS ON SEQUENCE SPACES

ERIK G.F. THOMAS

Abstract

We define the rank of a summable distribution and show that finite dimensional Fresnel distributions e^{iS} are summable distributions of rank 2, independently of the dimension of the space. This motivates the introduction of path distributions, which we discuss in the case of spaces of sequences (discrete time paths). For a continuous quadratic form $\langle K\xi, \xi\rangle$ on a nuclear sequence space E there exists a unique path distribution F_K on the dual sequence space E' such that

$$\int_{E'} e^{-i\langle x,\xi\rangle} F_K(dx) = e^{i\langle K\xi,\xi\rangle/2}, \quad \xi \in E.$$

If $K(i,j) = \min(i,j)$ this gives a discrete-time analogue of a Feynman integral.

1 Introduction

One of the most interesting problems in functional analysis is the appropriate mathematical definition of the Feynman path integral and related objects. Although it was not at the time (1948) obvious, the Feynman path integral [11] is an analogue of the Wiener measure on the space $C[0,T]$ of continuous functions, in which the Gaussian marginals $G_{t_1,\ldots,t_n}$ of the Wiener measure for the time instances $0 < t_1 < t_2 < \ldots, t_n \leqslant T$ are replaced by the Fresnel distributions $F_{t_1,\ldots,t_n} = G_{it_1,\ldots,it_n}$. The relation between the Wiener measure and the diffusion equation thus becomes a relation between the Feynman integral and the Schrödinger equation. In spite of considerable work on the matter, such as among others [1], [5], [7], [8], [4], [18], [20], [21], [22], [34], the problem remains to define the projective limit F of the marginal distributions $F_{t_1,\ldots,t_n}$ in such a way that $F(\Phi)$ makes sense for an appropriate linear topological vector space of functions Φ, containing the functions such as $\Phi(x) = \exp(-i\int_t^{t'} V(x(s))ds)$ at least for all smooth potentials V with bounded derivatives.

The finite dimensional Fresnel distributions are summable distributions in the sense of Schwartz [25], [26], [27]. These can be described as linear functionals, having a certain continuity property, on the space $\mathcal{B}$ of C^∞ functions which are bounded as well as all their derivatives. By a representation theorem of Schwartz they can also be characterized as sums of derivatives of bounded measures. The continuity implies that each summable distribution T on $\mathbb{R}^n$ has what we call its sum-order. This is the smallest integer m such that one has an estimate $|\langle T, \varphi\rangle| \leqslant M \sum_{|k|\leqslant m} ||D^k\varphi||_\infty$, where $|k| = k_1 + \cdots + k_m$. It can be shown that the sum-order of $F_{t_1,\ldots,t_n}$ is exactly $n+1$.

Summable distributions can also be defined on Banach spaces. A natural idea is to attempt to define F as a summable distribution on $C[0,T]$. But this cannot be, because the sum-order of the marginals would be majorized by the sum-order of F, whereas we see that they are unbounded. This precise obstacle leads to a new theory of path integrals, the first elements of which are proposed below.

For each multi-index $k = (k_1, \ldots, k_n)$ consider the maximum $|k|_\infty = \max_i k_i$. Then, since $|k|_\infty \leqslant |k| \leqslant n|k|_\infty$ each summable distribution on $\mathbb{R}^n$ has a finite rank, the rank being smallest integer m such that we have an estimate $|\langle T, \varphi\rangle| \leqslant M \sum_{|k|_\infty \leqslant m} ||D^k \varphi||_\infty$. The main motivating result (Theorem 5.1) on which the present paper is based is the following: in any dimension n the rank of $F_{t_1,\ldots,t_n}$ equals 2. More generally this is the case for any Fresnel distribution, i.e., a distribution whose Fourier transform is of the form e^{iK} where K is a real quadratic form.

The concept of rank is not invariant under linear transformations. For instance on $\mathbb{R}^2$ the operator $\frac{\partial^2}{\partial x \partial y}$ has rank 1 but is equivalent under a rotation to the wave operator which has rank 2. This means that in this context the path structure of $\mathbb{R}^n$ is important, and not just the linear structure or even the Euclidean structure.

We propose that path integrals, on general linear spaces of paths, are analogous to summable distributions, in as much as they will turn out to be objects representable in the form $D\mu$, with D an appropriate finite rank differential operator, but in general of infinite order, and μ a bounded Radon measure (of the type described in the work [28]).

In the present lecture we restrict attention to the case where time is a countable discrete set. We consider general spaces of paths, i.e. sequence spaces, but for simplicity we assume these to be nuclear or co-nuclear. No knowledge of nuclear spaces is required however, nuclear sequence spaces having a particularly simple direct definition. Examples are the space s of sequences rapidly going to zero, and the space s' of sequences of polynomial growth.

The purpose of this lecture is: 1. To discuss summable distributions on $\mathbb{R}^n$, i. following L. Schwartz, and ii. regarding $\mathbb{R}^n$ as the simplest discrete time path space. 2. To introduce path distributions on sequence spaces. 3. To prove the following theorem: If $\xi \mapsto \langle K\xi, \xi\rangle$ is a continuous quadratic form on a nuclear sequence space[1], then there exists a unique path-distribution on the dual space whose Fourier transform equals $e^{i\langle K\xi,\xi\rangle/2}$. Moreover this path distribution is a derivative of rank 2 of a bounded Radon measure. In particular, if $I = \mathbb{Z}_+$ and $K(i,j) = \min(i,j)$ and $\langle K\xi, \xi\rangle = \sum_{i\geqslant 0, j\geqslant 0} K(i,j)\xi_i\xi_j$ for $\xi \in s(\mathbb{Z}_+)$, then there is a unique path distribution on $s'(\mathbb{Z}_+)$ having the Fourier transform $e^{i\langle K\xi,\xi\rangle/2}$. This may be regarded as a discrete-time analogue of a Feynman integral. It seems Feynman himself has made use of discrete time path integrals when calculating path integrals using Fourier series [12] §3-11.

[1] assumed complete and assumed barreled, i.e., the uniform boundednes principle is valid: If ℓ_i, $i \in I$, are continuous linear forms such that $\sup_{i\in I} |\ell_i(\xi))| < +\infty$, for all $\xi \in E$, then they are equicontinuous.

2 Summable distributions on $\mathbb{R}^n$

Since one purpose of this paper is to define path distributions on sequence spaces we must first of all answer the question: precisely what are path distributions on the space of finite sequences $\mathbb{R}^n$. The answer we propose is that they are summable distributions in the sense of Schwartz [25], [26], [27].

In this section we essentially summarize Schwartz's theory of summable distributions, without proofs. We adopt the standard notations from distribution theory, such as $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$, $\mathcal{E}^{(m)}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$, $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ for the space of distributions. If φ is a function which is bounded as well as its derivatives up to order $m \in \mathbb{Z}_+$ we pose

$$p_m(\varphi) = \sup_{|k|\leqslant m} ||D^k\varphi||_\infty \tag{2.1}$$

where $|k| = k_1 + \cdots + k_n$ is the order of the operator $D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}$, and $D^0\varphi = \varphi$.

A summable distribution on $\mathbb{R}^n$ is a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ such that there exists an integer $m \geqslant 0$ and a number M such that

$$|\langle T, \varphi\rangle| \leqslant M p_m(\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \tag{2.2}$$

The smallest possible number m for which one has estimates such as (2.2) will be called the *summability order* of T or briefly the 'sum-order'.

The summable distributions form a linear subspace of $\mathcal{D}'(\mathbb{R}^n)$ which we denote by $\mathcal{D}'_L(\mathbb{R}^n)$. The summable distributions of sum-order $\leqslant m$ form a subspace of $\mathcal{D}'_L(\mathbb{R}^n)$ which we denote as $\mathcal{D}_L^{'(m)}(\mathbb{R}^n)$. (In Schwartz' work [25], [26], [27] $\mathcal{D}'_L$ is denoted as $\mathcal{D}'_{L^1}$ and is a particular case of the spaces $\mathcal{D}'_{L^p}$).

A summable distribution is of finite order in the usual sense, and we have

$$\text{order}(T) \leqslant \text{sum-order}\,(T), \tag{2.3}$$

with equality if T has compact support.

It immediately follows from the definition that a derivative of a summable distribution is summable with

$$\text{sum-order}\,\Big(\frac{\partial T}{\partial x_i}\Big) \leqslant \text{sum-order}\,(T) + 1 \tag{2.4}$$

The space $\mathcal{D}_L^{'(0)}$ of summable distributions of sum-order 0 coincides with the space $\mathcal{M}_b(\mathbb{R}^n)$ of bounded Radon measures on $\mathbb{R}^n$. If $\mu \in \mathcal{M}_b$ the distribution $D^k\mu$ is summable with sum-order $\leqslant |k| = k_1 + \cdots + k_n$.

Theorem 2.1. (L. Schwartz [25]). *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. Then the following conditions on T are equivalent:*

1. *T belongs to $\mathcal{D}'_L(\mathbb{R}^n)$.*

2. *T is a finite sum $\sum_{|k|\leqslant m} D^k\mu_k$ of derivatives of bounded measures μ_k.*

3. T is a finite sum $\sum_{|k|\leqslant m} D^k f_k$ of derivatives of L^1 functions f_k.

4. For every $\alpha \in \mathcal{D}$, $\alpha * T$ belongs to $\mathcal{M}_b(\mathbb{R}^n)$.

5. For every $\alpha \in \mathcal{D}$, $\alpha * T$ belongs to $L^1(\mathbb{R}^n)$.

The representation

$$T = \sum_{|k|\leqslant m} D^k \mu_k \tag{2.5}$$

where $\mu_k \in \mathcal{M}_b(\mathbb{R}^n)$, allows one to extend T to the space $\mathcal{B}^{(m)}(\mathbb{R}^n)$ of functions in $\mathcal{E}^{(m)}(\mathbb{R}^n)$ which are bounded as well as their derivatives up to order m, in particular to the space $\mathcal{B} = \mathcal{B}^{(\infty)} = \cap_m \mathcal{B}^{(m)}$, by the formula

$$\langle T, \varphi\rangle = \sum_{|k|\leqslant m} (-1)^{|k|} \langle \mu_k, D^k \varphi\rangle. \tag{2.6}$$

Measures $\mu \in \mathcal{M}_b$ being up to $\varepsilon > 0$ concentrated on a compact set, it follows that the extension has the following '*bounded convergence property of order m*': If φ_i is bounded in $\mathcal{B}^{(m)}$, i.e. $\sup_i p_m(\varphi_i) < +\infty$, and $\varphi_i \to \varphi$ in the space $\mathcal{E}^{(m)}$, then $\langle T, \varphi_i\rangle \to \langle T, \varphi\rangle$. One describes this kind of convergence by saying that φ_i pseudo-converges to φ in $\mathcal{B}^{(m)}$.

Similarly the map $T : \mathcal{B}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ has the 'bounded convergence property' (without specified order): i.e. the restriction of T to bounded subsets of $\mathcal{B}(\mathbb{R}^n)$ is continuous for the topology induced by $\mathcal{E}(\mathbb{R}^n)$.

In particular, if (φ_n) is a sequence in $\mathcal{D}$ pseudo-converging to $\varphi \in \mathcal{B}$ (e.g . $\varphi_n = \alpha_n \varphi$ where $\alpha_n(x) = \alpha(x/n)$, α being a test function equal to 1 in a neighborhood of 0) then $\langle T, \varphi\rangle = \lim_{n\to\infty} \langle T, \varphi_n\rangle$. This shows that the extension of T to $\mathcal{B}^{(m)}(\mathbb{R}^n)$ or $\mathcal{B}(\mathbb{R}^n)$ does not depend on the representation (2.5). We call it the canonical extension. In particular one can define the total mass

$$\langle T, 1\rangle \tag{2.7}$$

sometimes denoted by $\int T(dx)$ which accounts for the name 'summable distribution'.

