CATALYTIC AND MUTUALLY CATALYTIC BRANCHING

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Abstract

In this paper we present an overview of recent work on lattice and measure-valued models of catalytic reactions, in particular, on catalytic branching systems. The main phenomena exhibited by nearly critical branching systems are dimension-dependent clumping in small and large space and time scales. Special attention is given to the effects which occur when the catalyst is highly clumped, in particular, when in the continuum models the catalyst is a time-dependent singular measure. Finally, the interactive model of mutually catalytic branching is described and some recent results are reviewed. The basic tools include log-Laplace functionals, measure-valued martingale problems, collision local times, and duality.

1 Introduction: Catalytic reaction-diffusion systems

Reaction-diffusion partial differential equations have been studied for many years, and one of the applications is to model catalytic reactions. A catalytic reaction requires the presence of a catalyst which determines the rate of the reaction. For example, a single type catalytic reaction-diffusion is given by the partial differential equation:

$$\left.\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\Delta u(t,x) - \frac{1}{2}\Gamma_t(x)\, R\big(u(t,x)\big), \\ u_{|t=0+} &= \varphi \geqslant 0, \end{aligned}\right\} \tag{1.1}$$

where Δ is the d-dimensional Laplacian, $R : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz-continuous function, the initial function φ is continuous, and $(t,x) \mapsto \Gamma_t(x) \geqslant 0$ is also a continuous function representing the density of a catalyst at time t at x. However, there are some catalytic reactions in which the catalyst is concentrated on an x set of zero Lebesgue measure and equation (1.1) must be replaced by one with an irregular coefficient Γ. To illustrate this we consider an important example.

Example 1.1 (glycolysis). Glycolysis is a cellular reaction in which glucose is broken down into pyruvate, and energy is released. This is achieved by a chain of reactions taking place inside a cell which can be described by a system of reaction-diffusion equations. Each reaction in the chain is initiated by an enzyme catalyst. However, these enzyme catalysts may not be spatially distributed in a homogeneous way. Within the cell there is a network of "filamentous actin", and enzyme molecules can bind to filaments leading to highly concentrated catalytic regions (see [52]). ◊

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The example of glycolysis and other catalytic reactions (cf. [53]) lead to a study in which the catalyst is not given by a regular density function but rather by a measure-valued process:

$$\Gamma_t(\mathrm{d}x) = \text{ catalytic mass at time } t \text{ in the volume element } \mathrm{d}x. \tag{1.2}$$

In this case it is useful to reformulate the reaction-diffusion equation (1.1) by the corresponding mild form

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} \mathrm{p}_t(y-x)\,\varphi(y)\,\mathrm{d}y \\ &\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \mathrm{p}_{t-s}(y-x)\,R\left(u(s,y)\right)\Gamma_s(\mathrm{d}y)\,\mathrm{d}s, \end{aligned} \tag{1.3}$$

with the (standard) heat kernel

$$\mathrm{p}_t(x) := (2\pi t)^{-d/2} \exp\left[-\frac{|x|^2}{2t}\right]. \tag{1.4}$$

Such partial differential equations have been studied when $\Gamma_t(\mathrm{d}x) \equiv \Gamma(\mathrm{d}x)$ is a deterministic measure concentrated on an open set ([5]) or on a hypersurface ([16]). However, the example of glycolysis suggests generalizing this set-up to catalytic measures concentrated on fractal-like sets and also varying in time.

The study of reaction-diffusion equations with irregular reaction rates and their relation to catalytic branching models had been initiated in [13, 24, 14]. Subsequently there has been a considerable development in the mathematical literature. For earlier surveys, see [35, 36, 21], [27, Section 6.3], and [47]. For catalytic branching models in a discrete time setting, see [11, 12, 43, 44, 1, 6].

Before returning to describe probabilistic models which are related to equations of this type, we begin with models of catalytic reactions in discrete spaces.

