On the sections of a block of eightcells by a space rotating about a plane.

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INTRODUCTION.

1. If a space of three dimensions S_3 rotates in S_4 about a given fixed plane π the general case is that S_3 and π have only a line l in common. We will restrict ourselves here to the special case where S_3 passes through π.

If we start from a fourfold infinite block of eightcells and cut it by a space the polyhedra of intersection form a threedimensional space-filling. When the position of the intersecting space is an arbitrary one, the number of the polyhedra of different shape is infinite. Here we will restrict ourselves once more to the special cases where the number of the polyhedra of different shape is finite. These *commensurable* cases are characterized by the property that any space parallel to the considered position of the rotating space of intersection and passing through a vertex of one of the eightcells cuts the four edges through the opposite vertex in points the distances of which from that last vertex are commensurable with the length of the edge.

Finally we restrict ourselves to the case of a finite block consisting of $3^4 = 81$ eightcells, forming together an eightcell of three times the size, and we suppose that the fixed plane π passes through the centre of this block and is totally normal to a plane π' containing two opposite edges of it.

I. SECTIONS OF A SINGLE EIGHTCELL BY ANY CENTRAL SPACE NORMAL TO THE PLANE CONTAINING TWO OPPOSITE EDGES.

2. Let O be the centre of the eightcell C_8 and PQ one of the two edges situated in π'. Let R be the midpoint of PQ and $C_6^{(3)}$ the cube of intersection of C_8 with the space through OR normal to PQ. Then OR is a diagonal of $C_6^{(3)}$ and the plane of that space, bisecting that diagonal normally, is the fixed plane π; it cuts $C_6^{(3)}$ in a regular hexagon. So this hexagon is situated in the boundary of the solid that forms the intersection of C_8 by any space containing π, i. e. by any central space normal to the plane π' through O and PQ. This hexagon will be indicated by h_6.

3. We will try to smooth the way to an exact knowledge of the sections in question by considering the projection of C_8 on the plane π'. It consists (fig. 1) of a rectangle with sides $PQ = 1$ and $PP' = \sqrt{3}$ (where the length of the edge of C_8 is unity) divided by two parallels P_1Q_1 and P_2Q_2 to PQ into three equal rectangles. We indicate successively the projections of the 16 vertices, the 32 edges, the 24 faces and the 8 limiting cubes.

The vertices are

$$(P + 3\,P_1 + 3\,P_2 + P') + (Q + 3\,Q_1 + 3\,Q_2 + Q').$$

The edges are

$$(3\,PP_1 + 6\,P_1P_2 + 3\,P_2P') + (3\,QQ_1 + 6\,Q_1Q_2 + 3\,Q_2Q') \\ + (PQ + 3\,P_1Q_1 + 3\,P_2Q_2 + P'Q').$$

The faces are

$$(3\,PP_1P_2 + 3\,P_1P_2P') + (3\,QQ_1Q_2 + 3\,Q_1Q_2Q') \\ + (3\,PP_1Q_1Q + 6\,P_1P_2Q_2Q_1 + 3\,P_2P'Q'Q_2).$$

The limiting bodies are

$$PP_1P_2P' + QQ_1Q_2Q' + 3\,PP_1P_2Q_2Q_1Q + 3\,P_1P_2P'Q'Q_2Q_1.$$

We call the cubes projecting themselves in the lines PP_1P_2P' and QQ_1Q_2Q' the upper and lower cubes, the six other limiting bodies the side-cubes.

Any line TT_1T_2T' parallel to PP_1P_2P' is the projection of a cube

of intersection of C_8 with a space parallel to these bearing the upper and lower cubes. So any point U within the rectangle PP_1Q_1Q (or $P_2P'Q'Q_2$) is the projection of an equilateral triangle, which is the intersection of C_8 with the plane, in U entirely normal to the plane of the diagram. Likewise any point V within the rectangle $P_1P_2Q_2Q_1$ is the projection of an equiangular hexagon with alternately equal sides, which becomes a regular hexagon for the points V situated like O at equal distances from P_1Q_1 and P_2Q_2. Of these triangles and hexagons the projections U and V form the centres.

An intersecting space $S_3(\pi)$ through π projects itself on the plane of the diagram as a line l through O; therefore the section itself is represented in projection by the segment of l situated within the rectangle $PP'Q'Q$. According to the position of that line-segment we distinguish three different cases of intersection; if ϕ designates the absolute value of the acute angle between l and its position l_0 in which it is parallel to PP', which angle π we call the *angle of rotation*, the three cases are:

$$1^{st}.\ 0° < \phi < 30°, \quad 2^{nd}.\ 30° < \phi < 60°, \quad 3^{rd}.\ 60° < \phi < 90°.$$

In the first case the endpoints of the segment lie in the edges PQ, $P'Q'$ and the space $S_3(\pi)$ has no points in common with the upper and lower cubes: the section is limited by *three* pairs of parallel planes, whilst in the two other cases it is included by *four* pairs of parallel planes. In the second case the segment still contains points lying outside $P_1P_2Q_2Q_1$: by some of the planes normal to l the section is cut in semiregular hexagons, by others in equilateral triangles. In the third case no point of the segment lies without $P_1P_2Q_2Q_1$: the sections by planes normal to l are exclusively hexagons. We will consider each of these cases separately. But first we wish to make a general remark.

If we consider the line-segment that forms the projection of the solid of intersection as the locus of the points it contains, the section itself appears as built up of an infinite number of infinitely thin slices. Now in the space $S_3(\pi)$, bearing the section, the projection l normal to π in O cuts the plane of any of these slices at right angles in its centre. As the slices of the two different kinds (equilateral triangles and equiangular hexagons) equally admit of a rotation of $\pm 120°$ about their centres, in their planes, we find:

"By a rotation of $\pm 120°$ about its line of projection l the section is transformed into itself; in other words l is an axis of the section, of period three".

4. The case $\phi < 30°$. In this case the six side-cubes are cut

in congruent lozenges including a rhombohedron; for this is the solid bounded by three pairs of parallel planes admitting of an axis of period three.

We deduce the exact form of the lozenges limiting the section from another source. If $H_1H_2H_3H_4H_5H_6$ (fig. 2) is the hexagon h_6 situated in the plane π, OX_1 and OX_2 are two axes of symmetry of h_6 normal to each other and OX_3 is the axis of the rhombohedron bearing the endpoints A, A' at equal distances from O, then the planes containing the lozenges are

$$\begin{array}{lll} A(H_1H_2), & A(H_3H_4), & A(H_5H_6), \\ A'(H_4H_5), & A'(H_6H_1), & A'(H_2H_3). \end{array}$$

Now the line of intersection AD of the planes connecting A with H_1H_2 and H_3H_4 will meet π in the point of intersection P of H_1H_2 and H_3H_4, etc. So the edges through A and A' are found by joining A to the vertices of the triangle PQR and A' to the vertices of the triangle $P'Q'R'$. Then the figure is completed by drawing through the pairs of opposite vertices (H_1, H_4), (H_2, H_5), (H_3, H_6) of h_6 lines respectively parallel to AP, AQ, AR.

Now from $PH_2 = \frac{1}{3} PQ$ we deduce $DP = \frac{1}{3} AP$, etc.; so the three points B, D, F project themselves on the axis OA in the same point K for which $OK = \frac{1}{3} OA$ (in accordance with the relation $OB = \frac{1}{3} OA$ in fig. 1). Likewise C, E, G project themselves in the same point L for which $OL = \frac{1}{3} OA' = OK$, whilst O is the projection of the six points H_1, H_2, ..., H_6 on the axis. So we find $BH_1 = H_1C$, the projections KO, OL of these segments of the same line on the axis being equal, i. e. the vertices H_1, H_2, ..., H_6 of h_6 are the midpoints of the sides of the skew hexagon $BCDEFG$. This gives us the relation $BD = 2\, H_1H_2$. So the diagonals of the lozenges crossing the axis AA', or as we will say the *transverse* diagonals, are equal to $\sqrt{2}$, the edge of the eightcell being unity, for in this unit we have $H_1H_2 = \frac{1}{2}\sqrt{2}$. [1])

For the other diagonals intersecting the axis, the *axial* diagonals, we find

$$AC = \tfrac{4}{3} AN = \tfrac{4}{3}\sqrt{OA^2 + ON^2} =$$
$$= \tfrac{4}{3}\sqrt{OA^2 + (\tfrac{3}{2}\sqrt{3})^2} = \sqrt{(\tfrac{4}{3} OA)^2 + (2\sqrt{3})^2}.$$

Now in fig. 1 we have $B'A = \frac{4}{3} OA$; so AC is the hypotenuse of a rectangular triangle with the kathetæ $B'A$ and $B'C = 2\sqrt{3}$.

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[1]) We henceforward suppose the side of h_6 to be 3 *cm*; then $PQ = 3\sqrt{2}$ *cm*. and $PP' = 3\sqrt{6}$ *cm*.