It can be shown that conversely a linear map $T : \mathcal{B}(\mathbb{R}^n) \longrightarrow \mathbb{C}$, having the bounded convergence property, is the canonical extension of a unique summable distribution. This allows the introduction of a second equivalent definition of summable distributions : a summable distribution (of order m) is a linear form $T : \mathcal{B} \longrightarrow \mathbb{C}$ (resp. $\mathcal{B}^{(m)} \longrightarrow \mathbb{C}$) having the bounded convergence property (of order m).

This allows one to define several operations on summable distributions which are current for bounded measures:

Image distributions

Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a linear map. Then one defines the image of T under u by

$$\langle u(T), \varphi\rangle = \langle T, \varphi \circ u\rangle, \quad \varphi \in \mathcal{B}(\mathbb{R}^k) \tag{2.8}$$

Then

$$\text{sum-order}\,(u(T)) \leqslant \text{sum-order}\,(T) \tag{2.9}$$

Fourier transform

$$\mathcal{F}(T)(\xi) = \langle T, e_{-\xi}\rangle, \quad e_\xi(x) = e^{i\langle x,\xi\rangle} \tag{2.10}$$

$\mathcal{F}(T)$ is a continuous function of at most polynomial growth.

Direct products

The direct product of $T \in \mathcal{D}'_L(\mathbb{R}^p)$ and $S \in \mathcal{D}'_L(\mathbb{R}^q)$ is the distribution $T \otimes S \in \mathcal{D}'_L(\mathbb{R}^{p+q})$ defined for $\Phi \in \mathcal{B}(\mathbb{R}^p \times \mathbb{R}^q)$ by

$$\langle T \otimes S, \Phi\rangle = \langle T, \theta\rangle \tag{2.11}$$

where $\theta(x) = \langle S, \Phi_x\rangle$, with $\Phi_x(y) = \Phi(x,y)$. It is characterized by the relations

$$\langle T \otimes S, \varphi \otimes \psi\rangle = \langle T, \varphi\rangle\langle S, \psi\rangle \tag{2.12}$$

Convolution

Let $T, S \in \mathcal{D}'_L(\mathbb{R}^n)$. The convolution product $T * S$ is the image of $T \otimes S$ under the linear map $(x,y) \mapsto x+y$. Thus, for $\Phi \in \mathcal{B}(\mathbb{R}^n)$

$$\langle T * S, \Phi\rangle = \langle T \otimes S, \Psi\rangle \tag{2.13}$$

where $\Psi(x,y) = \Phi(x+y)$. Since $e_\xi(x+y) = e_\xi(x)e_\xi(y)$ one has, by formula 2.11

$$\mathcal{F}(T * S) = \mathcal{F}(T)\mathcal{F}(S) \tag{2.14}$$

The class $\mathcal{F}(\mathcal{O}_M)$.
In this section we identify a particularly useful class of summable distributions.

Let $\mathcal{O}_M$ be the space of functions $f \in \mathcal{E}(\mathbb{R}^n)$ such that f and the derivatives of f have at most polynomial growth. Then $\mathcal{O}_M$ operates by multiplication on the space $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$. Moreover, the functions in $\mathcal{O}_M$, having polynomial growth, themselves define temperate distributions.

Theorem 2.2. *Every $T \in \mathcal{F}(\mathcal{O}_M)$ is summable. More precisely, if P is a polynomial we have $PT \in \mathcal{D}'_L(\mathbb{R}^n)$. Conversely, if PT belongs to $\mathcal{D}'_L(\mathbb{R}^n)$ for all polynomials P, then we have $T \in \mathcal{F}(\mathcal{O}_M)$.*

Proof. If $T = \mathcal{F}(f)$ and $\alpha \in \mathcal{D}(\mathbb{R}^n)$ we have $\alpha * T = \mathcal{F}(\beta f)$ where $\beta \in \mathcal{S}$ is the inverse Fourier transform of α. Since βf belongs to $\mathcal{S}$ we have $\alpha * T \in \mathcal{S}$, a fortiori $\alpha * T \in L^1$. Therefore by Theorem 2.1 T is summable. Similarly if P is a polynomial $PT = \mathcal{F}(Df)$ for some differential operator D with constant coefficients, so $PT \in \mathcal{F}(\mathcal{O}_M)$. Conversely, if PT is summable for all polynomials P, $D\widehat{T}$ is continuous of polynomial growth for all differential operators D, and so $\mathcal{F}(T)$ belongs to $\mathcal{O}_M$. ■

3 Fresnel distributions

Let $S(x) = \frac{1}{2}(Ax, x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$, associated to the real symmetric invertible matrix A. Then we have the following:

Theorem 3.1 (Bijma, Thomas). *The Fresnel distribution e^{iQ} is summable and has summability order $n+1$. Moreover, for any polynomial P the product Pe^{iQ} is summable.*[2]

Proof. It is well known that the Fourier transform is proportional to $e^{ic(A^{-1}\xi,\xi)}$ for some $c \in \mathbb{R}$ and so belongs to the space $\mathcal{O}_M$. Thus Pe^{iQ} is summable for all P. Regarding the sum-order we only sketch the proof in the case of interest here, where A is positive definite.[3] The sum-order being invariant under linear transformations, we may assume $Q(x) = |x|^2$ is the square of the euclidean norm. In the case of dimension 1 it is easy to see by a direct calculation (partial integration) that e^{ir^2} has sum-order 2, and that $r^{n-1}e^{ir^2}$ has sum-order $n+1$. Using radial test functions and averaging we can then reduce $e^{i|x|^2}$ to $r^{n-1}e^{ir^2}$. ∎

The precise sum-order $n+1$, or rather the fact that the sum-order is an unbounded function of the dimension, is important for our motivation in the sequel, but will not be used otherwise.

We are particularly interested in the following Fresnel distribution:

Let $\sigma = \{t_1, \dots, t_n\}$ be such that $0 < t_1 < \cdots < t_n \leqslant T$. Let Let $\sigma = \{t_1, \dots, t_n\}$ be such that $0 < t_1 < \cdots < t_n \leqslant T$. Let

$$F_\sigma = \exp i\Big(\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})} + \cdots + \frac{(x_2 - x_1)^2}{2(t_2 - t_1)} + \frac{x_1^2}{2t_1}\Big)\frac{dx_n \dots dx_1}{\sqrt{2\pi i(t_n - t_{n-1}) \dots 2\pi i t_1}} \tag{3.1}$$

It follows that this is a summable distribution on $\mathbb{R}^\sigma = \mathbb{R}^n$ of sum-order $n+1$.

These distributions form a projective system:[4] if $\sigma \leqslant \sigma'$ (i.e. $\sigma \subset \sigma'$) then if $\pi_{\sigma,\sigma'}$ is the projection of $\mathbb{R}^{\sigma'}$ onto $\mathbb{R}^\sigma$ we have

$$F_\sigma = \pi_{\sigma,\sigma'}(F_{\sigma'}) \tag{3.2}$$

The Fourier transform of F_σ is $\widehat{F}_\sigma(\xi) = e^{i\langle K_\sigma \xi, \xi\rangle/2}$ where $\langle K_\sigma \xi, \xi\rangle = \sum_{1 \leqslant i,j \leqslant n} K(t_i, t_j)\xi_i \xi_j$, with $K(s,t) = \min(s,t)$.

4 Impossibility of defining Feynman integral as a summable distribution

The second definition of summable distributions lends itself to a generalization in which the space $\mathbb{R}^n$ is replaced by an infinite dimensional Banach space E, the space

[2] The fact that $e^{i\pi x^2}$ is summable is mentioned in [25] p. 271.

[3] Details for the general case can be found in F. Bijma's undergraduate thesis [3].

[4] Such projective, or compatible, systems of summable distributions are closely related to the pro-distributions of C. DeWitt-Morette [7].

$\mathcal{B}(E)$ having a natural definition and the structure of a Fréchet space with the norms

$$P_m(\Phi) = \sup_{||h_i|| \leqslant 1, i=1\dots,m} ||D_{h_1,\dots,h_m}\Phi||_\infty \tag{4.1}$$

$D_{h_1,\dots,h_m}$ denoting successive differentiation in the directions h_i, or if $m = 0$ the identity.

A summable distribution is by definition a linear form $T : \mathcal{B}(E) \longrightarrow \mathbb{C}$ with the bounded convergence property, i.e., whose restriction to the bounded subsets of $\mathcal{B}(E)$ is continuous with respect to the C^∞-topology. A generalization of this type has been considered by L. Schwartz (private communication). Summable distributions on locally convex spaces have been examined in detail by E. Cator in his doctoral dissertation [6].

In the case of the Banach space $E = C[0,T]$ let us denote $\pi_\sigma : C[0,T] \longrightarrow \mathbb{R}^\sigma$ the evaluation map $\pi_\sigma(x) = (x(t_1), \dots, x(t_n))$.

A natural question is whether the Feynman integral can be defined as a summable distribution on $C[0,T]$. More precisely the question is: Does there exist a summable distribution F on $C[0,T]$ such that we have

$$\pi_\sigma(F) = F_\sigma \qquad \forall\, \sigma \tag{4.2}$$

If such a distribution exists it is unique. But the answer to the question is:

Theorem 4.1. *There does not exist a summable distribution F on $C[0,T]$ such that we have* (4.2).

Proof. A summable distribution is continuous on the Fréchet space $\mathcal{B}(C[0,T])$ and so we have an estimate similar to (2.2):

$$|\langle F, \Phi\rangle| \leqslant M P_m(\Phi) \tag{4.3}$$

In particular this would imply that the images F_σ have summability order at most equal to m, which is contrary to the fact that the sum-orders of the distributions F_σ are unbounded. ∎

More generally one can show that it is in general impossible to define Fresnel distributions on infinite dimensional spaces as summable distributions. This obstacle motivates the theory of the next sections.

5 Summable distributions on $\mathbb{R}^n$ viewed as path-distributions

5.1 Rank or Max-order

For every multi-index $k = (k_1, \dots, k_n)$ we pose

$$|k|_\infty = \max_{i=1\dots n} k_i \tag{5.1}$$

A partial differential operator with constant coefficients is said to have rank $\leqslant m$ if it is of the form

$$D = \sum_{|k|_\infty \leqslant m} a_k D^k \tag{5.2}$$

We put

$$p_{(m)}(\varphi) = \sup_{|k|_\infty \leqslant m} ||D^k \varphi||_\infty \tag{5.3}$$

A summable distribution T will be said to have rank $\leqslant m$ if we have

$$|\langle T, \varphi\rangle| \leqslant M p_{(m)}(\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \tag{5.4}$$

Since $|k|_\infty \leqslant |k| \leqslant n|k|_\infty$ the distributions of finite rank on $\mathbb{R}^n$ are identical to the summable distributions.[5]

We denote $\mathcal{E}_{(m)}(\mathbb{R}^n)$ the space of functions φ such that $D^k\varphi$ is continuous for all k with $|k|_\infty \leqslant m$, and $\mathcal{B}_{(m)}(\mathbb{R}^n)$ the space of function $\varphi \in \mathcal{E}_{(m)}$ such that $D^k\varphi$ is bounded for all k with $|k|_\infty \leqslant m$.