2 Lattice models of catalytic branching

Catalytic reactions such as glycolysis take place at the molecular level. The goal of this research is to develop a class of mesoscopic models of these reaction-diffusions which include the spatial heterogeneities in the "geographical structure" of catalysts and reactants and temporal fluctuations which occur. Moreover, the heterogeneous distribution of catalysts and reactants has a profound effect on the rate of product formation – how can this be quantified?

In order to address these problems, we begin at the microscopic (molecular) level and with the simplifying assumption of a spatially discretized system. In this setting a chain of reactions such as occurs in glycolysis can be viewed as a kind of a multitype branching process

$$R_1 \xrightarrow{\Gamma_1} R_2 \xrightarrow{\Gamma_2} R_3 \xrightarrow{\Gamma_3} \ldots \xrightarrow{\Gamma_{n-1}} R_n \tag{2.1}$$

where $R_1, \ldots, R_n$ are a sequence of reactants, and $\Gamma_1, \ldots, \Gamma_{n-1}$ are catalysts. In a typical biochemical system such as glycolysis, molecules of reactant R_1 enter the system as a constant rate Poisson process, each molecule of reactant R_{j-1} is replaced by a random number of molecules of reactant R_j at a rate proportional to the time spent in the presence of a molecule of the catalyst Γ_{j-1}.

In order to simplify the exposition, we consider primarily the case of a single reactant type and the catalytic reaction $R_1 \xrightarrow{\Gamma_1} R_1$. The mathematical methods developed for the analysis of the latter system can be extended to the former multitype case.

2.1 Catalytic reactions of particle systems in $\mathbb{Z}^d$

We begin with a particle system on $\mathbb{Z}^d$. A particle of mass m located at a point x will be described by a point measure $m\delta_x$. A particle system in $\mathbb{Z}^d$ (with spatially varying masses) is then simply a measure μ on $\mathbb{Z}^d$. On the other hand, we will often identify a measure μ on $\mathbb{Z}^d$ with its density function $\mu : \mathbb{Z}^d \to \mathbb{R}_+$ with respect to the counting measure $\ell := \sum_{x \in \mathbb{Z}^d} \delta_x$, that is,

$$\mu(B) = \sum_{x \in B} \mu(x) = \int_B \mu(x)\, \ell(\mathrm{d}x), \qquad B \subset \mathbb{Z}^d. \tag{2.2}$$

Then $\mu(x)$ is the mass in x and, at the same time, the density of mass at x.

Given $N(t)$ reactant particles of mass m at time t located at $w_t^1, \ldots, w_t^{N(t)}$, we consider the point measure

$$X_t := m \sum_{j=1}^{N(t)} \delta_{w_t^j} \tag{2.3}$$

describing the state of the reactant at time t. For this to serve as a discrete space model of a reaction-diffusion system, we assume that the system dynamics involves spatial dispersion and that the reactant particles undergo catalytic branching. To model the spatial dispersion we assume first of all that the reactant particles undergo independent *simple random walks* w_j in $\mathbb{Z}^d$. Additionally, in the presence of the catalyst, a reactant particle can either produce a new reactant particle (of the same type) or die (*critical binary branching*). Splitting or death for a reactant particle with trajectory w will occur according to a Poisson process run with a clock $A_w^\Gamma(t)$, say, which measures the time spent in the presence of the catalyst. Here we assume that the catalyst at time t is given by a measure Γ_t on $\mathbb{Z}^d$. Then, by definition,

$$A_w^\Gamma(t) := \int_0^t \sum_{y \in \mathbb{Z}^d} \delta_{w(s)}(y)\, \Gamma_s(y)\, \mathrm{d}s, \qquad t \geqslant 0. \tag{2.4}$$

A_w^Γ is called the *collision (local) time* between the intrinsic random walk path w and the catalyst Γ. (Note that by our convention (2.2) there is no difference between the

notions of collision time and collision local time in the present $\mathbb{Z}^d$ case.) Examples of Γ and the related collision time A_w^Γ are listed below. Note that A_w^Γ is a continuous additive functional of the Markov process w.