Resuming we can say that for the case under consideration the projection on the plane π' (fig. 1) provides us with the means to construct the rhombohedrical section, OA being the length of its semiaxis, whilst AC and 6 *cm.* represent the diagonals of the limiting lozenges. We apply this in fig. 3, where the rectangle $PP'Q'Q$ with the two lines P_1Q_1, P_2Q_2 dividing it into three equal triangles is repeated in an altered position, to the two extreme cases of this first group of sections, i. e. for $\phi = 0$ and for $\phi = 30°$. By means of the rectangular triangles $A_1B_1'C_1$ and $A_1B_2'C_2$ the axial diagonals A_1C_1 and A_2C_2 are found and on these the lozenges $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ with transverse diagonals $B_1D_1 = B_2D_2 = 6$ *cm.* are constructed. In accordance with the fact that the section is a cube for $\phi = 0$, $A_1B_1C_1D_1$ is a square. In all other cases the axial diagonal AC is longer than the transverse diagonal BD; so, if we like, we may call AC the macrodiagonal and BD the brachidiagonal.

Now whilst ϕ increases from 0° to 30° the section (fig. 2) changes in a simple manner. Whilst the intersection of the rhombohedron with the central plane π normal to the axis is and remains the regular hexagon h_6, the two endpoints A, A' of the axis move away from O, starting from a distance $\frac{3}{2}\sqrt{6}$ *cm.*, to a distance $3\sqrt{2}$ *cm.* So we may say that the cube corresponding to the original position of the rotating space is stretched out in the direction of one of its diagonals into a rhombohedron.

If we suppose for a moment that the considered eightcell forms part of an infinite pile of eightcells built up in the direction of the edge PQ, so that the lower cube of an upper one coincides with the upper cube of the next lower one, and we disregard the limiting cubes common to two adjoining eightcells — as if we wished to change a range of cells into a long vessel by removing the interior diaphragms — we obtain a fourdimensional prism, the right section of which is a cube. This prism will be cut by *any* space in a rhombohedron, which only transforms itself into a hexagonal prism with h_6 as right section for $\phi = 90°$; this result will be applied directly.

5. The case $30° < \phi < 60°$. In this and in the following case the rhombohedrical section of the space $S_3(\pi)$ with the fourdimensional prism found above is truncated by two planes perpendicular to the axis AA' in the points W, W' where that axis meets the projections PP' and QQ' of the upper and lower cubes. Here the points W, W' lie on PP_1, Q_2Q'; so the endplanes of the truncated rhombohedron bear equilateral triangles and

the equal portions of the rhombohedron lying outside the eightcell are regular triangular pyramids. In other words: the distance of the endplanes to the plane π must exceed that of the vertices of the skew hexagon $BCDEFG$ to that selfsame plane, the sides of that hexagon not being cut by the endplanes.

In fig. 3 (in which we lay down the different results) the faces of the section are constructed for the two cases of this group we wish to consider, the cases where the endpoint W of the line-segment l situated on PP' is the point W_3 halfway between P and P_1 and where this point W_4 coincides with P_1. The figure shows the two lozenges $A_3B_3C_3D_3$ and $A_4B_4C_4D_4$, obtained in the way described. By means of the point W_3' where the normal in W_3 on A_3B_3' cuts A_3C_3 the line E_3F_3 normal to A_3C_3 is found, giving us in its turn the truncated lozenge $B_3C_3D_3E_3F_3$; the section is limited by six of these truncated lozenges and two equilateral triangles with E_3F_3 as side. In repeating this construction for $\phi = 60°$, where W_4 coincides with P_1 we find for W_4' the centre of the lozenge $A_4B_4C_4D_4$, i. e. the truncated lozenge becomes an equilateral triangle $B_4C_4D_4$, the section itself is an octahedron. This is as it should be: in the case $\phi = 60°$ the projection AA' (fig. 1) is normal to the diagonal $P'Q$ of the eightcell and therefore S_3 (π) is normal to that line.

6. The case $60° < \phi < 90$. Here the rhombohedron is truncated at both ends by planes normal to the axis AA' bearing hexagonal sections.

In fig. 3 the sections with the side-cubes are constructed for the two cases determined by the relations $P_2W_5 = 3\ W_5P_1$ and $P_2W_6 = 2\ W_6P_1$. The results, obtained in the indicated way, are the isosceles trapezia $E_5F_5G_5H_5$ and $E_6F_6G_6H_6$; the corresponding endplanes bear equiangular hexagons with the alternate sides equal to E_5F_5, G_5H_5 and to E_6F_6, G_6H_6. The manner in which the midpoints $\overline{W}_5'$, $\overline{W}_6'$ and W_5', W_6' of the lines EF and GH on the macrodiagonal AC are deduced from $\overline{W}_5$, $\overline{W}_6$ and W_5, W_6 is indicated in the figure.

In the extreme case $\phi = 90°$ the section becomes a hexagonal prism bounded by six rectangles with sides of $3\sqrt{2}$ *cm.* and 3 *cm.* and by two regular hexagons with sides of 3 *cm.*

7. After this explanation we think it will be evident to the reader that fig. 3 enables us to make cardboard models of the seven sections considered. [1]) But this figure can teach us quite as well

[1]) A more expeditious method will be given farther on.

how to find the images of the different sections in parallel projection, in the manner in which fig. 2 already represents the second case $\phi = 30°$. For the new problem, which makes its appearance in the third case of the pentagon $B_3C_3D_3E_3F_3$, viz. to find the triangles (and hexagons) situated in the planes truncating the rhombohedron, is easily solved. So in fig. 4, corresponding to that case, the line WN' through W parallel to ON meets AC in a point N' of EF and this construction always holds; in this special case it is more to the point however to remember that E is the midpoint of AD, etc.

In fig. 4 the angle ANO indicates the inclination of the faces of the section on the plane π of the hexagon h_6. It is easy to find the general relation between this angle ψ (fig. 1) and the angle of rotation ϕ. From the two relations

$$OR = OA \cos \phi, \quad OA = ON \, tang \, \psi$$

can be deduced

$$tang \, \psi = sec \, \phi \sqrt{2}.$$

We lay down what we have hitherto found in the following:

TABLE OF RESULTS.

Section.	FACES.	Tang ϕ	Tang ψ	Angle ϕ	Angle ψ
I	6 squares $A_1B_1C_1D_1$ (*cube*).....	0	$\sqrt{2}$	0°	54° 44′ 9″
II	6 lozenges $A_2B_2C_2D_2$.....	$\frac{1}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{6}$	30°	58° 30′ 54″
III	6 pentagons $B_3C_3D_3E_3F_3$ and 2 equilateral triangles (side E_3F_3).....	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{14}$	40° 53′ 36″	61° 52′ 28″
IV	8 equilateral triangles $B_4C_4D_4$ (*octahedron*)	$\sqrt{3}$	$2\sqrt{2}$	60°	70° 31′ 44″
V	6 trapezia $E_5F_5G_5H_5$ and two equiangular hexagons (sides E_5F_5, G_5H_5).....	$2\sqrt{3}$	$\sqrt{26}$	73° 53′ 52″	78° 54′ 15″
VI	6 trapezia $E_6F_6G_6H_6$ and two equiangular hexagons (sides E_6F_6, G_6H_6).....	$3\sqrt{3}$	$2\sqrt{14}$	79° 6′ 24″	84° 42′ 43″
VII	6 rectangles ($3\sqrt{2}$ *cm* and 3 *cm*) and 2 h_6 (*hexagonal prism*).....	∞	∞	90°	90°

8. Before we can pass to the study of the block of 81 eightcells we must say a word about the section of a single eightcell by a space normal to the plane π' but not passing through the

centre O; for it goes without saying that a space central to the central eightcell of the block is in general excentric with respect to the other eightcells of it.

If the projection l of the excentric intersecting space is inclined to the edge PQ, the section is either a whole rhombohedron or a part of it. The first case occurs when the two endpoints of the segment of l lying within the rectangle $PQQ'P'$ are points of the edges PQ, $P'Q'$; in the other case the section is a truncated rhombohedron, truncated at one end or at both ends according to whether one or neither of these endpoints lie on these edges. If it is truncated at one end it may happen that we have to deal with a rhombohedron of which only a triangular pyramid is cut off, or left; or that the endplane bears a hexagon, and the section is greater than, equal to, or less than half the rhombohedron. If it is truncated at both ends the two endplanes can be situated on different sides of the middle plane containing a regular hexagon equal to h_6, or on the same side; so we may even obtain a truncated triangular pyramid.