Theorem 5.1. *Let $S(x) = \frac{1}{2}(Ax, x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$, associated to the real symmetric invertible matrix A. Then the Fresnel distribution $E = e^{iS}$ has rank 2.*

Proof. Let

$$y_k = A_k(x) = \sum_{\ell=1}^{n} a_{k\ell} x_\ell. \tag{5.5}$$

Then $\dfrac{\partial}{\partial x_k} S(x) = A_k(x)$. Thus $\dfrac{\partial}{\partial x_k} E = iA_k E$ and $(\dfrac{\partial}{\partial x_k})^2 E = (ia_{kk} - A_k^2)E$. Put $D_k = 1 - (\frac{\partial}{\partial x_k})^2$ and $P_k = 1 + A_k^2 - ia_{kk}$. Then we have $D_k E = P_k E$ where the polynomial P_k is nowhere zero on $\mathbb{R}^n$. Therefore we can write:

$$E = \frac{1}{P_k} D_k E. \tag{5.6}$$

Repeating this for $k = 1$ to n we obtain:

$$E = \frac{1}{P_1} D_1 \dots \frac{1}{P_n} D_n E \quad \text{or} \quad E = \frac{1}{P_n} D_n \dots \frac{1}{P_1} D_1 E \tag{5.7}$$

Applying this to a test function and transposing we obtain:

$$\langle E, \varphi\rangle = \langle E, D_n(\frac{1}{P_n} \dots D_1(\frac{1}{P_1}\varphi)\rangle \tag{5.8}$$

[5] In the first version of this paper we used the expression 'max-order' instead of 'rank'. But according to L. Schwartz ([25] p. 189) the term 'rang', the French for 'rank', is an acceptable term for the maximum of the k_i in the differential operator D^k. The use of 'rank' as an attribute of a distribution is new.

The functions $\frac{1}{P_k}$ and their derivatives with respect to y_k are integrable over $\mathbb{R}$, and so by the chain rule, the derivatives with respect to the x_j are, as functions of y_k integrable over $\mathbb{R}$. e.g.

$$\frac{\partial}{\partial x_j}\frac{1}{1+A_k^2} = -\frac{2a_{kj}A_k}{(1+A_k^2)^2} = -\frac{2a_{kj}y_k}{(1+y_k^2)^2} \tag{5.9}$$

Thus we have:

$$\langle E,\varphi\rangle = \sum_{k=1}^{N}\int E(x)F_k(x)L_k\varphi(x)dx \tag{5.10}$$

where the F_k are in $L^1(\mathbb{R}^n)$, direct products of integrable functions of $y_1, y_2, \ldots, y_n$, and so integrable with respect to $y = Ax$, and A being invertible, integrable with respect to x. The L_k are differential operators of rank $\leqslant 2$. It follows that

$$|\langle E,\varphi\rangle| \leqslant \sum ||F_k||_1||L_k\varphi||_\infty \leqslant M\sum ||L_k\varphi||_\infty \tag{5.11}$$

which proves that the rank is $\leqslant 2$. Projection on some axis shows that the rank cannot be smaller than 2. ∎

Remark 5.2. Theorem 8.1 shows that more generally, if E has a Fourier transform of the form e^{iK}, with K a quadratic form $\neq 0$, the distribution E has rank 2.

5.2 Mollifiers

We define a *mollifier* of order m to be a measure $K \in \mathcal{M}_b$ having the following two properties:

1. $D^\ell K$ is a bounded measure for all indices ℓ with $|\ell|_\infty \leqslant m$.
2. There exists a differential operator D of rank m such that

$$DK = \delta \tag{5.12}$$

Theorem 5.3. *There exist mollifiers of order m. Let K be a mollifier of order m and let D be a differential operator of rank $\leqslant m$ such that $DK = \delta$. Then if T is any summable distribution of rank $\leqslant m$, the summable distribution $\mu = K * T$ is a bounded measure and*

$$T = D\mu \tag{5.13}$$

Thus, for summable distributions of rank $\leqslant m$ we get, without the use of Hahn-Banach theorem, a representation

$$T = \sum_{|k|_\infty \leqslant m} D^k\mu_k \tag{5.14}$$

analogous to (2.5), and actually something more precise.

In particular, a summable distribution of rank $\leqslant m$ has a canonical extension to the space $\mathcal{B}_{(m)}$, having the bounded convergence property of rank m: If $\varphi_i \to \varphi$ in the space $\mathcal{E}_{(m)}(\mathbb{R}^n)$ while remaining bounded in $\mathcal{B}_{(m)}(\mathbb{R}^n)$ then $\langle T, \varphi_i\rangle \to \langle T, \varphi\rangle$. This follows in the same way as in §2.

Conversely, if T has a representation (5.14) it follows that T has rank $\leqslant m$.

Proof. For order one we pose, if $\alpha > 0$

$$L = L_\alpha = \frac{1}{\alpha} Y e^{-x/\alpha} \tag{5.15}$$

(Y the Heaviside one step function) and

$$D = 1 + \alpha \frac{d}{dx} \tag{5.16}$$

so that

$$DL = \delta \tag{5.17}$$

and L and L' are bounded measures.

Next consider the case of rank 2, in dimension $m = 1$. For $\alpha > 0$ let $K_\alpha = L_\alpha * \check{L}_\alpha$, i.e.

$$K(x) = K_\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha} \tag{5.18}$$

$$D = 1 - \alpha^2 \left(\frac{d}{dx}\right)^2 \tag{5.19}$$

Then K, K' and K'' are bounded measures, and

$$DK = \delta, \quad \varphi = \varphi * \delta = \varphi * DK = K * D\varphi, \quad \varphi' = K' * D\varphi, \quad \varphi'' = K'' * D\varphi$$

and so there exists M such that

$$||\varphi||_\infty \leqslant M||D\varphi||_\infty, \; ||\varphi'||_\infty \leqslant M||D\varphi||_\infty \; ||\varphi''||_\infty \leqslant M||D\varphi||_\infty$$

$$P_2(\varphi) \leqslant M||D\varphi||_\infty$$

If T has sum-order 2 we therefore have, for some $M \geqslant 0$,

$$|\langle T, \varphi\rangle| \leqslant M||D\varphi||_\infty \tag{5.20}$$

It follows that

$$|\langle K * T, \varphi\rangle| = |\langle T, K * \varphi\rangle| \leqslant M||D(K * \varphi)||_\infty \leqslant M||\varphi||_\infty, \qquad \varphi \in \mathcal{D},$$

so that $K * T$ is a bounded measure by the Riesz Representation theorem.

Now in dimension $m > 1$ let

$$K = \overset{m}{\underset{i=1}{\otimes}} K_{\alpha_i} \tag{5.21}$$

Let

$$D = \prod_{i=1}^{m} 1 - \alpha_i^2 \Big(\frac{\partial}{\partial x_i}\Big)^2 \tag{5.22}$$

Then $D^\ell K$ is a bounded measure provided $|\ell|_\infty \leqslant 2$. By the same argument as before we see that if

$$P_{(2)}(\varphi) = \sup_{|\ell|_\infty \leqslant 2} ||D^\ell \varphi||_\infty \tag{5.23}$$

then

$$P_{(2)}(\varphi) \leqslant M||D\varphi||_\infty \tag{5.24}$$

and if T has rank $\leqslant 2$, i.e.

$$|\langle T, \varphi\rangle| \leqslant M P_{(2)}(\varphi) \tag{5.25}$$

we have

$$|\langle T, \varphi\rangle| \leqslant M||D\varphi||_\infty \tag{5.26}$$

so that $\mu = K * T$ is a bounded measure, and $T = D\mu$.

For higher rank m we take $H = K^{*n}$, if $m = 2n$ is even and otherwise if $m = 2n+1$ we take $H = K^{*n} * L$, and corresponding powers of the differential operator such as (5.22) and find in the same way: If T has rank m then $H * T$ is a bounded measure μ, and $T = D\mu$. ∎

One object of the theory is to generalize this, particularly (5.21) and (5.22) to infinite dimensional sequence spaces, co-nuclear for simplicity.

6 Path distributions on sequence spaces

6.1 Sequence spaces

Let I be a countable set. Consider a locally convex space[6] E over $\mathbb{R}$ with a continuous embedding

$$E \hookrightarrow \mathbb{R}^I$$

We call E a *sequence space* if moreover the following two conditions are satisfied:

[6] We assume E to be quasi-complete: i.e. all closed bounded sets are complete

a. If $|x_i| \leqslant |y_i|$ for all $i \in I$, and $y \in E$, then $x \in E$.

b. E has a fundamental system of continuous seminorms P such that $|x| \leqslant |y| \implies P(x) \leqslant P(y)$. In particular $P(|x|) = P(x)$.

In the sequel we consider only seminorms P with this property.

The conditions a. and b. imply that the complexification $E_{\mathbb{C}}$ of E can be identified to the set of sequences $x \in \mathbb{C}^I$ such that $|x| \in E$, the topology being defined by the seminorms $x \mapsto P(|x|)$.

The sequence space is said to be *regular* if moreover the following condition is satisfied:

c. The sequences with finite support are dense in E.

The spaces $\ell^p(I)$, $1 \leqslant p < \infty$ and $c_0(I)$ are regular sequence spaces, as is the space $s(\mathbb{Z})$ of sequences rapidly vanishing at infinity, and its dual $s'(\mathbb{Z})$ the space of polynomially bounded sequences.

If $\sigma \subset I$ is a finite subset we define three maps π_σ, j_σ and p_σ as follows: The projection or restriction map

$$\pi_\sigma : E \longrightarrow \mathbb{R}^\sigma \tag{6.1}$$

defined by $\pi_\sigma(x) = x_{|\sigma}$. The injection

$$j_\sigma : \mathbb{R}^\sigma \longrightarrow E \tag{6.2}$$

such that $j_\sigma(\xi)_i = 0$ for $i \in I \setminus \sigma$ and $\pi_\sigma j_\sigma(\xi) = \xi$ for all $\xi \in \mathbb{R}^\sigma$. The projector

$$p_\sigma : E \longrightarrow E \tag{6.3}$$

such that $p_\sigma(x)_i = x_i$ if $i \in \sigma$, $p_\sigma(x)_i = 0$ if $i \in I \setminus \sigma$. Equivalently:

$$p_\sigma = j_\sigma \pi_\sigma. \tag{6.4}$$

The condition c. above is equivalent to

c'. $\lim_\sigma p_\sigma(x) = x$ as σ increases indefinitely.[7]

By b. the maps p_σ are equicontinuous, so define an equicontinuous approximation of the identity by finite rank operators.

It follows that a subset $H \subset E$ is relatively compact iff $\lim_\sigma p_\sigma x = x$ uniformly for $x \in H$.

We assume from now on that E is a barreled space, i.e. the uniform boundedness principle is valid in E. It follows that if E is regular and barreled the dual of E may be identified to the space

$$E' = \left\{ y \in \mathbb{R}^I : \sum_{i \in I} |x_i y_i| < +\infty \ \forall\, x \in E \right\} \tag{6.5}$$

[7]These properties of the operators π_σ, j_σ and p_σ are characteristic of path spaces in general, but here we limit considerations to sequence spaces

the duality between E and E' being given by

$$\langle x, y\rangle = \sum_{i\in I} x_i y_i \tag{6.6}$$

Then E' with the strong dual topology is also a sequence space, in general not regular, as the case $\ell^1(I)$, $\ell^\infty(I)$ shows.

6.2 Differential calculus on sequence spaces

Let $X \subset E$ be an open subset. We define the space $\mathcal{E}^{(0)}(X) = \mathcal{E}_{(0)}(X)$ to be the space of $\mathcal{K}$-continuous functions, i.e. functions whose restriction to every compact subset is continuous. Equipped with the topology of uniform convergence on compact sets this space is just the completion of the space of continuous functions. If X is metrizable $\mathcal{K}$-continuous functions are continuous, convergent sequences being relatively compact.

If E is a barreled regular sequence space then a $\mathcal{K}$-continuous linear form $L : E \longrightarrow \mathbb{R}$ is continuous. Indeed, by the compactness of $\{y : |y| \leqslant |x|\}$ we have $L(x) = \lim_\sigma L(p_\sigma x)$ and this implies continuity by the uniform boundedness principle.