We thus have the catalytized reactive transitions:

$$\text{catalytic reproduction:} \quad \delta_x \longrightarrow \delta_x + \delta_x \text{ with clock } \frac{1}{2}A_w^\Gamma,$$

$$\text{catalytic killing:} \quad \delta_x \longrightarrow 0 \text{ with clock } \frac{1}{2}A_w^\Gamma.$$

Example 2.1 (catalysts and their collision time).

(i) (uniform catalyst) In this case

$$\Gamma_t(y) \equiv \gamma, \quad \text{hence } A_w^\Gamma(t) \equiv \gamma t, \tag{2.5}$$

and the reactant process $X = X^\Gamma$ is a classical branching random walk with branching rate γ.

(ii) (single point catalyst) Here $\Gamma_t \equiv \delta_0$, thus

$A_w^\Gamma(t) =$ occupation time (or density) at 0 by time t for the random walk w.

(iii) (constant number of catalytic random walk particles) In the case

$$\Gamma_t = \sum_{j=1}^{M} \delta_{\gamma_t^j}, \tag{2.6}$$

where $\gamma^1, \ldots, \gamma^M$ are independent simple random walks in $\mathbb{Z}^d$ also independent of the intrinsic reactant path w, we have

$$A_w^\Gamma(t) = \sum_{j=1}^{M} \ell(t, \gamma^j, w) \tag{2.7}$$

where $\ell(t, \gamma^j, w)$ denotes the occupation time at 0 *by time* t for the random walk $w - \gamma^j$.

(iv) (autonomous branching random walk catalyst) In this case the catalytic particles are assumed to perform autonomously a critical binary branching simple random walk in $\mathbb{Z}^d$. In low dimensions branching particle systems exhibit spatial clumping, and therefore they provide an example in which we can study the effect of a random heterogeneous catalytic medium on the distribution of reactant mass. Such a study has recently been carried out in [45]. ◊

2.2 Mutually catalytic branching particle systems

In a catalytic reactant cycle the reactant R_{j-1} serves as the catalyst for the production of the reactant R_j, $j = 2, \ldots, M$, and R_M serves as the catalyst for the production of R_1.

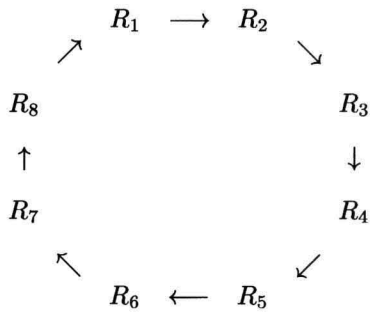

Figure: Catalytic reaction cycle ($M = 8$)

In the special case $M = 2$, we have two *mutually catalytic reactants.* At the particle level, letting the two reactant types be called "Red" and "Blue", and let all of them move in the lattice $\mathbb{Z}^d$ as simple random walks, we have in the case of critical branching

$$\begin{aligned} \text{Red+Blue} &\to \text{Blue offspring} \\ \text{Red+Blue} &\to \text{Red offspring} \\ \text{Red+Blue} &\to \text{Blue death} \\ \text{Red+Blue} &\to \text{Red death} \end{aligned}$$

each of them with clock $\frac{1}{4}A_{\mathrm{RB}}$, where $A_{\mathrm{RB}} = A_{\mathrm{RB}}(\mathrm{d}t)$ should refer to the collision time between a red and a blue particle.

2.3 Continuous-state catalytic branching in $\mathbb{Z}^d$

The particle models introduced above are intuitively appealing and are used in simulation experiments (cf. `http://www.mi.uni-erlangen.de/~klenke`). However, the continuous-state processes which arise in the limit of a large number of particles of small mass have the advantage that they can be studied by powerful tools of stochastic calculus.