9. The method of investigating the sections of an eightcell by a space rotating about a central plane π by means of the projection in the plane π' in the centre totally normal to it has been brought most to the front in the preceding pages, as indeed it is our opinion that this method is preferable for its generality to any other [1]), which opinion is strengthened by its applicability to a block of eightcells. But still we must acknowledge that, with regard to the construction of the side-faces of the rhombohedrical sections, a more direct method, more suitable to the wants and needs of the maker of cardboard models, may be obtained by cutting one

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[1]) If we suppose in space S_n of n dimensions a measure-polytope P_{2n} to be given and we have to consider the sections of this polytope by a pencil of central spaces S_{n-1} normal to the plane π' passing through two opposite edges PQ, $P'Q'$, the same method can be used. In this case the projection of the given polytope P_{2n} on to the plane π is a rectangle divided into n—1 equal rectangles by n—2 lines parallel to PQ, $P'Q'$. The section is either a measure-polytope $P_{2(n-1)}$ of n—1-dimensional space, stretched out in the direction of one of the diagonals figuring as axis with the period n—1, which may be called a rhombotope, or a mutilated rhombotope truncated at both ends. With regard to the section of P_{2n} with the central space S_{n-2} to which the plane π' is entirely normal, which n—2-dimensional solid is common to all the n—1-dimensional sections, we can only remark here, that its vertices are vertices of P_{2n} or midpoints of edges of P_{2n} according to n being even or odd. We add that a new method of dealing with this n—2-dimensional section, which is indeed a generalisation of the fact that in fig. 2 the hexagon h_6 is the part of the plane π common to the two equilateral triangles PQR, $P'Q'R'$, will appear shortly in the *Proceedings* of this Academy (*Verslagen*, Dec. 1907, p. 467, *Proceedings*, Jan. 1908, p. 485).

of the side-cubes of the eightcell by the rotating space. We will close this first section by explaining this.

We consider that side-cube of the eightcell of which the lozenge $ABCD$ (fig. 2) is a section, one of the three side-cubes projecting themselves on the rectangle PP_2Q_2Q (fig. 1). The space of that cube meets the plane π in the line H_1H_2 (fig. 2); this side of the hexagon h_6 joins the centres H_1, H_2 of the two adjacent faces of the cube projecting themselves on to the rectangle $P_1P_2Q_2Q_1$ (fig. 1) the centre O of which forms the projection of π and therefore of H_1H_2.

We project the chosen side-cube on to the plane passing through PQ and the centre of the cube — and containing therefore the opposite edge which again may be called P_2Q_2 (fig. 5) — as a rectangle PP_2Q_2Q with the sides $PQ = 3\sqrt{2}$ *cm.*, $PP_2 = 6$ *cm.* and remark that the centre O of $P_1P_2Q_2Q_1$ — where P_1 and Q_1 bisect PP_2 and QQ_2 — is still the projection of H_1 and H_2, and therefore of H_1H_2. This projection PP_2Q_2Q ($PP_2 = 6$ *cm.*) differs from the projection PP_2Q_2Q ($PP_2 = 2\sqrt{6}$ *cm.*) of fig. 1 and fig. 3, which difference is due to the fact that the planes of projection differ. Indeed the plane of projection π' of fig. 1 and fig. 3 passes through PQ and the centre of the eightcell, whilst that of fig. 5 is determined by PQ and the centre of the chosen side-cube; the first plane π' indicates the angles of rotation of the intersecting space turning about π, the second cannot do this. So the question arises: „how can we find the lines of intersection l_1', l_2', ... l_7' of the new plane of projection with the seven considered positions of the rotating space?" To obtain an answer we remark that any line of this new plane of projection, e. g. the diagonal PP_2 of the face of the cube which projects itself on PP_2, is divided by the different points of intersection with the lines l_1', l_2', ... l_7' we are bent on determining, in the same proportion as the projection PP_2 of that selfsame diagonal on π' is divided by the corresponding lines l_1, l_2, ... l_7 of fig. 3. After having obtained these lines (fig. 5) the construction of the side-faces is very easy. Indeed we know that the normal erected in the threedimensional space of the cube on the plane PP_2Q_2Q in any point U within that rectangle, reckoned from U to either of the points of intersection with the boundary of the cube, is equal to the distance of that point U from the nearer of the two edges PQ, P_2Q_2. So in the case III the half $C_3D_3E_3W_3$ of the pentagon $B_3C_3D_3E_3F_3$ of fig. 3 is obtained by erecting in M_3 and W_3 normals $M_3D_3 = P_1P$ and $W_3E_3 = W_3P$ on l_3'; so are found the halves $A_1C_1D_1$, C_2D_2P,

$C_3D_3E_3W_3$, $P_1Q_2D_4$, $F_5G_5W_5W_5'$, $F_6G_6W_6W_6'$ of the six side-faces already given in fig. 3 and moreover in $P_1W_7W_7'Q_1$ even the half of the side-face $P_1P_2Q_2Q_1$ of the hexagonal prism of case VII.

It goes nearly without saying that also by this new simple method the shape and size of *all* the faces can be found. So in the case V the endplanes of the section are equiangular hexagons with alternately equal sides E_5F_5, G_5H_5. And by constructing the unknown katheta of a right-angled triangle, the hypotenuse of which is the segment of l_5' within PP_2Q_2Q while one katheta is the difference of the distances of the sides E_5F_5, G_5H_5 to the centre of the hexagon, we find the distance of the endplanes, which will enable us to make a drawing of the corresponding solid in parallel perspective, etc.

II. SECTIONS OF THE BLOCK OF EIGHTCELLS.

10. Before entering upon the subject of the sections it will be well to say a few words about the block itself. We suppose 81 eightcells to be built up into a rectangular block having three along each edge and, in the fourdimensional space containing the block that is itself an eightcell of three times the size, we imagine a plane π' passing through two opposite edges of this large block and therefore containing its centre, which may still be indicated by O. Now the block is to be cut by a space $S_3(\pi)$ normal to π' in O and containing therefore the plane π in O normal to π'. This plane π cuts the large block in a regular hexagon, the sides of which are three times the side of h_6; we will call it h_6'.

Now in the initial position of the intersecting space $S_3(\pi)$, the case $\phi = 0$, the section evidently consists of a block of 27 cubes, central sections of 27 of the 81 eightcells, forming the "middle layer" of the block, while the other two layers lie beyond $S_3(\pi)$ in opposite directions (parallel to the two edges situated in π') which may be called "above" and "below" or "plus" and "minus". This section of a very simple character is represented in fig. 6 [1]) in parallel perspective, in the same manner as fig. 2; but in order not to obtain too large figures in future we will suppose the sides of h_6 to be 1 *cm.*, those of h_6' 3 *cm.* As fig. 6 shows, the cubes of this section have been numbered in a definite way 1, 2, 3, . . .,

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[1]) The shading of the figs. 6, 7, 10 will be explained later on.

27, which are in reality the numbers we assign to the eightcells of the middle layer; in the same manner the eightcells of the layer above are designated by $+1, +2, +3, \ldots, +27$, those of the layer below by $-1, -2, -3, \ldots, -27$, so that above and below 1 are $+1$ and -1, etc.

It may be remembered that, while each of the 27 cubes of the block is a central section of an eightcell parallel to two of its opposite limiting cubes (the upper cube and the lower cube), each face is a section of a side-cube, each edge a section of a face and each vertex a section of an edge. Also the cubes which are in plane, line, or point contact — that means the cubes which have a plane, line, or point in common — are sections of eightcells respectively in space, plane, or line contact.

11. In order to identify the limiting cubes of each eightcell as their sections with the space $S_3(\pi)$ take different forms and positions we distinguish them by letters. We suppose the cubes a, b, c, d, to meet at one of the vertices of the eightcell and represent the parallel cubes meeting at the diagonally opposite vertex by a', b', c', d'; of these d and d' are the upper cube and the lower cube. The visible faces of the cube 1 (fig. 6) correspond in this manner to the side cubes a, b', c of the eightcell 1.

The three pairs of parallel planes dividing the large cube of three times the size into the 27 separate cubes are the intersections of the initial position of the space $S_3(\pi)$ with three of the four pairs of parrallel spaces dividing the large eightcell of three times the size into the 81 separate eightcells. The fourth pair of dividing parallel spaces is parallel to $S_3(\pi)$ and divides the fourdimensional block into the three layers which we have called middle layer, plus layer and minus layer. We may illustrate the positions of the cubes limiting the eightcells, and the manner in which the eightcells are numbered and their limiting cubes lettered, by giving a small table of the space contact of the eightcell 14, the central one of the block, with the surrounding ones. The eightcell 14 is:

in	aa'	contact	with	the	eightcell	11,
„	bb'	„	„	„	„	23,
„	cc'	„	„	„	„	13,
„	$a'a$	„	„	„	„	17,
„	$b'b$	„	„	„	„	5,
„	$c'c$	„	„	„	„	15,
„	dd'	„	„	„	„	$+14$,
„	$d'd$	„	„	„	„	-14.

Here we mean that the a cube of eightcell 13 coincides with the a' cube of eightcell 11, etc. [1]).