In the calculus on general (quasi-complete) locally convex spaces it is convenient to take the spaces $\mathcal{E}^{(0)}$ as point of departure [33]. Thus the space $\mathcal{E}^{(1)}(X)$ is the set of functions Φ such that for all $h \in E$ and $x \in X$ the directional derivative $D\Phi(x,h) = \frac{d}{dt}\Phi(x+th)_{|_{t=0}}$ exists and $D\Phi \in \mathcal{E}^{(0)}(X \times E)$. Then $D\Phi(x,h) = D_h\Phi(x)$ depends linearly on h, and the calculus is analogous to the calculus on Fréchet spaces in [16].

More generally one defines the spaces $\mathcal{E}^{(m)}(E)$ and $\mathcal{B}^{(m)}(E)$, $0 \leqslant m \leqslant +\infty$, in analogy with the finite dimensional case.

We omit details, but we now note what this becomes in the case of (regular, barreled) sequence spaces E.

If $X \subset E$ is an open subset we have the following characterization of the space $\mathcal{E}^{(1)}(X)$ where we put

$$\varphi_\sigma = \Phi \circ j_\sigma \tag{6.7}$$

A function Φ belongs to $\mathcal{E}^{(1)}(X)$ if and only if

1. The functions φ_σ have continuous partial derivatives of order 1.
2. The limit
$$D_h\Phi(x) = \lim_\sigma\, (D_{\pi_\sigma h}\,\varphi_\sigma)(\pi_\sigma x) \tag{6.8}$$
exists for all $x \in X$ and $h \in E$.
3. The map $D\Phi : (x,h) \mapsto D_h\Phi(x)$ belongs to $\mathcal{E}^{(0)}(X \times E)$.

This implies that one has

$$D_h\Phi(x) = \sum_{i\in I} h_i \frac{\partial\Phi}{\partial x_i}(x) \tag{6.9}$$

Note that the same expression then also exists for $h \in E_{\mathbb{C}}$ and that the function $(x, h) \mapsto D_h\Phi(x)$ belongs to $\mathcal{E}^{(0)}(X \times E_{\mathbb{C}})$.

6.3 Path derivatives on sequence spaces

For $h \in E_{\mathbb{C}}$ let formally

$$D_{(h)} = \prod_{i \in I}(1 + h_i \frac{\partial}{\partial x_i}) \tag{6.10}$$

For I finite this is a rank 1 partial differential operator on $\mathbb{R}^I$.

We define the space $\mathcal{E}_{(1)}(E)$ as follows:

A function $\Phi : X \longrightarrow \mathbb{C}$ belongs to $\mathcal{E}_{(1)}(X)$ if

1. The functions φ_σ have continuous partial derivatives of rank 1.

2. The limit

$$D_{(h)}\Phi(x) = \lim_\sigma (D_{(\pi_\sigma h)}\varphi_\sigma)(\pi_\sigma x) \tag{6.11}$$

   exists for all $x \in X$ and $h \in E_{\mathbb{C}}$.

3. The map $(x, h) \mapsto D_{(h)}\Phi(x)$ belongs to $\mathcal{E}_{(0)}(X \times E_{\mathbb{C}})$.

To distinguish it from the directional derivative the expression (6.11) will be called a 'path derivative'. It is clear that $\mathcal{E}_{(1)}(X)$ is a linear space. We equip it with the topology defined by the seminorms

$$P_{(1),K,H}(\Phi) = \sup_{x \in K, h \in H} |D_{(h)}\Phi(x)| \tag{6.12}$$

in which $K \subset X$ is compact, and $H \subset E_{\mathbb{C}}$ is compact and balanced: i.e., $\lambda H \subset H$ for $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$.

Roughly speaking the proposed 'path calculus' is analogous to the 'ordinary' calculus, but with directional derivatives replaced by path derivatives. We summarize some results without proof.

If $m > 1$ the space $\mathcal{E}_{(m)}(X)$ is defined as the set of functions $\Phi \in \mathcal{E}_{(0)}(X)$ such that for all $x \in X$ and $(h_1, \ldots, h_m) \in E_{\mathbb{C}}^m$ the derivative $D_{(h_m)}D_{(h_{m-1})} \cdots D_{(h_1)}\Phi(x)$ exists and belongs to $\mathcal{E}_{(0)}(X \times E_{\mathbb{C}}^m)$. The space is equipped with the seminorms

$$P_{(m),K,H}(\Phi) = \sup_{x \in K, h_i \in H, i=1\ldots m} |D_{(h_m)}D_{(h_{m-1})} \cdots D_{(h_1)}\Phi(x)| \tag{6.13}$$

where $K \subset X$ is compact and $H \subset E_{\mathbb{C}}$ is compact and balanced.

We define $\mathcal{E}_{(\infty)}(X) = \cap_{m \geqslant 0}\mathcal{E}_{(m)}(X)$ and equip this space with the topology defined by the seminorms (6.13), $m \geqslant 1$.

The functions in $\mathcal{E}_{(m)}(X)$ could be said to be path differentiable to rank m.

Proposition 6.1. *The spaces $\mathcal{E}_{(m)}(X)$ are complete. The functions in $\mathcal{E}_{(m)}(X)$ belong to $\mathcal{E}^{(m)}(X)$ and we have the continuous inclusion*

$$\mathcal{E}_{(\infty)}(X) \hookrightarrow \mathcal{E}_{(m+1)}(X) \hookrightarrow \mathcal{E}_{(m)}(X) \hookrightarrow \mathcal{E}^{(m)}(X) \tag{6.14}$$

We omit the proof (which exploits the fact that the map $h \mapsto D_{(h)}\Phi(x)$ is holomorphic).

The space $\mathcal{B}_{(m)}(E)$ is defined in analogy with the space $\mathcal{B}^{(m)}$ as the space of functions $\Phi \in \mathcal{E}_{(m)}(E)$ for which the quantities

$$P_{(m),H}(\Phi) = \sup_{x\in E, h_i\in H, i=1,\ldots,m} |D_{(h_m)}\ldots D_{(h_1)}\Phi(x)| \tag{6.15}$$

are finite, H being compact and balanced in $E_{\mathbb{C}}$. Also $\mathcal{B}_{(\infty)}(E) = \cap_{m\geqslant 0}\mathcal{B}_{(m)}(E)$. These spaces have their natural topologies defined by the semi-norms (6.15). We have the continuous inclusions

$$\mathcal{B}_{(\infty)}(E) \hookrightarrow \mathcal{B}_{(m+1)}(E) \hookrightarrow \mathcal{B}_{(m)}(E) \hookrightarrow \mathcal{B}^{(m)}(E) \tag{6.16}$$

Here $\mathcal{B}_{(0)}(E) = \mathcal{B}^{(0)}(E)$. For $m > 0$ the last inclusion is proper.

Proposition 6.2. *The functions $e_y : x \mapsto e^{i\langle x,y\rangle}$ belong to $\mathcal{B}_{(\infty)}(E)$ for all $y \in E'$. Moreover the map $y \longrightarrow e_y \in \mathcal{E}_{(\infty)}(E)$ is continuous on the space E'_c i.e., the dual equipped with the topology of uniform convergence on compact sets. For $y \in K$, a bounded subset of E'_c, the functions e_y remain bounded in $\mathcal{B}_{(\infty)}(E)$.*

Proof. We have

$$D_{(h)}e_y = \Lambda(h)e_y, \qquad \Lambda(h) = \prod_{k\in I}(1 + ih_k y_k) \tag{6.17}$$

Similarly

$$D_{(h_m)}\ldots D_{(h_1)}e_y = \Lambda(h_m)\ldots\Lambda(h_1)e_y \tag{6.18}$$

Since $\Lambda(h)$ is uniformly bounded for $h \in H$ compact balanced in $E_{\mathbb{C}}$, it follows that e_y belongs to $\mathcal{B}_{(m)}(E)$ for all m. The other statements follow routinely. ∎

6.4 Path distributions

Definition 6.3. *A* path distribution *is a linear form $T : \mathcal{B}_{(\infty)}(E) \longrightarrow \mathbb{C}$ having the bounded convergence property:*

$\Phi_i \in B$, B bounded in $\mathcal{B}_{(\infty)}(E)$, $\Phi_i \to \Phi$ in $\mathcal{E}_{(\infty)}(E)$ $\implies$ $\langle T, \Phi_i\rangle \to \langle T, \Phi\rangle$.

A path distribution T of rank $\leqslant m$ *is a linear form $T : \mathcal{B}_{(m)}(E) \longrightarrow \mathbb{C}$ having the bounded convergence property of rank m:*

$\Phi_i \in B$, B bounded in $\mathcal{B}_{(m)}(E)$, $\Phi_i \to \Phi$ in $\mathcal{E}_{(m)}(E)$ $\implies$ $\langle T, \Phi_i\rangle \to \langle T, \Phi\rangle$.

If $0 \leqslant m \leqslant \infty$ a net (Φ_i) bounded in $\mathcal{B}_{(m)}$ and converging in $\mathcal{E}_{(m)}$ (to a limit function necessarily belonging to $\mathcal{B}_{(m)}$) will be said to pseudo-converge *in $\mathcal{B}_{(m)}$.*

We denote $\mathcal{D}'_{(m)}(E)$ the linear space of path distributions of rank $\leqslant m$, and $\mathcal{D}'_{(\infty)}(E)$ or $\mathcal{D}'_P(E)$ the space of all path distributions.

We equip $\mathcal{D}'_{(m)}(E)$ with the topology of uniform convergence on bounded subsets of $\mathcal{B}_{(m)}(E)$, $0 \leqslant m \leqslant \infty$.

Proposition 6.4. *For $0 \leqslant m \leqslant \infty$ the space $\mathcal{D}'_{(m)}(E)$ is a complete locally convex space. We have the continuous inclusions*

$$\mathcal{M}_b(E) \hookrightarrow \mathcal{D}'_{(m)}(E) \hookrightarrow \mathcal{D}'_{(m+1)}(E) \hookrightarrow \mathcal{D}'_{(\infty)}(E) \tag{6.19}$$

and regarding summable distributions

$$\mathcal{D}_L'^{(m)}(E) \hookrightarrow \mathcal{D}'_{(m)}(E) \tag{6.20}$$

Proof. It is clear that the limit of a Cauchy net is a linear form whose restriction to bounded sets in $\mathcal{B}_{(\infty)}(E)$ is continuous for the topology induced by $\mathcal{E}_{(m)}(E)$. The continuity and injectivity in (6.19) and (6.20) follows by transposition from the fact that the inclusions (6.16) are pseudo continuous and have pseudo dense image. ∎

Proposition 6.5. *For every $T \in \mathcal{D}'_{(\infty)}(E)$ the following formula defines a path distribution $D_{(h)}T$*

$$\langle D_{(h)}T, \Phi\rangle = \langle T, D_{(-h)}\Phi\rangle \tag{6.21}$$

The map $D_{(h)}$ is continuous from $\mathcal{D}'_{(m)}(E)$ to $\mathcal{D}'_{(m+1)}(E)$ and from $\mathcal{D}'_{(\infty)}(E)$ to $\mathcal{D}'_{(\infty)}(E)$.

The Fresnel distribution we wish to construct will be of the form $D_{(-h)}D_{(h)}\mu$ with $\mu \in \mathcal{M}_b(E')$ and $h \in E'$ so will belong to $\mathcal{D}'_{(2)}(E')$.

6.4.1 Convolution

As to the convolution product, we only define the product of $T \in \mathcal{D}'_{(m)}(E)$, $0 \leqslant m \leqslant \infty$, with $\mu \in \mathcal{M}_b(E)$ obtaining a path distribution $T * \mu \in \mathcal{D}'_{(m)}(E)$.