We next introduce the continuous-state analogs of the particle systems discussed above. Recall that $X_t(x)$ and $\Gamma_t(x)$ denote the amount of reactant, respectively catalyst, at site $x \in \mathbb{Z}^d$ at time t. Then $\{X_t : t \geqslant 0\}$ is required now to satisfy the following system of Itô stochastic differential equations

$$X_t(x) = X_0(x) + \int_0^t \frac{1}{2}\Delta X_s(x)\,\mathrm{d}s + \int_0^t \sqrt{\Gamma_s(x)\,X_s(x)}\,\mathrm{d}W_s(x), \tag{2.8}$$

$t \geqslant 0$, $x \in \mathbb{Z}^d$, where Δ is the discrete Laplacian (note that $\frac{1}{2}\Delta$ is the generator of a simple random walk on $\mathbb{Z}^d$), and $\{W(x) : x \in \mathbb{Z}^d\}$ is a system of independent

one-dimensional (standard) Brownian motions. This system is also referred to as *simple super-random walk* $X = X^\Gamma$ *in* $\mathbb{Z}^d$ *with catalyst* Γ. In the special case of Example 2.1 (i) of a uniform catalyst $\Gamma_t(x) \equiv \gamma$, the process $X = X^\Gamma$ is the well-understood simple super-random walk with constant branching rate γ. On the other hand, analogously to the Example 2.1 (iv), one can take an autonomous simple super-random walk with branching rate γ as catalyst Γ. For this Γ, the solution $X = X^\Gamma$ to (2.8) is called the *simple super-random walk with simple super-random walk catalyst* Γ.

2.4 Continuous-state mutually catalytic branching in $\mathbb{Z}^d$

Consider a pair of mutually catalytic reactants denoted by $\mathbf{X} = (X^1, X^2)$. By definition, they satisfy the following system of stochastic differential equations:

$$X_t^j(x) = X_0^j(x) + \int_0^t \frac{1}{2}\Delta X_s^j(x)\,\mathrm{d}s + \int_0^t \sqrt{\gamma\, X_s^1(x)\, X_s^2(x)}\,\mathrm{d}W_s^j(x) \tag{2.9}$$

$j = 1, 2$, $t \geqslant 0$, $x \in \mathbb{Z}^d$, and $\{W^j(x) : x \in \mathbb{Z}^d,\ j = 1, 2\}$ is a system of independent one-dimensional Brownian motions. The constant $\gamma > 0$ is called the *collision rate.* We will come back to this model in Section 5.

3 Continuum space models of catalytic branching

In this section we pass from $\mathbb{Z}^d$ to $\varepsilon\mathbb{Z}^d$ and let $\varepsilon \downarrow 0$. (This, in particular, provides insight into the clumping structure of the lattice system when ε is small.)

3.1 Lattice approximation to the continuum space model

In order to explore the possibility of a continuum space analog of the continuous-state catalytic branching models in $\mathbb{Z}^d$ we replace

- $\mathbb{Z}^d$ by $\varepsilon\mathbb{Z}^d \subset \mathbb{R}^d$,
- the discrete Laplacian Δ by the scaled Laplacian Δ^ε :

$$(\Delta^\varepsilon f)(x) := \varepsilon^{-2} \sum_{y:\ |y-x|=\varepsilon} [f(y) - f(x)], \qquad x \in \varepsilon\mathbb{Z}^d, \tag{3.1}$$

- the counting measure $\ell = \sum_{x\in\mathbb{Z}^d} \delta_x$ on $\mathbb{Z}^d$ by the scaled counting measure

$$\ell^\varepsilon := \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} \delta_x \tag{3.2}$$

(approaching the Lebesgue measure on $\mathbb{R}^d$ as $\varepsilon \downarrow 0$),

and modify our convention (2.2) by identifying measures $\mu(\mathrm{d}x)$ on $\varepsilon\mathbb{Z}^d$ with their density function $x \mapsto \mu(x)$ with respect to the scaled counting measure ℓ^ε:

$$\mu(B) \;=\; \varepsilon^d \sum_{x\in B} \mu(x) \;=\; \int_B \mu(x)\,\ell^\varepsilon(\mathrm{d}x), \qquad B \subset \varepsilon\mathbb{Z}^d. \tag{3.3}$$

Similarly, we replace the earlier system $\{W(x) : x \in \mathbb{Z}^d\}$ of independent Brownian motions by the

$$\text{"approximating white noise"} \quad \left\{\varepsilon^{-d/2} W(x) : x \in \varepsilon\mathbb{Z}^d\right\} \tag{3.4}$$

where now $\{W(x) : x \in \varepsilon\mathbb{Z}^d\}$ is a system of independent one-dimensional Brownian motions.