The section of the block of cubes by the plane π is shewn in fig. 7, the letters a, b', c, a', b, c' refering to the limiting cubes of the block itself. It will be seen that, as to their sections with this plane, the intersected cubes can be arranged in three groups. The first consists of 2, 4, 12, 14, 16, 24, 26 cut in regular hexagons, the second of 1, 11, 13, 21, 23, 25 cut in equal and similarly placed equilateral triangles, the third of 3, 5, 7, 15, 17, 27 also cut in equal and similarly placed equilateral triangles, oppositely placed with respect to those of the second group. And the non-intersected cubes, eight in number, consist, as to the distance of their centres from π, of four different groups

19; 10, 20, 22; 6, 8, 18; 9.

So we find seven groups in all; they will reappear directly in a more important point of view.

12. Let us now consider the projection of the 81 eightcells on to the plane π'. In the above adopted reduced scale the projection of the block (fig. 8) is a rectangle with sides $3\sqrt{2}$ *cm.* and $3\sqrt{6}$ *cm.*; in this large rectangle the concentric similar and similarly placed rectangle $PP'Q'Q$, the sides of which are $\sqrt{2}$ *cm.* and $\sqrt{6}$ *cm.*, represents the projection of the central eightcell 14. By producing PP' and QQ' the large rectangle is divided into three strips, an upper one, a middle one and a lower one, which are evidently the projections of the three sets of 27 eightcells, forming respectively the plus layer, the middle layer and the minus layer. Now the manner in which the 27 eightcells of the middle layer project themselves on the middle strip $P_0P_0'Q_0'Q_0$ can be deduced from that in which the 27 cubes of the section by the initial position of the space $S_3(\pi)$ project themselves on the line AA' which forms the projection of $S_3(\pi)$ and is at the same time (fig. 6) a diagonal of the block of the 27 cubes. So we have now to consider for a moment the projection of the 27 cubes on the diagonal AA'.

If we first direct our attention to the vertices and begin (fig. 6) with those of the large cube we find as projections the two extremities A, A' and two points dividing AA' into three equal parts. If then we pass to the cubes 9, 14, 19 we see that the projec-

[1]) In an infinite block of eightcells the numbers of eightcells respectively in space, plane, line and point contact with any eightcell considered as the central one are equal to the numbers of limiting bodies, faces, edges, vertices of that eightcell, i. e. 8, 24, 32, 16. So we find, the central eightcell included, 81 eightcells in all, i. e. all the eightcells of our block are wanted in order to include the central one entirely.

tions of their vertices consist of A', A and eight points dividing AA' into nine equal parts. So we easily convince ourselves of the fact that the 64 vertices of the 27 cubes project themselves into the indicated ten points [1]) and that the projection of any of the 27 cubes on AA' covers a third part of that line limited by two of these ten points. Returning to the projection of the 27 eightcells of

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[1]) If the three edges of the block of cubes concurring in A are taken as positive axes of coordinates and we suppose the edge of the cubes to be unity, the coordinates of the vertices are (p, q, r), where p, q, r are to be chosen from 0, 1, 2, 3. So, if $n(p, q, r)$ represents the number n of permutations with definite p, q, r and we join together the numbers $n(p, q, r)$ for which $p+q+r$ has the same value, we find groups of vertices lying in the same plane $x+y+z=$ constant, i. e. points with a common projection on AA'. In this manner we get

$$1(000),\ 3(100),\ 3(200)+3(110),\ \ 3(300)+6(210)+1(111),$$
$$6(310)+3(220)+3(211),\ \text{etc.}$$

i. e. groups of 1, 3, 6, 10, 12, 12, 10, 6, 3, 1 vertices projecting themselves in the ten different points on AA'.

This result admits of the following general extension:

"If the $(p+1)^d$ vertices of a block of p^d measure-polytopes of d-dimensional space (arranged in the form of a measure-polytope of p times the size) are projected on to a central diagonal of the block, the projections are the endpoints of that diagonal and the $pd-1$ points that divide it into pd equal parts. Moreover the numbers of the projections coinciding respectively with these $pd+1$ points are the coefficients of the successive powers of x in the polynomial $(1+x+x^2+\ldots+x^p)^d$."

No doubt this theorem can be proved in different ways, e. g. by the induction from $p-1$ to p and $d-1$ to d, or more simply still by the generation of the configuration of the vertices, proceeding from a row of $p+1$ equidistant points to a row of $p+1$ squares, etc. Of the demonstration connected with the manner in which the theorem was found we can only give a summary sketch here. It consists of three parts. In the first part we consider the special case p infinite, in which the block fills up the $\frac{1}{2^d}$ th part of d-dimensional space corresponding to the positive sense of the d axes OX_1, $OX_2, \ldots OX_d$; in that case the result depends, in the way indicated in the theorem, on the form $(1-x)^{-d}$. In the second part we show that the finite block of the theorem may be considered as the algebraic sum of 2^d infinite blocks equipollent to the infinite block of the first part but showing with regard to the position of the vertex at finite distance that originally occupied the origin the effect of a parallel translation, any of the vertices of the finite block forming the origin of one of these 2^d infinite ones; in this algebraic sum the sign of any of the 2^d infinite blocks is positive or negative according to the number of coordinates of its origin differing from nought being even or odd. In the third part we prove that this sum of infinite blocks can be represented by $(1-B)^d$, if B^k represents one of these blocks, the system of coordinates of which origin consists of k units and $d-k$ noughts (1 standing for B^0, i. e. for the infinite block in its original position) and then it is clear that this geometric composition of the 2^d infinite blocks to the finite one is to be translated algebraically by multiplying the result $(1-x)^{-d}$ by $(1-x^{p+1})^d$. So we find $\left(\frac{1-x^{p+1}}{1-x}\right)^d$ or $(1+x+x^2+\ldots x^p)^d$.

We may add that the general theorem given above is also an extension of the problem in how many ways the word *abracadabra* written in a triangle may be read, or in how many ways the king of the chessboard can march from a given row to any row k squares higher (see E. Lucas, "*Théorie des nombres*", p. 13, 14).

the middle layer on the middle strip of fig. 8 we find that the projection of any of these eightcells covers a third part of the strip, consisting of three of the nine parts into which this strip can be divided by lines parallel to P_0Q_0. Now, if we arrange (fig. 6) the 27 cubes in groups, and reckon as belonging to the same group these cubes that cover in projection on AA' the same third part, we alight evidently on the seven groups found above. So in order to apply this result to the projection of the block of eightcells, we have repeated under the projection (fig. 8) the line AA' in seven parallel positions A_1A_1', A_2A_2', . . ., A_7A_7', indicated on each of them the position of the third part that forms the projection and inscribed at each of these third parts the numbers

19; 10, 20, 22; 1, 11, 13, 21, 23, 25; etc.

of the eightcells projecting themselves on the corresponding third part of the strip. For convenience we will designate these groups of eightcells by G_1, G_3, G_6, G_7, G_6', G_3', G_1' indicating by the subscript the number of the eightcells, and we will add the signs + and — to the corresponding groups of the plus layer and the minus layer [1]).

13. Now we are quite provided with the means of determining, for any arbitrary position of the intersecting space $S_3(\pi)$, the number, form and size of the different solids, sections of separate eightcells which fill up the section of the large block of eightcells, which is itself an eightcell of three times the size. We may even assert that this process, complicated as it seems, in reality is easier than another to which we are accustomed and which we perform daily: "to see what o'clock it is." For our dial — the plane π' on to which the block of eightcells has been projected — has one hand only — the projection of the intersecting space turning round O. But in order to facilitate the enumeration of the results it will be well to introduce beforehand a simple notation for the different kinds of solids we obtain.

As we have seen the section of any of the 81 eightcells is a

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[1]) If we replace the 27 cubes by their centres, the projection of these centres on a diagonal of the block of cubes gives us a range of seven points coinciding with the centres of the rectangles P_0PQQ_0, $P'P_0'Q_0'Q'$ and the five points dividing the segment of AA' limited at these two points into six equal parts. So the numbers 1, 3, 6, 7, 6, 3, 1 are found — compare the preceding note — as the coefficients of the powers of x in $(1+x+x^2)^3$.

In the extension of the problem to the measure-polytope $P_{2n}^{(p)}$ with edges equal to p units of space S_n divided into p^n measure-polytopes $P_{2n}^{(1)}$ with edges unity, the numbers of polytopes of the groups G coinciding in projection on π' (compare the preceding note) are the coefficients of the powers of x in $(1+x+x^2+\ldots+x^{p-1})^n$.

rhombohedron or a part of it, obtained by truncation at one end or at both ends by planes normal to the axis; if we except the special case $\phi = 90°$, which will be treated separately, this rule is general. Now this section can be designated by the symbol $a(p, q)$, where a represents the length of the axis of the unmutilated rhombohedron expressed in *cm.*, whilst p and q indicate the fractions of the axis cut off by the endplanes, the first of these parameters refering to that endplane the external normal of which corresponds in sense to $A'A$. For the unmutilated rhombohedron is determined by the hexagonal section normal to the axis in the centre, which section is the same for all the rhombohedra, and the length of the axis. So $a(0, 0)$ stands for a whole rhombohedron, $a(p, 0)$ or $a(0, p)$ for a rhombohedron truncated at one end, $a(p, p)$ for a rhombohedron truncated at both ends admitting of a plane normal to the axis dividing it into two congruent halves; so $a(p, q)$ transforms itself into $a(q, p)$ by a rotation through 180° about any line normal to the axis, etc.