We need the following Lemma:

Lemma 6.6. *If $T \in \mathcal{D}'_{(m)}(E)$ and B is a bounded subset of $\mathcal{B}_{(m)}(E)$ then $T(B)$ is bounded.*

Proof. If Φ_n is any sequence in B and the sequence (λ_n) in $\mathbb{R}$ converges to 0, the sequence $\lambda_n\Phi_n$ is bounded in $\mathcal{B}_{(m)}(E)$ and converges to zero in $\mathcal{E}_{(m)}(E)$. Therefore $\lambda_n\langle T, \Phi_n\rangle = \langle T, \lambda_n\Phi_n\rangle$ goes to zero. This being the case for all such sequences (λ_n), it follows that $\sup_n |\langle T, \Phi_n\rangle| < +\infty$, which proves the assertion, the sequence (Φ_n) being arbitrary in B. ∎

For $a \in E$ let $(\tau_a\Phi)(x) = \Phi(x - a)$. Then the translation τ_a leaves invariant the function spaces $\mathcal{E}_{(m)}(E)$ and $\mathcal{B}_{(m)}(E)$, and commutes with the operators $D_{(h)}$. Moreover, if $B \subset \mathcal{B}_{(m)}(E)$ is a bounded set the union $\cup_{a\in E}\tau_a(B)$ is bounded in $\mathcal{B}_{(m)}(E)$.

Proposition 6.7. *For each $T \in \mathcal{D}'_{(m)}(E)$, $0 \leqslant m \leqslant \infty$, and $\mu \in \mathcal{M}_b(E)$ the equation*

$$T * \mu = \int_E \tau_a(T)\mu(da) \tag{6.22}$$

defines a path distribution $T * \mu \in \mathcal{D}'_{(m)}(E)$.

Proof. We put for $\Phi \in \mathcal{B}_{(m)}(E)$

$$\langle T * \mu, \Phi \rangle = \int_E \langle \tau_a(T), \Phi \rangle \mu(da) = \int_E \langle T, \tau_{-a}\Phi \rangle \mu(da) \tag{6.23}$$

If Φ_i is bounded in $\mathcal{B}_{(m)}$ and converges to Φ in the space $\mathcal{E}_{(m)}(E)$, it follows that $\tau_{-a}\Phi_i$ is bounded in $\mathcal{B}$ uniformly with respect to i and $a \in E$ and converges to $\tau_{-a}\Phi$ in $\mathcal{E}_{(m)}(E)$ uniformly for $a \in K$, a compact set in E. By the Lemma and the bounded convergence property of T it follows that $\langle T, \tau_{-a}\Phi_i \rangle$ is bounded with respect to i and $a \in E$ and converges to $\langle T, \tau_{-a}\Phi \rangle$ uniformly for $a \in K$. By the bounded convergence property of μ it now follows that $\int \langle T, \tau_{-a}\Phi_i \rangle \mu(da)$ converges to $\int \langle T, \tau_{-a}\Phi \rangle \mu(da)$, which proves that $T * \mu$ belongs to $\mathcal{D}'_{(m)}(E)$. ∎

6.5 Fourier transform

Since the functions $e_y : x \mapsto e^{i\langle x,y \rangle}$ belong to $\mathcal{B}_{(\infty)}(E)$ for all $y \in E'$ we can define the Fourier transform as follows

$$\widehat{T}(y) = \langle T, e^{-i\langle x,y \rangle} \rangle \tag{6.24}$$

Proposition 6.8. *The Fourier transform $\widehat{T}$ is a $\mathcal{K}$-continuous function on E'_c. If E is a Fréchet space, or E'_c is a Fréchet space, then $\widehat{T}$ is continuous on E'_c.*

Proof. The first statement follows directly from Proposition 6.2. The second, where E is assumed to be a Fréchet space, is a consequence of the theorem of Banach-Dieudonné ([15] ch. 4, part 2, Thm. 2). The third statement is trivial, $\mathcal{K}$-continuous functions on metrizable spaces being continuous. ∎

Proposition 6.9. *The path distribution T is completely determined by its Fourier transform.*

Proof. The distribution $\pi_\sigma(T)$ has the following Fourier transform, in which ${}^t\pi_\sigma$ denotes the transpose of π_σ:

$$\widehat{\pi_\sigma(T)}(\xi) = \widehat{T}({}^t\pi_\sigma(\xi))$$

Since the Fourier transform is known to be one-one in the finite dimensional case it follows that $\widehat{T}$ determines the marginal distributions $\pi_\sigma(T)$, as well as therefore the distributions $T_\sigma = p_\sigma(T) = j_\sigma(\pi_\sigma(T))$. The proof follows if we prove

$$\langle T, \Phi \rangle = \lim_\sigma \langle T_\sigma, \Phi \rangle$$

i.e.

$$\langle T, \Phi\rangle = \lim_{\sigma}\langle T, \Phi \circ p_\sigma\rangle$$

for all $\Phi \in \mathcal{B}_{(\infty)}(E)$. But this is a consequence of the bounded convergence property of T, since $\Phi_\sigma = \Phi \circ p_\sigma$ converges to Φ in the space $\mathcal{E}_{(\infty)}(E)$ while remaining bounded in $\mathcal{B}_{(\infty)}(E)$ (This is almost identical to the proof in the case of summable distributions such as in [6]). ∎

Proposition 6.10. *If $S = T * \mu$ then $\widehat{S} = \widehat{T}\widehat{\mu}$.*

We omit the obvious proof.

7 Construction of Fresnel distributions on conuclear sequence spaces

7.1 Nuclear sequence spaces

We assume that E is a regular sequence space which is barreled and nuclear[8]. This is equivalent to the following: There exists a set

$$\Omega \subset \mathbb{R}^I_+ \tag{7.1}$$

which is directed : for $a, b \in \Omega$ there exists $c \in \Omega$ with $a \leqslant c$, $b \leqslant c$, and such that

1. E is the set of $\xi \in \mathbb{R}^I$ for which the quantities

   $$p_a(\xi) = \sup_i a_i|\xi_i| \tag{7.2}$$

   are finite for all $a \in \Omega$.

   The space E is equipped with the topology associated with the seminorms p_a, $a \in \Omega$.

2. For every $a \in \Omega$ there exists $b \in \Omega$ such that $a/b \in \ell^1(I)$ (we put $0/0 = 0$ by definition).

This implies that the seminorms

$$q_a(\xi) = \sum_i a_i|\xi_i| \tag{7.3}$$

with $a \in \Omega$, form a fundamental system of continuous seminorms:

$$q_a(\xi) = \sum_i (a_i/b_i)b_i|\xi_i| \leqslant ||a/b||_1 p_b(\xi). \tag{7.4}$$

[8] As always quasi-complete. The example $s \oplus [1]$, of the rapidly converging sequences, shows that nuclearity does not imply regularity, i.e. the density of the finite sequences.

The dual of E is the set of linear forms $\xi \mapsto \sum_i x_i \xi_i$ such that there exists $a \in \Omega$ and $M \geqslant 0$ with $|x| \leqslant Ma$, and we identify it with the corresponding subspace of $\mathbb{R}^I$:

$$E' = \{x \in \mathbb{R}^I : \exists a \in \Omega, M \geqslant 0 : |x_i| \leqslant Ma_i \; \forall \, i \in I\} \tag{7.5}$$

and pose for $x \in E'$ and $\xi \in E$:

$$\langle x, \xi \rangle = \sum_i x_i \xi_i \, . \tag{7.6}$$

Equipped with the strong dual topology, E' is a conuclear space ([28] Thm 1, p. 231).

E is a Fréchet space iff there exists a fundamental sequence of weights $\omega(k) \in \Omega$, $k \in \mathbb{N}$, such that

i. $E = s(\omega) = \{\xi \in \mathbb{R}^I : \sup_i \omega_i(k)|\xi_i| < +\infty \; \forall \, k \in \mathbb{N}\}$

ii. For every $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that $\omega(k)/\omega(\ell) \in \ell^1(I)$.

It then follows that

iii. $E' = s'(\omega) = \{x \in \mathbb{R}^I : \exists k \in \mathbb{N}, M \geqslant 0 : |x_i| \leqslant M\omega_i(k) \; \forall \, i \in I\}$.

For example if $I = \mathbb{N}$ or $\mathbb{Z}$ and $\omega_i(k) = (1+i^2)^k$ we obtain the Schwartz space $s(I)$ of rapidly decreasing sequences, and the space $s'(I)$ of sequences of polynomial growth.

For those readers not familiar with the theory of nuclear spaces [14], [28], it will for the present purpose be enough to take the conditions 1. and 2. (or i., ii.) above as axioms. We shall need no other part of the theory.

If $b = (b_i)_{i \in I}$ is a strictly positive sequence the space bE of products $(b_i \xi_i)_{i \in I}$, $\xi \in E$, with the weights a/b, $a \in \Omega$, is a nuclear sequence space (path) isomorphic to E, and its dual is the space of quotients E'/b.

For simplicity of the notations in the proofs we shall assume $I = \mathbb{N}$, but this can of course be accomplished by ordering the set I.

7.2 The quadratic form

The quadratic form we consider is associated to a linear operator $K : E \longrightarrow E'$ which is assumed to be symmetric:

$$\langle K\xi, \eta \rangle = \langle K\eta, \xi \rangle \tag{7.7}$$

Contrary to the case of autocorrelation operators for gaussian measures, we do not assume K to be positive.

The symmetry implies that the bilinear form $(\xi, \eta) \mapsto \langle K\xi, \eta \rangle$ is separately continuous. If E is a Fréchet space or the strong dual of a Fréchet space (reflexive by the nuclearity), or if K is positive, this implies that the form is continuous.[9] In general we *assume* it to be continuous.

[9]This is a consequence of the uniform boundedness principle, of the theorem of Dieudonné-Schwartz ([9] p. 96, théorème 9) and of a theorem of Schwartz ([29], Cor. p. 157) respectively.

Therefore there exists a weight $a \in E'_+$, such that

$$|\langle K\xi, \eta\rangle| \leqslant \sum_i a_i|\xi_i| \sum_j a_j|\eta_j| \tag{7.8}$$

Let e_i be the sequence with zeros except at the i^{th} place where there is a 1. Let

$$K_{ij} = \langle Ke_i, e_j\rangle \tag{7.9}$$

Then

$$|K_{ij}| \leqslant a_i a_j, \quad i,j \in I \tag{7.10}$$

Without restricting the generality we may and shall assume that $a_i > 0$ for all $i \in I$. For if a vanishes on the set $N \subset I$, we put $J = I \setminus N$, and replace E by the set of restrictions $E_{|J}$ (isomorphic to the quotient of E, by the space of sequences in E that vanish on J) and we replace E' by the subspace ${}_JE'$ of elements supported by J. Then if we construct a Fresnel distribution on ${}_JE'$ we will get the distribution to be constructed as image under the injection, a path morphism, ${}_JE' \hookrightarrow E'$. In particular, if J is finite, there is nothing more to prove, and so we assume $I = J$ to be countably infinite.

By the above estimate it follows that we have, for $\xi, \eta \in E$:

$$\langle K\xi, \eta\rangle = \sum_{i,j} K_{ij}\xi_i\eta_j \tag{7.11}$$

the sum being absolutely convergent[10]

$$\sum_{i,j} |K_{ij}||\xi_i||\eta_j| < +\infty. \tag{7.12}$$

If $a/b \in \ell^1$, we have

$$\sum_{i,j} \frac{|K_{ij}|}{b_i b_j} < +\infty \tag{7.13}$$

Going over to the spaces bE and E'/b we may assume

$$||K||_1 = \sum_{i,j} |K_{ij}| < +\infty \tag{7.14}$$

This implies that the matrix defines a trace class operator in $\ell^2(I)$. [11]

Note that in the new space E'/b one of the weights is the constant sequence $1 \in E'$.

[10] Here we benefit from the nuclearity. For a Banach space such as c_0 continuous bilinear forms do not have such a simple description.