Instead of (2.8) we then consider, for a fixed $\varepsilon > 0$, the system of stochastic integral equations

$$X_t^\varepsilon(x) \;=\; X_0^\varepsilon(x) + \int_0^t \frac{1}{2}\Delta^\varepsilon X_s^\varepsilon(x)\,\mathrm{d}s + \int_0^t \sqrt{\Gamma_s^\varepsilon(x)\, X_s^\varepsilon(x)}\,\varepsilon^{-d/2}\,\mathrm{d}W_s(x), \tag{3.5}$$

$t \geqslant 0$, $x \in \varepsilon\mathbb{Z}^d$, for the reactant's density $X_t^\varepsilon(x)$ given the catalyst's density $\Gamma_s^\varepsilon(x)$ in x at time t.

In order to study the limit as $\varepsilon \downarrow 0$, we observe that, for ε fixed, the measure-valued process X^ε associated with the solution of this system satisfies a *measure-valued martingale problem,* denoted by $\mathbf{MP}_{\varepsilon\mathbb{Z}^d}$. Write $\langle \mu, f\rangle$ for the integral $\int f(x)\,\mu(\mathrm{d}x)$.

$\mathbf{MP}_{\varepsilon\mathbb{Z}^d}$ For φ in the domain $D(\Delta)$ of the Laplacian Δ in $\mathbb{R}^d$,

$$M_t^\varepsilon(\varphi) \;:=\; \langle X_t^\varepsilon, \varphi\rangle - \langle X_0^\varepsilon, \varphi\rangle - \int_0^t \left\langle X_s^\varepsilon, \frac{1}{2}\Delta^\varepsilon\varphi\right\rangle \mathrm{d}s, \qquad t \geqslant 0, \tag{3.6}$$

is a continuous martingale with increasing process

$$\langle\langle M^\varepsilon(\varphi)\rangle\rangle_t \;=\; \int_{[0,t]\times\varepsilon\mathbb{Z}^d} \varphi^2(x)\, L_{[\Gamma^\varepsilon, X^\varepsilon]}\left(\mathrm{d}[s,x]\right), \qquad t \geqslant 0. \tag{3.7}$$

and $L_{[\Gamma^\varepsilon, X^\varepsilon]}$ is defined by

$$\int_{\mathbb{R}_+\times\varepsilon\mathbb{Z}^d} \psi(s,x)\, L_{[\Gamma^\varepsilon, X^\varepsilon]}\left(\mathrm{d}[s,x]\right) \;=\; \int_0^\infty \int_{\mathbb{Z}^d} \psi(s,x)\, X_s^\varepsilon(x)\,\Gamma_s^\varepsilon(x)\,\ell^\varepsilon(\mathrm{d}x)\mathrm{d}s,$$

for all continuous functions $\psi : \mathbb{R}_+ \times \varepsilon\mathbb{Z}^d \to \mathbb{R}_+$ with compact support.

We interpret $L_{[\Gamma^\varepsilon, X^\varepsilon]}$ as the *collision local time* of the catalyst Γ^ε and the reactant X^ε. [Note that the additional term "local" is justified in the sense of (3.3) since, for each t fixed,

$$x \;\mapsto\; L_{[\Gamma^\varepsilon, X^\varepsilon]}\left([0,t]\times\{x\}\right) \tag{3.8}$$

is a *density* function with respect to the scaled counting measure ℓ^ε of (3.2).]