14. We consider the case III ($\tang \phi = \frac{1}{2}\sqrt{3}$) in detail, and to that end we reproduce in fig. 9 the projection of the block of eightcells on π' and the projection $l = AB_1C_1 \ldots A'$ of the intersecting space. Here within the rectangle $PP'Q'Q$ that is the projection of any eightcell of the group G_7, let us say the central one 14, lies the segment C_2C_2' of l, whilst the points B_3, B_3' on the edges PQ, $P'Q'$ are the endpoints of the axis of the rhombohedron. So, as $B_3C_2 = C_2'B_3'$ is a sixth part of B_3B_3', we find for the section $\frac{1}{2}\sqrt{42}\,(\frac{1}{6}, \frac{1}{6})$.

In general the value of a is $\sec \phi \sqrt{6}$; in the following table, which gives us for all the groups of intersected eightcells 1st. the segment of l lying within the projection, 2nd. the segment of l representing the axis of the unmutilated rhombohedron, 3rd. the symbol $a(p, q)$, this value of a is indicated at the head, which allows us to use the simplified symbol (p, q).

TABLE OF RESULTS FOR CASE III,

$$\operatorname{tang}\phi = \tfrac{1}{2}\sqrt{3}\,,\ \operatorname{tang}\psi = \tfrac{1}{2}\sqrt{14}\,,\ a = \tfrac{1}{2}\sqrt{42}$$

G_7	$C_2\,C_2'$	$B_2\,B_2'$	$(\frac{1}{6}, \frac{1}{6})$	$+G_7$	B_2C_2	B_2B_2'	$(0, \frac{5}{6})$	$-G_7$	$C_2'B_2'$	$B_2\,B_2'$	$(\frac{5}{6}, 0)$
G_6	$C_2\,B_4'$	$B_2\,B_4'$	$(\frac{1}{2}, 0)$	$+G_6$	B_2C_2	B_2B_4'	$(0, \frac{1}{2})$	$-G_6'$	$C_2'B_2'$	$B_4\,B_2'$	$(\frac{1}{2}, 0)$
G_6'	$B_4\,C_2'$	$B_4\,B_2'$	$(0, \frac{1}{2})$	$+G_3$	C_1C_2	B_1B_4	$(\frac{1}{6}, \frac{1}{6})$	$-G_3'$	$C_2'C_1'$	$B_4'B_1'$	$(\frac{1}{6}, \frac{1}{6})$
G_3	$C_2\,B_4$	$B_1\,B_4$	$(\frac{5}{6}, 0)$	$+G_1$	C_1B_3	$A\,B_3$	$(\frac{1}{2}, 0)$	$-G_1'$	$B_3'C_1'$	$B_3'A'$	$(0, \frac{1}{2})$
G_3'	$B_4'C_2'$	$B_4'B_1'$	$(0, \frac{5}{6})$								

So if we represent the really different pieces $(\frac{1}{6}, \frac{1}{6})$, $(\frac{1}{2}, 0)$, $(\frac{5}{6}, 0)$ by A, B, C we find:

G_7, $+G_3$, $-G_3'$	$(\frac{1}{6}, \frac{1}{6})$	13	pieces A
G_6, $+G_1$, $-G_6'$	$(\frac{1}{2}, 0)$	26	„ B
G_6', $+G_6$, $-G_1'$	$(0, \frac{1}{2})$		
G_3, $-G_7$	$(\frac{5}{6}, 0)$	20	„ C
G_3', $+G_7$	$(0, \frac{5}{6})$		
		59	pieces altogether.

So in case III 59 of the 81 eightcells are cut.

15. In the following table giving the results for the seven cases considered Q, O, T, H, P denote respectively cube, octahedron, tetrahedron, hexagonal prism, triangular prism, whilst A, B, C, D, E represent less regular sections of eightcells.

GENERAL TABLE OF RESULTS.

Number of the case	Tangent of ϕ	Tangent of ψ	Value of a	Number of pieces	Enumeration of the different sections.
I	0	$\sqrt{2}$	$\sqrt{6}$	27	all G : 27 Q.
II	$\frac{1}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{6}$	$2\sqrt{2}$	45	$G_7, +G_1, -G_1'$ $(0, 0)$ 9 A; $G_6, -G_3'$ $(\frac{1}{3}, 0)$, $G_6' + G_3$ $(0, \frac{1}{3})$ 18 B; $G_3, -G_6'$ $(\frac{2}{3}, 0)$, $G_3, +G_6$ $(0, \frac{2}{3})$ 18 C
III	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{14}$	$\frac{1}{2}\sqrt{42}$	59	$G_7, +G_3, -G_3'$ $(\frac{1}{6}, \frac{1}{6})$ 13 A; $G_6, +G_1, -G_6'$ $(\frac{1}{2}, 0)$, $G_6', +G_6, -G_1'$ $(0, \frac{1}{2})$ 26 B; $G_3, -G_7$ $(\frac{5}{6}, 0)$, $G_3', +G_7$ $(0, \frac{5}{6})$ 20 C
IV	$\sqrt{3}$	$2\sqrt{2}$	$2\sqrt{6}$	51	$G_7, +G_6, -G_6'$ $(\frac{1}{3}, \frac{1}{3})$ 19 O; $G_6, +G_3, -G_7$ $(\frac{2}{3}, 0)$, $G_6' + G_7, -G_3'$ $(0, \frac{2}{3})$ 32 T
V	$2\sqrt{3}$	$\sqrt{26}$	$\sqrt{78}$	63	G_7 $(\frac{5}{12}, \frac{5}{12})$, G_6 $(\frac{3}{4}, \frac{1}{12})$, G_6' $(\frac{1}{12}, \frac{3}{4})$ 12 B 7 A; $+G_7, -G_6'$ $(\frac{1}{4}, \frac{7}{12})$, $+G_6, -G_7$ $(\frac{7}{12}, \frac{1}{4})$ 26 C; $+G_3, -G_6$ $(\frac{11}{12}, 0)$, $+G_6', -G_3'$ $(0, \frac{11}{12})$ 18 D
VI	$3\sqrt{3}$	$2\sqrt{14}$	$2\sqrt{42}$	57	G_7 $(\frac{4}{9}, \frac{4}{9})$, G_6 $(\frac{7}{9}, \frac{1}{9})$, G_6' $(\frac{1}{9}, \frac{7}{9})$ 12 B 7 A; $+G_7$ $(\frac{1}{3}, \frac{5}{9})$, $-G_7$ $(\frac{5}{9}, \frac{1}{3})$ 14 C; $+G_6'$ $(0, \frac{8}{9})$, $-G_6$ $(\frac{8}{9}, 0)$ 12 D; $+G_6$ $(\frac{2}{3}, \frac{2}{9})$, $-G_6'$ $(\frac{2}{9}, \frac{2}{3})$ 12 E
VII	∞	∞	∞	57	$G_7, +G_7, -G_7$ 21 H; $G_6, +G_6, -G_6$, $G_6', +G_6', -G_6'$ 36 P

It goes without saying that the *different* A, B, C, D mentioned in this table stand for *different* sections.

16. Now we have still to solve this problem: "if we are given, for any of the six cases II, III, ..., VII considered, a bag containing the set of models representing the corresponding sections of the eightcells of the block, how can these pieces be put together so as to obtain the corresponding section of the block itself?"

This problem can be easily solved in the following way. We begin by arranging the set of models into classes according to their shape, subdivide these classes into groups according to the last table of results and assign to each model of any group one of the numbers of the eightcells belonging to that group, without sign, with the sign $+$ or with the sign $-$ according to the layer the group belongs to. Then we inscribe on the faces of any model originating from the upper cube and the lower cube of the corresponding eightcell the letters d and d', on the three or the six side-faces originating from the side-cubes the letters a, b, c or a', b', c' or both triplets, taking care to put the right triplet at the right place and to write the three constituents of each triplet in due order of succession. After having prepared the work in this way we have only to consult a table of contact, of which we gave above only the part refering to the central eightcell 14, in order to be able to bring to coincidence the faces of the models that are to coincide, i. e. in order to be able to build up the section of the block of eightcells in the way desired.

The result of this solution is given in the six groups of figures 10_{II}, 10_{III}, . . . 10_{VII} corresponding to the six cases II, III, . . . VII. Each of these groups consists of:

1st. one relative large figure in parallel perspective representing the section of the block of eightcells built up of the pieces that are sections of the separate eightcells,

2nd. two or more small figures in parallel perspective marked A, B, C, D, E, representing these different pieces,

3rd. one or two plane sections of the solids of intersection obtained with planes on which we will fix attention afterwards.