[11] This is well known: The rank one operator $K_{ij}\bar{e}_i \otimes e_j$ has trace norm equal to $|K_{ij}|$. Hence the operator $K = \sum_{ij} K_{ij}e_i \otimes \bar{e}_j$ is of trace class with trace norm at most equal to $||K||_{\ell^1(I\times I)}$

7.3 Infinite dimensional Mollifiers

Define probability measures on $\mathbb{R}$

$$M_\alpha = \frac{e^{-|x|/\alpha}}{2\alpha} dx, \quad \alpha > 0 \tag{7.15}$$

and if $\alpha = (\alpha_i)_{i \in I}$ is strictly positive, let

$$M = \underset{i \in I}{\otimes} M_{\alpha_i}, \tag{7.16}$$

be the infinite product measure.

Note that

$$\widehat{M}_{\alpha_i}(\xi) = \frac{1}{1 + \alpha_i^2 \xi^2} \tag{7.17}$$

so that

$$\Big(1 - \alpha_i^2 \big(\frac{d}{dx_i}\big)^2\Big) M_{\alpha_i} = \delta \tag{7.18}$$

We shall also make use of the measure

$$\nu = \underset{i \in I}{\otimes} \nu_i \tag{7.19}$$

where

$$\nu_i = Y e^{-s_i/\beta_i} \frac{ds_i}{\beta_i} \tag{7.20}$$

We denote finite tensor products as

$$M^{(n)} = \overset{n}{\underset{i=1}{\otimes}} M_{\alpha_i}; \quad \nu^{(n)} = \overset{n}{\underset{i=1}{\otimes}} \nu_i \tag{7.21}$$

We also note

$$D^{(n)} = \prod_{i=1}^{n} \Big(1 - \alpha_i^2 \big(\frac{d}{dx_i}\big)^2\Big) \tag{7.22}$$

so that

$$D^{(n)} M^{(n)} = \delta^{(n)} \tag{7.23}$$

the Dirac measure in $\mathbb{R}^n$.

We denote $K^{(n)}$ the matrix obtained by restricting the matrix K to $\{1, \ldots, n\}^2$, and we denote $\langle K^{(n)}\xi, \xi\rangle$ the corresponding quadratic form on $\mathbb{R}^n$.

From the finite dimensional theory we know that the function $\xi \mapsto e^{i\langle K^{(n)}\xi,\xi\rangle/2}$ being in the space $\mathcal{O}_M$ of multipliers, there exists a summable distribution $F^{(n)}$ on $\mathbb{R}^n$ such that

$$\widehat{F}^{(n)}(\xi) = e^{i\langle K^{(n)}\xi,\xi\rangle/2} \tag{7.24}$$

Also, we know from Theorem 5.1 that, if $K^{(n)}$ is non-degenerate, the rank of $F^{(n)}$ is two. This implies that

$$\mu^{(n)} = M^{(n)} * F^{(n)} \tag{7.25}$$

is a bounded measure. But it is a consequence of (7.38) and of the representation (7.33) that, for arbitrary symmetric $K^{(n)}$, μ_n is a bounded measure, and consequently $F^{(n)}$ has rank $\leqslant 2$.

Moreover, we have

$$F^{(n)} = D^{(n)}\mu^{(n)} \tag{7.26}$$

The measures $M^{(n)}$ and the distributions $F^{(n)}$ form a projective system, as do consequently, the measures $\mu^{(n)}$. We shall prove that with the appropriate choice of the α_n we have, the norms denoting the total variation,

$$\sup_n ||\mu^{(n)}|| < +\infty. \tag{7.27}$$

The measures M_α can be written as a superposition of gaussian measures

$$\frac{1}{2\alpha}e^{-|x|/\alpha} = \frac{1}{2\alpha^2}\int_0^{+\infty} e^{-s/2\alpha^2}\frac{e^{-x^2/2s}}{\sqrt{2\pi s}}ds \tag{7.28}$$

or briefly

$$M_\alpha = \frac{1}{\beta}\int_0^{+\infty} e^{-s/\beta}G_s ds\,, \quad \beta = 2\alpha^2 \tag{7.29}$$

as is easily seen by taking Fourier transforms.

Taking direct products it follows that:

$$M^{(n)} = \int_0^{+\infty}\cdots\int_0^{+\infty}\frac{e^{-s_1/\beta_1}}{\beta_1}\cdots\frac{e^{-s_n/\beta_n}}{\beta_n}G_S^{(n)}ds_1\dots ds_n \tag{7.30}$$

where $G_S^{(n)} = G_{s_1}\otimes\cdots\otimes G_{s_n}$ and

$$\widehat{G}_S^{(n)}(\xi) = e^{-\sum s_i\xi_i^2/2} = e^{-\langle S^{(n)}\xi,\xi\rangle/2} \tag{7.31}$$

$S^{(n)}$ being the diagonal matrix with $(s_1,\dots,s_n)$ on the diagonal.

We can now write $\mu^{(n)}$ as a superposition of Gauss-Fresnel distributions

$$G_S^{(n)} * F_K^{(n)} = G_S^{(n)} * G_{iK}^{(n)} = G_{S+iK}^{(n)} \tag{7.32}$$

$$\mu^{(n)} = M^{(n)} * F_K^{(n)} = \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{e^{-(s_1/\beta_1+\cdots+s_n/\beta_n)}}{\beta_1 \ldots \beta_n} G_{S+iK}^{(n)} ds_1 \ldots ds_n \tag{7.33}$$

or briefly

$$\mu^{(n)} = \int G_{S+iK}^{(n)} \nu^{(n)}(dS) \tag{7.34}$$

This implies the following inequality for the variation measures:

$$|\mu^{(n)}| \leqslant \int |G_{S+iK}^{(n)}| \nu^{(n)}(dS) \tag{7.35}$$

7.4 Estimates of determinants

In this section A, K, S, without superscripts, denote $n \times n$ matrices, S diagonal with strictly positive diagonal $(s_1, \ldots, s_n)$. If A is a symmetric matrix with positive definite real part it follows that A is invertible ([17] §3.4) and that A^{-1} also has positive definite real part[12]. Let G_A be the distribution such that

$$\widehat{G}_A(\xi) = e^{-\langle A\xi,\xi\rangle/2}. \tag{7.36}$$

i.e.

$$G_A = \frac{1}{(2\pi)^{n/2}\sqrt{\det(A)}} e^{-\langle A^{-1}x,x\rangle/2} dx \tag{7.37}$$

where $\sqrt{\det(A)} > 0$ if A is real (cf. [17] §3.4).

Then

$$\int_{\mathbb{R}^n} |G_A(x)| dx = \sqrt{\frac{|\det(A)|}{\det(\mathrm{Re}\,(A))}} \tag{7.38}$$

and

$$\int_{\mathbb{R}^n} G_A(x) dx = 1 \tag{7.39}$$

Applying this to $A = S + iK$, with S diagonal, strictly positive, and K real, we obtain

$$||G_{S+iK}|| = \sqrt{\frac{|\det(S+iK)|}{\det S}} \tag{7.40}$$

[12] G_A being the inverse Fourier transform of $e^{-\langle A\xi,\xi\rangle/2}$ this is a consequence of the Riemann-Lebesgue lemma.

or more conveniently:

$$||G_{S+iK}|| = \sqrt{|\det(I+iS^{-1}K)|} = \sqrt{|\det(I+iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})|} \tag{7.41}$$

To estimate this we use Hadamard's inequality:

$$|\det(A)| \leqslant \prod_{i=1}^{n} L_i \tag{7.42}$$

where $L_i = \sqrt{\sum_{j=1}^{n} |A_{ij}|^2}$ is the length of the i^{th} row.

In the case $A = I + iS^{-1}K$

$$L_i^2 = 1 + \frac{1}{s_i^2}\sum_{j=1}^{n} k_{ij}^2 = 1 + \frac{k_i^2}{s_i^2} \tag{7.43}$$

$$|\det(I+iS^{-1}K)| \leqslant \prod_{i=1}^{n} \sqrt{1+\frac{k_i^2}{s_i^2}} \tag{7.44}$$

k_i being the length of the i^{th} row of the matrix K.

We make use of the following estimate

$$\frac{1}{\beta}\int_0^{+\infty} e^{-s/\beta}(1+\frac{1}{s^2})^{\frac{1}{4}} ds \leqslant 1 + \frac{2}{\beta} \tag{7.45}$$

The integral equals:

$$1 + \frac{1}{\beta}\int_0^{+\infty} e^{-s/\beta}\Big((1+\frac{1}{s^2})^{\frac{1}{4}} - 1\Big) ds \leqslant 1 + \frac{k}{\beta} \tag{7.46}$$

where

$$k = \int_0^{+\infty} (1+\frac{1}{s^2})^{\frac{1}{4}} - 1\, ds \tag{7.47}$$

$$\leqslant \int_0^1 \frac{(s^2+1)^{\frac{1}{4}}}{\sqrt{s}} - 1 ds + \int_1^{+\infty} \frac{1}{4}\frac{1}{s^2} ds < 2\, 2^{\frac{1}{4}} - 1 + \frac{1}{4} < 2. \tag{7.48}$$

Now if we apply this to the the matrices $K^{(n)}$ and $S^{(n)}$ we obtain:

$$\begin{aligned} ||\mu^{(n)}|| &\leqslant \int ||G_{S^{(n)}+iK^{(n)}}|| \nu^{(n)}(dS) \\ &\leqslant \prod_{i=1}^{n}\int_0^{+\infty}\Big(1+\frac{k_i^2}{s_i^2}\Big)^{\frac{1}{4}} e^{-s_i/\beta_i}\frac{ds_i}{\beta_i} \\ &\leqslant \prod_{i=1}^{n}(1+2\frac{k_i}{\beta_i}) \leqslant e^{2M} \end{aligned} \tag{7.49}$$

if

$$\sum_{i=1}^{n} \frac{k_i}{\beta_i} \leqslant M. \tag{7.50}$$

Since we have $k_i \leqslant \sum_j |K_{ij}|$ it follows that $\sum_i k_i \leqslant \sum_{i,j} |K_{ij}|$ and the required uniform estimate for $||\mu^{(n)}||$ holds provided β is bounded away from zero.

Thus, under the condition $||K||_1 < +\infty$ and $\beta_i \geqslant c > 0$ we have

$$\sup_n ||\mu^{(n)}|| \leqslant e^{2||K||_1/c} < +\infty, \tag{7.51}$$

which implies the existence of the projective limit

$$\mu = \varprojlim{}_n \mu_n \tag{7.52}$$

on the space $\mathbb{R}^I$ ([32]).

7.5 Strong concentration

Let X and Y be topological Hausdorff spaces such that X is included in Y with continuous inclusion.

$$X \underset{i}{\hookrightarrow} Y \tag{7.53}$$

Denote $\mathcal{K}(X)$ the set of compact subsets of X.

A bounded positive Radon measure μ on Y will be said to be *strongly concentrated* on X if it is the image of a bounded Radon measure ν under the injection i.

In practice this means μ can be regarded as a measure on X, or more accurately, may be replaced by ν, forgetting Y.

Proposition 7.1.

1. *Let μ be a positive bounded Radon measure on Y. Then μ is strongly concentrated on X iff*

$$\sup_{K \in \mathcal{K}(X)} \mu(K) = \mu(Y) \tag{7.54}$$

2. *Let μ_1 and μ_2 be bounded positive Radon measures on Y, such that $\mu_1 \leqslant \mu_2$. Then if μ_2 is strongly concentrated on X, so is μ_1.*

Proof. 1. is a particular case of a theorem of Schwartz ([28], Thm. 11, p. 37), but can also be proved easily by use of Choquet's representation theorem characterizing Radon measures on X as set functions on $\mathcal{K}(X)$. 2. Is an obvious consequence of 1. ∎

Remark 7.2. Strong concentration of μ on X implies that μ is concentrated on X (i.e. X is μ measurable and $\mu(Y \setminus X) = 0$, but the converse of this is in general false.