Now we identify each measure μ on $\varepsilon\mathbb{Z}^d$ with a measure on $\mathbb{R}^d$ supported by $\varepsilon\mathbb{Z}^d$ and denoted by the same symbol μ. Let $\mathcal{C} = \mathcal{C}\left([0,\infty), \mathcal{M}_f(\mathbb{R}^d)\right)$ denote the space of all continuous functions from $[0,\infty)$ into the set $\mathcal{M}_f(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ (the latter equipped with the topology of weak convergence, and $\mathcal{C}$ with the compact-open topology). The law [3] of the process X^ε can be characterized as the unique probability distribution $\mathbf{P}^\varepsilon$ on $\mathcal{C}$ with initial condition X_0^ε such that all X_t^ε, $t \geqslant 0$, are supported by $\varepsilon\mathbb{Z}^d$ and such that the martingale problem $\mathbf{MP}_{\varepsilon\mathbb{Z}^d}$ is solved (cf. [34, Chapter 4]).

We can then ask for the weak convergence of the laws $\mathbf{P}^\varepsilon$ on $\mathcal{C}$ as $\varepsilon \downarrow 0$ provided that $X_0^\varepsilon \to X_0$ in law.

3.2 Super-Brownian motion (SBM) in $\mathbb{R}^d$

In the case of a uniform medium, $\Gamma_s^\varepsilon(x) \equiv \gamma$, standard arguments can be used to prove tightness of the laws of the measure-valued processes $\{X^\varepsilon : 1 \geqslant \varepsilon > 0\}$ and to show that any limit point $X = X^\Gamma$ solves the following *martingale problem*:

MP(SBM) For $\varphi \in D(\Delta)$,

$$M_t(\varphi) = \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \int_0^t \left\langle X_s, \frac{1}{2}\Delta\varphi\right\rangle \mathrm{d}s \tag{3.9}$$

is a continuous martingale with increasing process

$$\left\langle\!\left\langle M.(\varphi)\right\rangle\!\right\rangle_t = \int_{[0,t]\times\mathbb{R}^d} \varphi^2(x)\, L_{[\Gamma,X]}\left(\mathrm{d}[s,x]\right) \tag{3.10}$$

where

$$L_{[\Gamma,X]}\left(\mathrm{d}[s,x]\right) = \gamma X_s(\mathrm{d}x)\,\mathrm{d}s \tag{3.11}$$

is well-defined.

In this case the martingale problem can be shown to have a unique solution by proving that any limit point $X = X^\Gamma$ is related to the log-Laplace equation via log-Laplace transforms:

$$\left.\begin{aligned} -\log \mathbf{E}\left\{\exp\langle X_t, -\varphi\rangle \;\middle|\; X_0\right\} &= \langle X_0, u(t)\rangle, \\ \frac{\partial u}{\partial t} &= \frac{1}{2}\left(\Delta u - \gamma u^2\right), \quad u_{|t=0+} = \varphi \geqslant 0, \end{aligned}\right\} \tag{3.12}$$

for a suitable set of test functions φ. The resulting process is the well-known continuous *super-Brownian motion (SBM)* X with branching rate γ.

Although the SBM exists in any dimension, the qualitative properties are highly dimension dependent and an understanding of this provides insight into the phenomenon of spatial clumping. Some of the key properties are:

[3] Here are our general rules for the choice of notation of probability distributions: The letter P (with or without any index) refers to a particle's motion law, $\mathbb{P}$ to a distribution of a random medium, and $\mathbf{P}$ to a law of a measure-valued process under consideration, given the medium (if any). Expectations with respect to such distributions are denoted by $E, \mathbb{E}$, and $\mathbf{P}$, respectively.