17. It will be necessary now to say a word or two about the shading and the colouring of the groups of fig. 10; we begin by the shading.

The shading of the figures 10 is based on the supposition that the eightcells of the block are alternately black and white in the manner which may be considered as the consequent fourdimensional

extension of the shading of the chessboard squares. So the shading of the eightcells of the middle layer is shown in that of the cubes (fig. 6) forming their sections with the space $S_3(\pi)$ in its initial position I, and in the sections of these cubes by the plane π (fig. 7); to obtain the shading of the eightcells of the plus and the minus layer we have to invert black and white of the corresponding eightcells, i. e. of the eightcells bearing the same number, of the middle layer. Then two eightcells in space contact or in line contact are alike, two eightcells in plane contact or in point contact are different with regard to the shading. By adding that we suppose the central eightcell to be white the shading is quite determined by the stated rule of contacts.

But there is more. For we can prove easily by this rule that any two eightcells the projections of which on the plane π' coincide are alike with regard to the shading. So the three eightcells of G_3 projecting themselves (fig. 8) on the rectangle equal to P_0PQQ_0 and overlapping this for two thirds must coincide in shade, as they are in space contact with the unique eightcell of G_1 that projects itself on P_0PQQ_0 and therefore differ in shade from that eightcell; the cube common to this eightcell and any of the three belonging to G_3 projects itself on the overlapped part of P_0PQQ_0. So we can go on and assert that the six eightcells of G_6 must correspond in shade, each of them being in space contact with at least one of the three eightcells of G_3 and therefore differing in shade with these, etc. [1]) So we find easily the following result:

$$\text{White are the 41 eightcells} \ldots \left| \begin{array}{cccc} + G_1 & + G_6 & + G_6' & + G_1' \\ & G_3 \quad G_7 & & G_3' \\ - G_1 & - G_6 & - G_6' & - G_1' \end{array} \right|,$$

$$\text{black are the 40 eightcells} \ldots \left| \begin{array}{cccc} & + G_3 & + G_7 & + G_3' \\ G_1 & & G_6 \quad G_6' & G_1' \\ & - G_3 & - G_7 & - G_3' \end{array} \right|.$$

Now the question arises: "from the fact that the eightcells of the same group correspond in shade and their sections are equal the possibility of *all* equal pieces corresponding in shade presents itself; in which of the cases I, II, ... VII is this possibility realised?" To answer it we have only to consult the general table

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[1]) We can give an analogue to the "rule of contact" for the projections on π' in the form: "Two eightcells the projections of which overlap for one third or are in point contact are alike, two eightcells the projection of which overlap for two thirds or are in line contact are different with regard to the shading."

of results. We find that the answer is alternately no, yes, no, etc. So in the cases II, IV, VI all equal pieces correspond in shade. And of the other cases the first behaves differently from III, V, VII, in this sense that in the cases III, V, VII all equal pieces will correspond in shade, when the shading of the eightcells of both the plus layer and the minus layer have been inverted, an alteration which does not affect the first case.

18. If in the solution of the problem of reconstruction of the section of the block by putting together the different pieces, we confine ourselves to the pieces belonging to the groups of the middle layer, the result is a part of the whole section, included between two planes. These planes are evidently the planes of intersection of the intersecting space $S_3(\pi)$ with the two spaces separating the middle layer from the plus layer and the minus layer; as the two separating spaces are parallel to and equidistant from the space containing the centres of the eightcells of the middle layer, the two new planes, which we call $+\pi$ and $-\pi$, are parallel to and equidistant from π. In the initial position I the intersecting space S_3 (π) coincides with the space bisecting the distance between the separating spaces; for this reason the planes $+\pi$ and $-\pi$ are at infinity in that case and cannot appear in fig. 6. For which value of the angle of rotation ϕ, supposed to increase from 0° to 90° do these planes $+\pi$, $-\pi$ begin to intersect the section? A single glance at fig. 8 shows us that this angle P_0OA is characterized by its tangent $\frac{1}{9}\sqrt{3}$, so it amounts to 10° 53′ 36″; in the rhombohedrical section of the block corresponding to this angle the planes $+\pi$, $-\pi$ would make their appearance in the extremities A, A' of the axis of the rhombohedron. For any acute angle that surpasses this value the planes $+\pi$ and $-\pi$ divide the section into three parts, the extreme ones of which — equal to one another — are left uncoloured, while the middle slice is coloured yellow. On the side-faces of the sections the polygons of intersection with the planes $+\pi$, $-\pi$ separate parts of the surface differing in shade. What appears in the planes $+\pi$, $-\pi$ themselves is shown in the case II for both the planes, in the other cases for the plane $+\pi$ only; in these sections can be seen how the polygons are built up of the d-faces and the d' faces of the pieces of the middle layer. The other plane sections, added to the figure-groups 10_{III}, 10_{IV}, . . . 10_{VII}, refer to the endplane not visible in the principal figure.

After what has been said it will not be necessary farther to explain the meaning of the numbers without sign or with one of

the signs $\pm$ on the side-faces of the sections of the block, on the endplanes of these sections and on the sections of these sections with the planes $+\pi$, $-\pi$. With regard to the letters A, B, C, D, E, indicating the sections of the separate eightcells of different shape, inscribed also in the endplanes and the sections with the planes $+\pi$, $-\pi$ it may suffice to remember that the forms A and B meet the invariable plane π respectively in regular hexagons and equilateral triangles marked on them, whilst the other forms C, D, E have no point in common with that plane.

19. A mere inspection of the groups of figures 10_{II}, 10_{III}, ..., 10_{VII} can show how the section of the block composed of the sections A, B, C, ... of the separate eightcells changes, when the angle of rotation ϕ varies from 0° to 90°, from which ensues that the angle ψ of the side-faces of the section with the plane π increases from 54° 44′ 9″ to 90°. If we now push the rotation still farther and suppose that ϕ varies from 90° to 180° we pass at the supplementary values of these considered above by stadia VI′, V′, ..., I′ closely connected to the sections VI, V, ..., I. For, if we consider the two positions ϕ and $180°-\phi$ of the intersecting space S_3 (π) as each other's mirror-image with regard to the initial position $\phi = 0$ of that space as a threedimensional looking-glass, and remark that the reflection of the block itself in that looking-glass interchanges only the signs $+$ and $-$ of the eightcells of the plus layer and the minus layer, then it is evident that the two sections corresponding to two supplementary values ϕ, $180°-\phi$ of the angle of rotation are each other's looking-glass image with regard to the plane π, the interchange of the signs $\pm$ included.

We finish our considerations by a rapid survey of the different cases.

Case II. The sections differ but slightly from the initial case (fig. 6). If we invert the shading of the pieces corresponding to the plus layer and the minus layer, and we glue together the corresponding pieces $(1, +1)$, etc. of the groups G_6, $+G_6$ and $(3, -3)$, etc. of the groups G_6', $-G_6'$, we hit upon a figure which can be derived from the solid (fig. 6) by stretching in the direction AA'. As we already remarked the shading of the equilateral triangles in $+\pi$, $-\pi$ correspond to the d-faces and the d'-faces of the middle layer; if the shades were inverted they would refer to the d'-faces of the plus layer and the d-faces of the minus layer.

As a glance at the figure shows, the section with the plane $+\pi$

passes into that with the plane $-\pi$ by a rotation through 180° about the centre of the figure, succeeded by the substitution of $28 - k$ for any number k; as this simple rule holds in general, we give in the groups 10_{III}, 10_{IV}, ..., 10_{VII} the section with the plane $+\pi$ only.

If R represents a rhombohedrical section the solids A, B, C satisfy the relations $R = A = B + C$; A and C are always white, B is always black.

Case III. The small triangle marked e represents the endplane below, invisible in the principal figure, as seen from within the section, i. e. from above; in the four following cases the polygons marked e must be interpreted in the same way.

Here we have $R = A + 2\,C = 2\,B$; A and C are white in the middle layer and black elsewhere, while B is black in the middle layer and white elsewhere.

Case IV. Here the relation holds $R = O + 2\,T$; the octahedra are white, the tetrahedra black.