If X is a Suslin space strong concentration is equivalent to concentration ([28], p. 107 Lemma 16, p. 122 Thm. 10). This allows a shortcut in our proof if E is a Fréchet space (necessarily separable) or the strong dual of a Fréchet space, in which case E' is a Suslin space ([28], p. 112 Thm. 7).

Proposition 7.3. *Let E and F be locally convex hausdorff spaces, such that $F \hookrightarrow E$ with continuous and dense inclusion, and consequently*

$$E' \hookrightarrow F' \tag{7.55}$$

the dual spaces being equipped with their strong dual topologies. Assume E to be a barreled nuclear space. Let μ be a positive bounded Radon measure on F'. Then μ is strongly concentrated on E' iff its Fourier transform Φ is continuous on F with respect to the topology induced by E.

Proof. The condition is obviously necessary. It is sufficient because Φ is then uniformly continuous on F with respect to the induced topology, and so extends to a continuous function on E, which again is positive definite. Therefore the statement is an immediate consequence of Minlos' theorem ([2], Ch. IX, §6 no. 10). ∎

7.6 Construction of auxiliary measures

In the case of a finite dimensional Gauss-Fresnel distribution G_A, having the symmetric autocorrelation matrix A with $\mathrm{Re}\,(A)$ positive definite, the variation measure has the following expression

$$|G_A| = ||G_A||G_B \tag{7.56}$$

where $B^{-1} = \mathrm{Re}\,(A^{-1})$. If $A = S + iK$ with S a strictly positive diagonal matrix and K real, $B = B(S)$ turns out to be

$$B(S) = S + KS^{-1}K \tag{7.57}$$

as is seen by writing $(S + iK)(B^{-1} + iT) = I$, with T real, identifying real and imaginary parts and eliminating T. With this abbreviation we therefore have

$$|G_{S+iK}| = \sqrt{|\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})|}G_{B(S)} \tag{7.58}$$

We shall construct similar measures in the infinite dimensional case, and arrange that for ν a.a. S they are strongly concentrated on the space E'.

Lemma 7.4. *Let $\alpha = (\alpha_i) \in E'$ be strictly positive. Then the quadratic form $\xi \mapsto \langle S\xi, \xi\rangle$ is continuous on E for ν-almost all $S = (s_i)$. Moreover, for every $\varepsilon > 0$, there is a set A_ε such that $\nu(A_\varepsilon^c) \leqslant \varepsilon$ and such that the set of maps $\xi \mapsto \langle S\xi, \xi\rangle$ with $S \in A_\varepsilon$, is equicontinuous.*

Proof. We know that the continuity on E of the form $\xi \mapsto \sum_i s_i \xi_i^2$ is equivalent to having an estimate

$$|\langle S\xi, \xi\rangle| \leqslant M p_a(\xi)^2 \tag{7.59}$$

for some weight $a \in E'_+$, or equivalently

$$s_i = O(a_i^2) \tag{7.60}$$

for some $a \in E'_+$. Let a be such that $\alpha/a \in \ell^1(I)$. Since $\beta_i = 2\alpha_i^2$ we have

$$\int \sum_i \frac{s_i}{a_i^2} \nu(ds) = c \sum_i \frac{\beta_i}{a_i^2} < +\infty \tag{7.61}$$

Therefore we have

$$\sum_i \frac{s_i}{a_i^2} < +\infty \quad \nu \ a.a\ s \tag{7.62}$$

A fortiori $s_i = O(a_i^2)$ for ν *a.a.* S. ■

Let $A_\varepsilon = \left\{s \in (0, +\infty)^I : \sum_i \frac{s_i}{a_i^2} \leqslant M\right\}$. Then

$$\nu(A_\varepsilon^c) \leqslant \frac{1}{M} \int \sum_i \frac{s_i}{a_i^2} \nu(ds) \leqslant \varepsilon \tag{7.63}$$

if M is chosen large enough. For $S \in A_\varepsilon$ we have

$$s_i \leqslant M a_i^2 \tag{7.64}$$

and so

$$|\langle K\xi, \xi\rangle| \leqslant M q_a(\xi)^2 \tag{7.65}$$

Recall that if T is an operator in a Hilbert space $\mathcal{H}$, the determinant $\det(I + T)$ is well defined if T is trace class, and the map $T \mapsto \det(I + T)$ is continuous on the Banach space $\mathcal{L}_1(\mathcal{H})$ of trace class operators ([23], Ch XIII).

Lemma 7.5. *If $\beta_i \geqslant c > 0$ is bounded away from 0 we have*

$$\sum_{i,j} \frac{|K_{ij}|}{\sqrt{s_i}\sqrt{s_j}} < +\infty \quad \nu \ a.a.S \tag{7.66}$$

Proof. We have

$$\int \sum_{i,j} \frac{|K_{ij}|}{\sqrt{s_i}\sqrt{s_j}} \nu(ds) = \pi \sum_{i,j} \frac{|K_{ij}|}{\sqrt{\beta_i}\sqrt{\beta_j}} \leqslant \frac{\pi}{c} \sum_{i,j} |K_{ij}| < +\infty \tag{7.67}$$

and the integrand is finite almost everywhere. ∎

Note that since the map $(K_{ij}) \mapsto K$ from $\ell^1(I \times I)$ to $\mathcal{L}_1(\ell^2(I))$ is continuous, the infinite dimensional determinant

$$\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}}) \tag{7.68}$$

is the limit of finite dimensional determinants obtained by restriction to $\{1, \dots, n\}^2$.

We use, without explicit mention, the following fact: If $(A_{ij})_{i,j\in\mathbb{N}}$ is a matrix whose support is $\{1, \dots, n\}^2$, and $A^{(n)}$ is its restriction to $\{1, \dots, n\}^2$ we have $\det(I + A) = \det(I^{(n)} + A^{(n)})$, the operators $I + A$ and $I^{(n)} + A^{(n)}$ having the same eigenvalues and multiplicities, except for 1 ([23] XIII.106).

The density in $\ell^1(I \times I)$ of the matrices with finite support also shows that for every S for which $S^{-\frac{1}{2}}KS^{-\frac{1}{2}}$ is trace class, $S^{-1}K$ is also trace class and

$$\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}}) = \det(I + iS^{-1}K) \tag{7.69}$$

It follows that the square roots of the finite dimensional determinants, and the infinite dimensional determinant, are in absolute value majorized by the function F, integrable with respect to ν,

$$F(S) = \prod_{i=1}^{\infty}\Big(1 + \frac{k_i^2}{s_i^2}\Big)^{\frac{1}{4}} \tag{7.70}$$

where $k_i^2 = \sum_j |K_{ij}|^2$. This will allow us to use Lebesgue's theorem later on.

Lemma 7.6. *Let α be strictly positive and such that $\sum_i \frac{1}{\alpha_i} < +\infty$. Then*

a. The matrix $H(S) = KS^{-1}K$ is well defined (absolutely convergent) and belongs to $\ell^1(I \times I)$ for ν a.a. S.

b. The quadratic form $\xi \mapsto \langle H(S)\xi, \xi\rangle$ is continuous on E for ν a.a. S.

c. Moreover, for every $\varepsilon > 0$, there is a set B_ε such that $\nu(B_\varepsilon^c) \leqslant \varepsilon$ and such that the set of maps $\xi \mapsto \langle H(S)\xi, \xi\rangle$ with $S \in B_\varepsilon$, is equicontinuous.

Proof. a. The matrix coefficients of $H(S)$ are $H_{j\ell}(S) = \sum_i \frac{1}{s_i}K_{ij}K_{i\ell}$. We prove that this sum converges absolutely for ν almost all S, and that $H(S)$ belongs to $\ell^1(I \times I)$ for ν almost all S.

$$\sum_{j,\ell}|H_{i,j}(S)| \leqslant \sum_{i,j,\ell}\frac{1}{s_i}|K_{ij}|\,|K_{i\ell}| = \sum_i \frac{1}{s_i}\sum_j |K_{ij}|\sum_\ell |K_{i\ell}| \leqslant ||K||_1^2 \sum_i \frac{1}{s_i} \tag{7.71}$$

so it is sufficient to prove that this last sum converges almost everywhere. We have

$$\int \sum_i \frac{1}{\sqrt{s_i}}\nu(ds) = \sum_i \int_0^{+\infty} \frac{e^{-s_i/\beta_i}}{\sqrt{s_i}}\frac{ds_i}{\beta_i} = \sum_i \sqrt{\frac{\pi}{\beta_i}} < +\infty. \tag{7.72}$$

Therefore

$$\sum_i \frac{1}{\sqrt{s_i}} < +\infty \tag{7.73}$$

ν-almost everywhere. A fortiori the expression (7.71) is finite for ν-almost all S.

b. This assertion is now trivially satisfied. Since 1 is one of the weights, the expression $||\xi||_\infty = \sup_i |\xi_i|$ is a continuous seminorm on E, and

$$|\langle H_{(S)}\xi, \xi\rangle| \leqslant ||H(S)||_1 ||\xi||_\infty^2 \tag{7.74}$$

c. Let $C_M = \{S \in (0, +\infty)^{\mathbb{N}} : \sum_i \frac{1}{\sqrt{s_i}} \leqslant \sqrt{M}\}$. Then $\nu(C_\varepsilon^c) = O(\frac{1}{M}) \leqslant \varepsilon$ for M large enough. Now let $B_\varepsilon = \{S \in (0, +\infty)^{\mathbb{N}} : \sum_i \frac{1}{s_i} \leqslant M\}$. Then $C_\varepsilon \subset B_\varepsilon$ and so $\nu(B_\varepsilon^c) \leqslant \varepsilon$. For $S \in B_\varepsilon$ we have $||H(S)||_1 \leqslant M||K||_1^2$, which implies the equicontinuity. ∎

Now choose $\alpha = (\alpha_i) \in E'$ so that $\sum_i \frac{1}{\alpha_i} < +\infty$. This is possible because $1 \in E'$ is a weight.

Then α and $\beta = 2\alpha^2$ are bounded away from zero and satisfy the assumptions of the previous lemmas. Thus for almost all S the non-negative form

$$\xi \mapsto \langle B(S)\xi, \xi\rangle = \langle S\xi, \xi\rangle + \langle S^{-1}K\xi, K\xi\rangle \tag{7.75}$$

is continuous on E. By Minlos' theorem there then exists a Gaussian measure $G_{B(S)}$ on E' such that

$$\widehat{G}_{B(S)}(\xi) = e^{-\langle B(S)\xi,\xi\rangle/2}, \quad \xi \in E. \tag{7.76}$$

Also for almost all S we know that $\det(I+iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})$ exists. Therefore, for ν-almost all S we can define the measure[13]

$$|G_{S+iK}| = \sqrt{|\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})|}G_{B(S)} \tag{7.77}$$

For the moment we regard this as a measure on $\mathbb{R}^I$. We shall prove that

$$|\mu| \leqslant \int |G_{S+iK}|\nu(ds) \tag{7.78}$$

To prove the above inequality it will be sufficient to prove, for every continuous function φ on $\mathbb{R}^I$ depending on only a finite number of coordinates, and bounded (by 1 for definiteness) that

$$|\mu(\varphi)| \leqslant \int |G_{S+iK}|(|\varphi|)\nu(dS) \tag{7.79}$$

[13] Despite this notation we do not claim to have defined a measure G_{S+iK}

Now we know that

$$|\mu^{(n)}(\varphi)| \leqslant \int |G_{S^{(n)}+iK^{(n)}}|(|\varphi|)\nu^{(n)}(dS) = \int |G_{S^{(n)}+iK^{(n)}}|(|\varphi|)\nu(dS) \tag{7.80}$$

the measure $\nu^{(n)}$ being the image of ν under the projection $S \mapsto S^{(n)}$, and it is sufficient to justify passing to the limit. For those S such that the expressions (7.62) and (7.71) are finite, and so for ν-almost all S, we have, by the absolute convergence,

$$\begin{aligned}\langle B(S)\xi,\xi\rangle &= \sum_{i=1}^{\infty} s_i\xi_i^2 + \sum_{i=1}^{\infty}\frac{1}{s_i}\sum_{j=1}^{\infty}K_{ij}\xi_j\sum_{\ell=1}^{\infty}K_{i\ell}\xi_\ell \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} s_i\xi_i^2 + \sum_{i=1}^{n}\frac{1}{s_i}\sum_{j=1}^{n}K_{ij}\xi_j\sum_{\ell=1}^{n}K_{i\ell}\xi_\ell = \lim_{n\to\infty}\langle B(S^{(n)})\xi,\xi\rangle\end{aligned} \tag{7.81}$$