- Fix $t > 0$. If $d = 1$, then X_t is an absolutely continuous measure. If $d \geqslant 2$, then X_t is almost surely singular, and the Hausdorff dimension of the *support* $\mathrm{supp}X_t$ of X_t is 2.
- Two independent SBMs in $\mathbb{R}^d$ collide if and only if $d < 6$. Here two measure-valued processes X^1, X^2 are said to *collide* if

$$\mathbf{P}\left(\mathrm{gsupp}X^1 \cap \mathrm{gsupp}X^2) \neq \varnothing\right) > 0 \tag{3.13}$$

  where $\mathrm{gsupp}X^j$ denotes the *global support* of X^j :

$$\mathrm{gsupp}X^j := \mathrm{closure}\left(\bigcup_{t>0} \mathrm{supp}X_t^j\right). \tag{3.14}$$

- A Brownian particle δ_w with path w collides with an independent SBM if and only if $d < 4$.

Remark 3.1 (tempered measures). The SBM can be extended to an $\mathcal{M}_p(\mathbb{R}^d)$-valued process with $p > d$, where $\mathcal{M}_p(\mathbb{R}^d) := \{\mu : \langle \mu, \phi_p \rangle < \infty\}$ with reference function $\phi_p(x) := (1 + |x|^2)^{-p/2}$, $x \in \mathbb{R}^d$. ◊

Remark 3.2 (regular media, averaging effect). If the present constant catalytic medium $\Gamma_t(\mathrm{d}x) \equiv \gamma\,\mathrm{d}x$ is replaced by a time-independent *regular* medium $\Gamma_t(\mathrm{d}x) \equiv \gamma(x)\,\mathrm{d}x$, where $x \mapsto \gamma(x)$ is now a spatially homogeneous ergodic field of *finite* expectation, then the large scale structure of the related catalytic SBM $X = X^\Gamma$ will depend only on the expectation of $\gamma(0)$. See [19] for an example of an *averaging* effect of this type. There, large scale fluctuations are studied for a branching particle system in supercritical dimensions leading to generalized Ornstein-Uhlenbeck processes which depend only on the constant *mean* of the catalytic medium $x \mapsto \gamma(x)$. These results once more motivate us to restrict the further study to *irregular* catalysts, and to extend our consideration to media with infinite mean. ◊

3.3 Catalytic super-Brownian motion in $\mathbb{R}^d$

We now turn to the question of the existence of a continuum space catalytic process $X = X^\Gamma$ for an irregular catalyst Γ. In the case of a general catalytic medium $\Gamma_t(\mathrm{d}x)$ the formal limit should solve the following measure-valued *martingale problem:*

$\mathbf{MP}_{\mathbb{R}^d}(\Gamma)$ For $\varphi \in D(\Delta)$,

$$\left.\begin{aligned} M_t(\varphi) &:= \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \int_0^t \left\langle X_s, \tfrac{1}{2}\Delta\varphi \right\rangle \mathrm{d}s, \\ \langle\langle M_\cdot(\varphi)\rangle\rangle_t &= \int_{[0,t]\times\mathbb{R}^d} \varphi^2(x)\, L_{[\Gamma,X]}\left(\mathrm{d}[s,x]\right), \end{aligned}\right\} \tag{3.15}$$

where $L_{[\Gamma,X]}\left(\mathrm{d}[s,x]\right)$ is the collision local time between the catalyst Γ and the reactant $X = X^\Gamma$ according to the following definition.

Definition 3.3 (collision local time). ([2]) The collision local time $L_{[X^1,X^2]} = L_{[X^1,X^2]}(\mathrm{d}[s,x])$ between two continuous measure-valued processes X^1, X^2 on $\mathbb{R}^d$ is said to exist, if the limit

$$\begin{aligned} &\int_{\mathbb{R}_+\times\mathbb{R}^d} \psi(s,x)\, L_{[X^1,X^2]}\left(\mathrm{d}[s,x]\right) \\ &:= \lim_{\delta\downarrow 0} \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi\Big(s, \frac{x^1+x^2}{2}\Big)\, \mathrm{p}_\delta(x^1-x^2)\, X_s^1(\mathrm{d}x^1)\, X_s^2(\mathrm{d}x^2)\,\mathrm{d}s \end{aligned} \tag{3.16}$$

exists in probability, for all continuous functions $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ which vanish for all sufficiently large s and satisfy

$$\sup_s \psi(s,x) \;\leqslant\; c_\psi\, \phi_p(x). \tag{3.17}$$

(Recall that the symbol p_δ refers to the heat kernel, and that ϕ_p is the reference function introduced in Remark 3.1.) ◊