This case is by far the most remarkable one; it solves the question: "how to divide an octahedron $O^{(3)}$ with edges equal to three units into octahedra $O^{(1)}$ and tetrahedra $T^{(1)}$ with edges unity?" If we place the $O^{(3)}$ with one of its diagonals vertically, the solution can be given as follows. Divide the vertical diagonal into six equal parts. Cut the octahedron $O^{(3)}$ by five horizontal planes passing through the points of division. Divide the square of the middle section into nine and the squares of the adjacent sections into four squares equal to the squares of the extreme sections. Then these $1 + 4 + 9 + 4 + 1 = 19$ equal squares

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[1]) In connexion with the space-filling properties of octahedra $O^{(1)}$ and tetrahedra $T^{(1)}$ in the two different positions it is evident that it must be possible to fill an octahedron $O(p)$ and a tetrahedron $T(p)$, both with edges p, by $O^{(1)}$ and $T^{(1)}$. We only mention the results here. In the case of $O(p)$ $\frac{p}{3}(2p^2+1)\,O^{(1)}$ and $\frac{2p}{3}(p^2-1)\,T^{(1)}$ of each of the two positions are required; in the case of $T(p)$ we want $\frac{p}{6}(p^2-1)\,O^1$, $\frac{1}{6}p\,(p+1)\,(p+2)\,T^{(1)}$ corresponding in position with $T(p)$ and $\frac{1}{6}p\,(p-1)\,(p-2)\,T^1$ in the opposite position. These results verify the relations in volume

$$\frac{p}{3}(2p^2+1)\,O^{(1)} + \frac{4p}{3}(p^2-1)\,T^{(1)} = p^3\,O^{(1)},\quad \frac{p}{6}(p^2-1)\,O^{(1)} + \frac{p}{3}(p+2)\,T^{(1)} = p^2\,T^{(1)}$$

based on the fact that $4\,T^{(1)}$ correspond to one $O^{(1)}$, as they ought to do.

In threedimensional space divided into $O^{(1)}$ and $T^{(1)}$ there is plane contact between two polyhedra of different kind. So an $O^{(1)}$ and two $T^{(1)}$ in plane contact with it on two opposite faces form a rhombohedron, an $O^{(1)}$ and the eight $T^{(1)}$ in plane contact with it form the well-known figure of the two equal but oppositely placed tetrahedra penetrating one another in an octahedron.

represent the horizontal middle sections of the 19 $O^{(1)}$. Moreover the $4 + 12 + 24 + 12 + 4$ line-segments equal to unity, forming together the sides of the 19 squares, represent the horizontal edges of the tetrahedra $T^{(1)}$. Any of these 56 segments belongs to two tetrahedra if there are two segments lying in two adjacent planes crossing it at right angles and having their centres in the vertical through the centre of the chosen segment. If there is only one such segment the chosen segment belongs to only one tetrahedron. So we find in the layers between the planes successively $4 + 12 + 12 + 4 = 32\ T^{(1)}$, 16 right-handed ones and 16 left-handed ones.

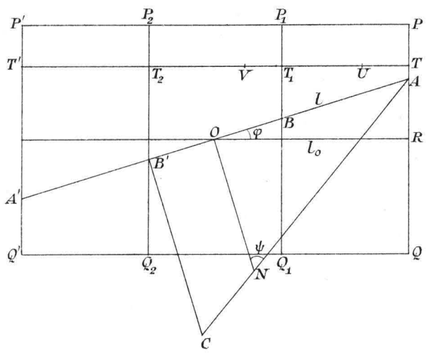
Case V. Here we find $R = A + 2\,C + 2\,B + 2\,D$. Of these different pieces A and B occur in the middle layer only, C and D in the extreme layers only. The forms A are all white, the forms B all black. The forms C and D show this particularity that not even the equal forms belonging to the same extreme layer correspond in shade. So the upper layer contains 13 C, six white ones $(\frac{7}{12}, \frac{1}{4})$ and seven black ones $(\frac{1}{4}, \frac{7}{12})$, etc.

Case VI. Here $R = A + 2\,C + 2\,E + 2\,B + 2\,D$. Moreover all A, E, D are white, all C, B black.

Case VII. This case leads us back to the well-known plane-filling by regular hexagons and equialateral triangles.

Liscard
Groningen } November, 1907.

Fig. 1.



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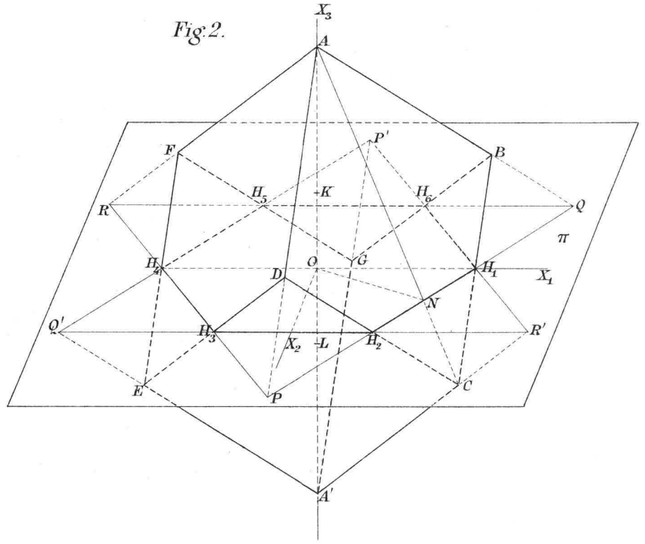

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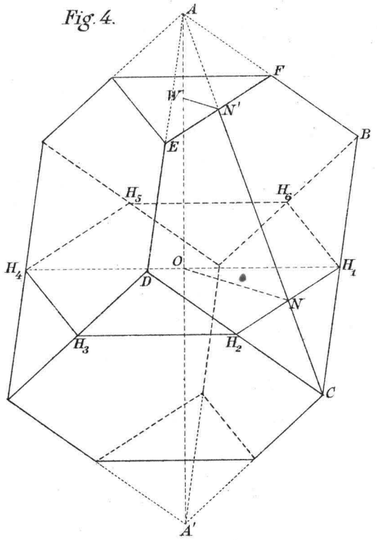

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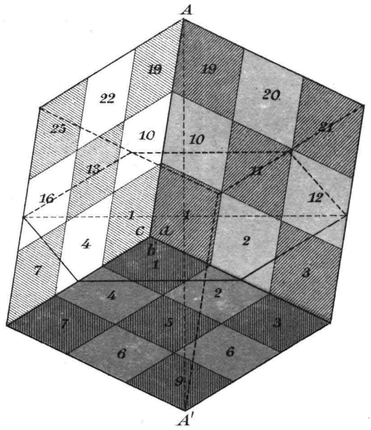

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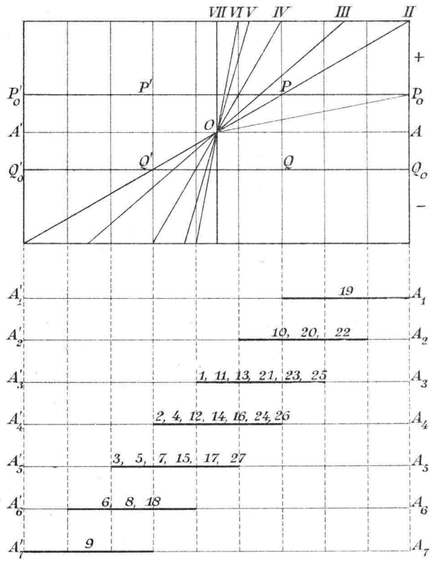

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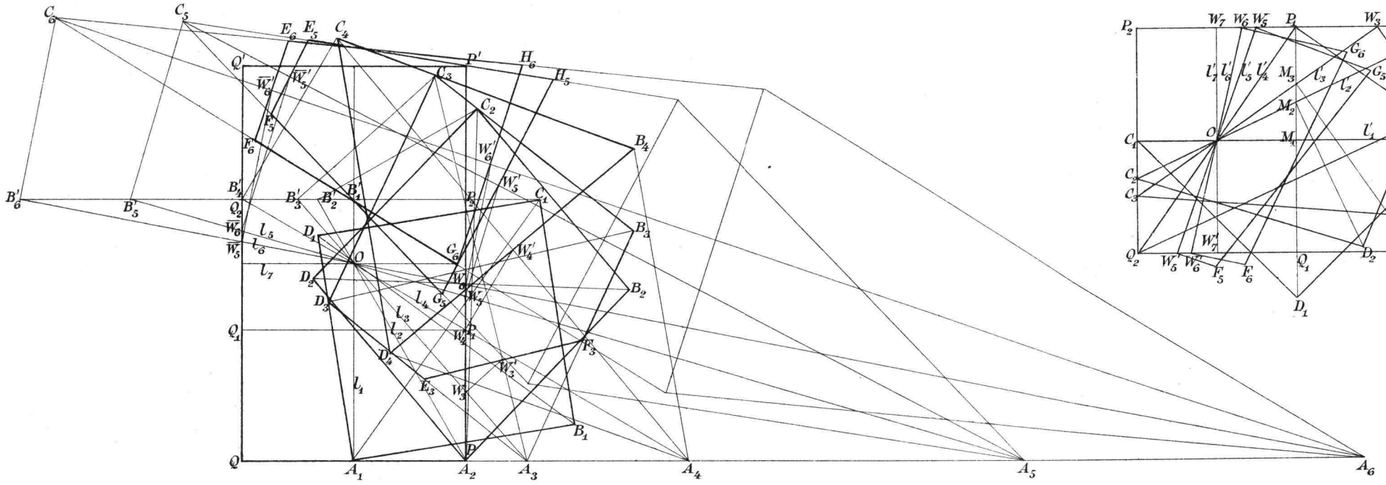