Therefore, by the Levy convergence theorem (in finite dimensions), it follows that

$$G_{B(S)}(|\varphi|) = \lim_{n\to\infty} G_{B(S^{(n)})}(|\varphi|) \tag{7.82}$$

We also know that for ν-almost all S

$$\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}}) = \lim_{n\to\infty}\det(I^{(n)} + iS^{(n)-\frac{1}{2}}K^{(n)}S^{(n)-\frac{1}{2}}) \tag{7.83}$$

the matrix $S^{(n)-\frac{1}{2}}K^{(n)}S^{(n)-\frac{1}{2}}$ being the restriction of $S^{-\frac{1}{2}}KS^{-\frac{1}{2}}$ to $\{1,\ldots,n\}^2$. Thus we have

$$|G_{S+iK}(|\varphi|) = \lim_{n\to\infty}|G_{S^{(n)}+iK^{(n)}}|(|\varphi|) \tag{7.84}$$

for ν-almost all S. Since

$$|G_{S^{(n)}+iK^{(n)}}|(|\varphi|) \leqslant F(S) \tag{7.85}$$

we may pass to the limit under the integral sign, and conclude

$$|\mu(\varphi)| \leqslant \int |G_{S+iK}(|\varphi|)\nu(dS) \tag{7.86}$$

By standard approximation techniques from measure theory one can now justify this for bounded continuous or Borel functions, and conclude as to (7.78).

Next we wish to prove that the measure $\int |G_{S+iK}|\nu(dS)$ is strongly concentrated on E'. Its characteristic function, defined on $\mathbb{R}^{(I)}$, is

$$\int e^{-\langle B(S)\xi,\xi\rangle/2}\sqrt{|\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})|}\nu(dS) \tag{7.87}$$

$$= \lim_{D}\int_D e^{-\langle B(S)\xi,\xi\rangle/2}\sqrt{|\det(I + iS^{-\frac{1}{2}}KS^{-\frac{1}{2}})|}\nu(dS) \tag{7.88}$$

where D is a set such that the functions $\xi \mapsto e^{-\langle B(S)\xi,\xi\rangle}$ are, for $S \in D$, equicontinuous with respect to the topology induced on $\mathbb{R}^{(I)}$ by E. Since the integrals converge uniformly with respect to ξ, this proves the continuity of the characteristic function with respect to this induced topology, and so, by Minlos' theorem, the fact that $\int |G_{S+iK}|\nu(dS)$ and a fortiori $|\mu|$, is strongly concentrated on E'.

This essentially ends the proof of the main theorem in the next section.

8 Main theorem

We summarize the argument:

By a diagonal transformation, which leaves invariant the path structure, we accomplish that the matrix K belongs to $\ell^1(I \times I)$. We choose a positive sequence $\alpha \in E'$ such that $\sum_i \frac{1}{\alpha_i} < +\infty$ and we construct a bounded measure μ on E'. Let $D = D_{(-\alpha)}D_{(\alpha)}$ so that

$$D = \prod_{i\in I}\Big(1 - \alpha_i^2\big(\frac{\partial}{\partial x_i}\big)^2\Big) \tag{8.1}$$

is a multiplicative analogue of a Laplacian, and define $F = D\mu$. This is functional defined on the space $\mathcal{B}_{(2)}(E)$ of functions having bounded path derivatives of rank 2. By (7.26) the projection of F onto $\mathbb{R}^n$ is the Fresnel distribution with Fourier transform (7.24). The finite sequences being strongly (even sequentially) dense in E, it follows that $\widehat{F}(\xi) = e^{i\langle K\xi,\xi\rangle/2}$. This uniquely determines F (but not of course μ or D). Thus we obtain our main result, in which we denote $F_K = F$ and $F_K(e_y)$ as an integral:

Theorem 8.1. *Let E be a complete nuclear sequence space*[14]*, and let E' be its dual sequence space, equipped with the strong dual topology. Let $\xi \mapsto \langle K\xi,\xi\rangle$ be a continuous quadratic form on E. Then there exists a unique path distribution F_K on E' such that*

$$\int_{E'} e^{-i\langle x,\xi\rangle} F_K(dx) = e^{i\langle K\xi,\xi\rangle/2}, \quad \xi \in E.$$

Moreover $F_K = D\mu$ is a derivative of rank 2 of a bounded Radon measure.

Given the previous arguments, it is sufficient to note that a diagonal transformation $u : x \mapsto x/b$ leaves invariant the path structure and, by the chain rule $D_{(h)}(\Phi \circ u)(x) = (D_{(uh)}\Phi)(ux)$, the form of the representation $D\mu$.

[14]As always assumed barreled and regular.

Corollary 8.2. *If $\langle K\xi,\xi\rangle \geqslant 0$ for all $\xi \in E$, there exists for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geqslant 0$ a unique path distribution F_λ on E' such that*

$$\int_{E'} e^{-i\langle x,\xi\rangle} F_\lambda(dx) = e^{-\lambda\langle K\xi,\xi\rangle/2}, \quad \xi \in E. \tag{8.2}$$

Proof. If $\lambda = \alpha - i\beta$, with $\alpha, \beta \in \mathbb{R}$ it suffices to take $F_\lambda = F_{\beta K} * G_{\alpha K}$ (cf. 6.7). ∎

Example 8.3. Let $I = \mathbb{Z}$ or $\mathbb{N}$ $\langle K\xi,\xi\rangle = \sum_{i\in I} \sigma_i \xi_i^2/2$. Then the product Fresnel distribution F_K exists on $s'(I)$ (resp. $s(I)$) iff $(\sigma_i)_{i\in I} \in s'(I)$ (resp. $s(I)$).

Example 8.4. Let $I = \mathbb{Z}_+$, $K_{i,j} = \min(i,j)$. Then the quadratic form

$$\langle K\xi,\xi\rangle = \sum_{i\geqslant 0, j\geqslant 0} K_{i,j}\xi_i\xi_j \tag{8.3}$$

is continuous on the space $s(\mathbb{Z}_+)$. Thus there exists a unique summable distribution F_K on the space $s'(\mathbb{Z}_+)$ of sequences of polynomial growth such that $\widehat{F_K}(\xi) = e^{i\langle K\xi,\xi\rangle/2}$.

Given the analogy with the Feynman integral one would expect F_K to be (strongly) concentrated on a much smaller space, but this will have to be left to another occasion.

References

[1] Albeverio, S. and R. Hoegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics, Vol. 523, Springer, 1976.

[2] Bourbaki, N., *Livre VI Intégration*, Hermann, Paris, 1969.

[3] Bijma, F., *Generalized Fresnel Distributions*, Undergraduate Thesis, University of Groningen, 1999.

[4] Cameron, R.H. and D.A. Storvick, A simple definition of the Feynman integral, with applications, Memoirs of the AMS, 46, No. 288, 1983, 1–43.

[5] Cartier, P. and C. DeWitt-Morette, A new perspective on Functional Integration, *J. of Math. Physics* **36** (1995), 2137–2340.

[6] Cator, E., *Distributions on Locally Convex Spaces*, Ph.D. Thesis, University of Utrecht, 1997.

[7] DeWitt-Morette, C., Feynman's Path integrals. Definition without limiting procedure, *Comm. Math. Physics* **28** (1972), 47–67.

[8] DeWitt Morette, C., A. Maheshwari and B. Nelson, Path Integration in Non-relativistic Quantum Mechanics, *Physics Reports* **50** (1979), 256–372.

[9] Dieudonné, J. and L. Schwartz, La dualité dans les espaces $\mathcal{F}$ et $\mathcal{LF}$, *Annales de l'Institut Fourier* **1** (1949), 61–101.

[10] Dineen, S., *Complex Analysis in Locally Convex Spaces*, North Holland Math. Studies, Vol. 57, 1981.

[11] Feynman, R.P., The development of the space-time view of Quantum Electrodynamics, *Science* **153** (1966), 699–708.

[12] Feynman, R.P. and A.R. Hibbs, *Quantum Mechanics and Path Integrals,* McGraw Hill, New York, 1965.

[13] Grosche, C., *Path Integrals, Hyperbolic Spaces, and Selberg Trace Formula*, World Scientific, 1995.

[14] Grothendieck, A., *Produits Tensoriels Topologiques et Espaces Nucléaires*, Amer. Math. Soc. Memoir 16, 1955.

[15] Grothendieck, A., *Topological Vector Spaces*, Gordon and Breach, 1992.

[16] Hamilton, R.S., The Inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* **7** (1982), 65–222.

[17] Hörmander, L., *The Analysis of Linear Partial Differential Operators I*, Second Edition, Springer-Verlag, 1990.

[18] Johnson, G.W. and M.L. Lapidus, *The Feynman integral and Feynman's operational calculus*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[19] Kaku, M., *Quantum Field Theory*, Oxford University Press, 1993.

[20] Khandekar, D.C., S.V. Lawande and K.V. Bhagwat, *Path-Integral Methods and Their Applications*, World Scientific, 1993.

[21] Kree, P., Introduction aux théories des distributions en dimension infinie, in: *Journées Gém. Dimens. Infinie*, Lyon (1975), Bull. Soc. Math. France, Mémoire 46, 1976, 143–162.

[22] Muldowney, P., *A General Theory of Integration in Function Spaces*, Pittman Research Notes, Vol. 153, John Wiley & Sons, Inc., New York, 1987.

[23] Reed, M. and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, 1978.

[24] Ryder, L.H., *Quantum Field Theory*, Cambridge University Press, 1996.

[25] Schwartz, L., *Théorie des Distributions*, Hermann, Paris, 1950.

[26] Schwartz, L., Espaces de fonctions differentiables à valeurs vectoriëlles, *Jour. d'Anal. Math. de Jerusalem* **4** (1954-55), 1–61 and 88–148.

[27] Schwartz, L., Distributions à valeurs vectorielles, *Annales de l'Institut Fourier*, tome 7, 1957, tome 8, 1959.

[28] Schwartz, L., *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Oxford University Press, London, 1973.

[29] Schwartz, L., Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés, *Jour. Anal. Math.* **13** (1964), 115–256.

[30] Thomas, E.G.F., Path Integrals on Finite Sets, *Acta Applicandae Math.* **43** (1996), 191–232.

[31] Thomas, E.G.F., Finite Path Integrals, in: *Proceedings of the Norbert Wiener Centenary Congress* (East Lansing, MI, 1994), Proceedings of Symposia in Applied Mathematics, Vol. 52, Amer. Math. Soc., Providence, RI, 1997, 225–232.

[32] Thomas, E.G.F., Projective limits of complex measures and martingale convergence, preprint, 1997.

[33] Thomas, E.G.F., Calculus on locally convex spaces, report, 1996.

[34] Truman, A., The polygonal path formulation of the Feynman path integrals, Lecture Notes in Physics, Vol. 106, Springer-Verlag, 1979, 73-102.

Erik G.F. Thomas
Mathematics Institute
University of Groningen
P.B. 800, 9700 AV Groningen, The Netherlands
E-mail: E.Thomas@math.rug.nl