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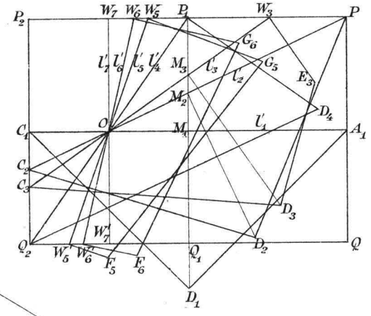

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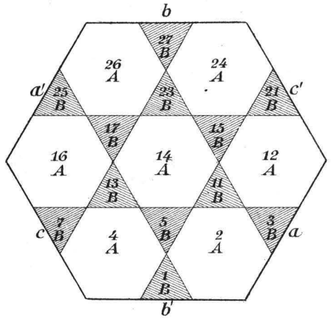

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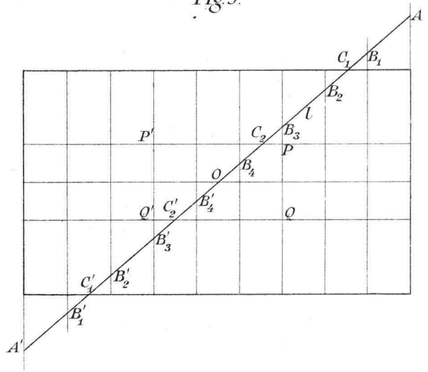

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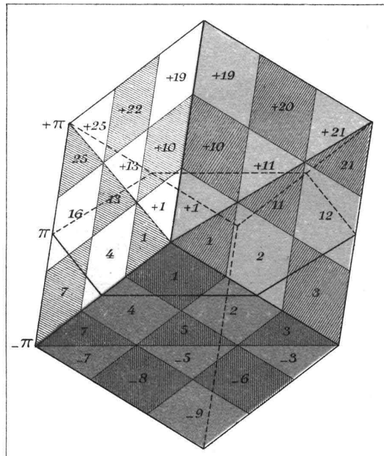

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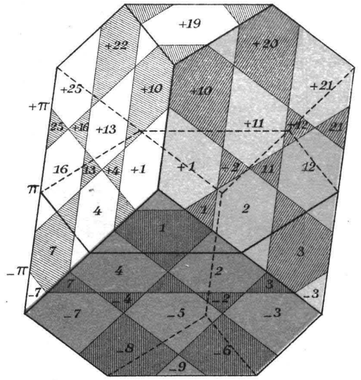

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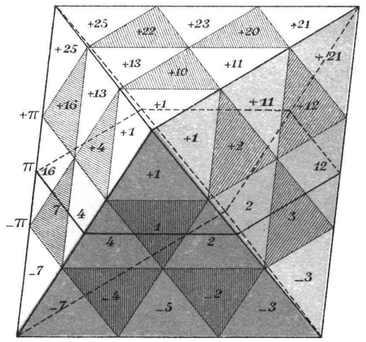

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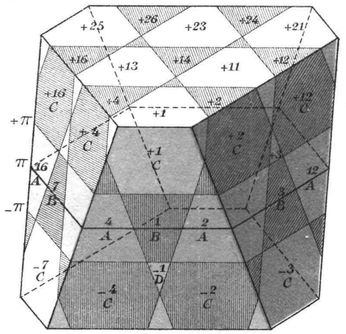

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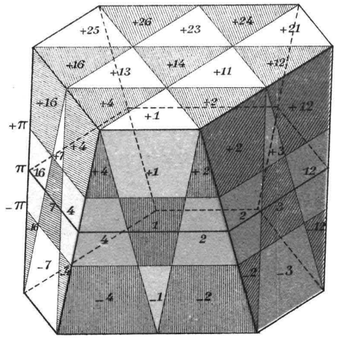

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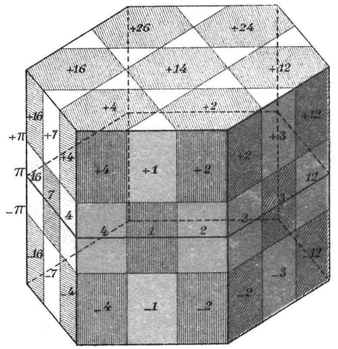

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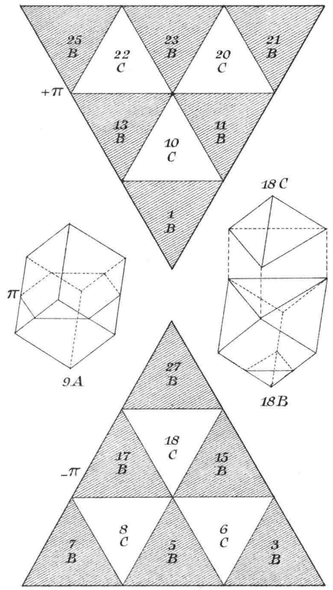

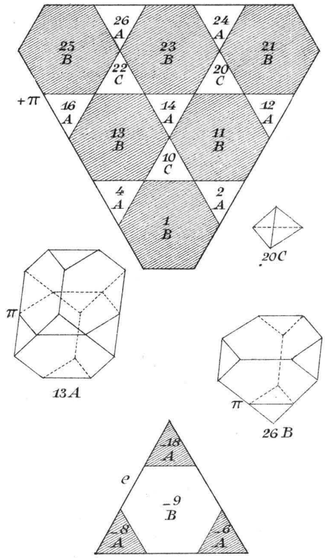

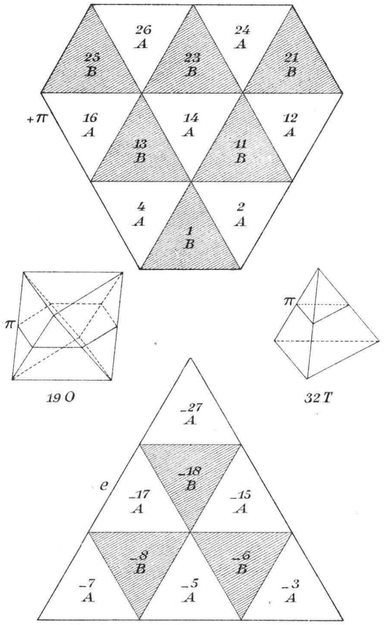

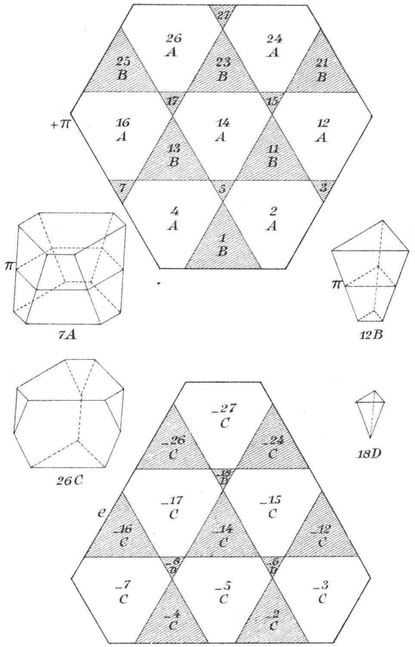

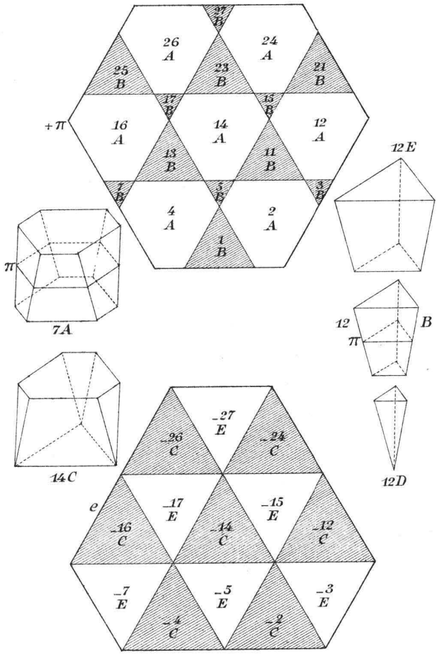

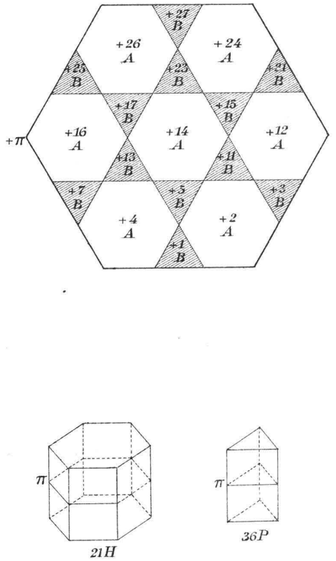

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