In order to establish existence of $X = X^\Gamma$ for a class of irregular catalysts Γ, we first consider formally the corresponding log-Laplace functional:

$$-\log \mathbf{E}\left\{\exp\langle X_t, -\varphi\rangle \mid X_0\right\} \;=\; \langle X_0, u(0,\cdot\,|\,t)\rangle, \tag{3.18}$$

$t \geqslant 0$, $\varphi \geqslant 0$, where for t fixed, $u = u(\cdot,\cdot\,|\,t)$ is the solution of the *log-Laplace catalytic reaction-diffusion equation:*

$$-\frac{\partial u}{\partial r} \;=\; \frac{1}{2}\Delta u - \Gamma_r(\mathrm{d}x)u^2, \quad 0 \leqslant r < t, \quad u(t-,\cdot\,|\,t) = \varphi(\cdot) \geqslant 0. \tag{3.19}$$

(Note that compared with (1.1) we now used a *backward formulation* with a terminal condition φ to take into account the duality nature of the relationship between X and u given by (3.18) together with the time-inhomogeneity of Γ.) Equation (3.19) can be reformulated to capture the microscopic particle perspective via Dynkin's function space equation ([31]):

$$u(r,x\,|\,t) \;=\; E_{r,x}\left[\varphi(w_t) - \int_r^t u^2(s,w_s\,|\,t)\, A_w^\Gamma(\mathrm{d}s)\right]. \tag{3.20}$$

Here $E_{r,x}$ refers to expectation related to the law $P_{r,x}$ of (standard) Brownian motion w in $\mathbb{R}^d$ starting from x at time r, and the continuous additive functional $A_w^\Gamma(\mathrm{d}s)$ of Brownian motion w is the special case $L_{[\Gamma,\delta_w]}(\mathrm{d}s \times \mathbb{R}^d)$ of the collision local time $L_{[\Gamma,\delta_w]}$ between the catalyst Γ and the intrinsic Brownian reactant particle δ_w. We call A_w^Γ the *Brownian collision local time* with Γ.

The essential question now is for which continuous measure-valued paths Γ the Brownian collision local time A_w^Γ exists.

Example 3.4 (time-independent catalyst). In ([28]) the case $\Gamma_t \equiv \Gamma_0$ is treated, where Γ_0 is a deterministic measure on $\mathbb{R}^d$, $d \geqslant 2$. The condition

$$\sup_{x \in \mathbb{R}^d} \int_{\|y-x\| \leqslant 1} |y-x|^{-d+2-\beta} \, \Gamma_0(\mathrm{d}y) < \infty, \tag{3.21}$$

where $\beta \in (0,2)$ is a fixed constant, guarantees the existence of the Brownian collision local time A_w^Γ. In particular, (3.21) is satisfied if the Hausdorff dimension of the support $\mathrm{supp}\Gamma_0$ of Γ_0 is at least $d-2+\beta$. ◊

Remark 3.5 (branching functionals). The Brownian collision local time A_w^Γ is a special case of a continuous additive functional of Brownian motion w. More generally, continuous additive functionals of Markov processes are used to control the branching in related superprocesses; see, for instance, [30, 31, 17, 20, 54]. ◊

Now we return to a time-dependent catalyst Γ, which, for the moment, is deterministic.

Theorem 3.6 (existence of the catalytic SBM). ([17]) *Assume that the collision local time A_w^Γ exists and satisfies the local admissibility condition*

$$\sup_{x \in \mathbb{R}^d} E_{r,x} \int_r^t \phi_p(w_s) \, A_w^\Gamma(\mathrm{d}s) \xrightarrow[[r,t] \downarrow \{s_0\}]{} 0, \qquad s_0 \geqslant 0. \tag{3.22}$$

Then the equation (3.20) *is well-posed, and there is a measure-valued Markov process $X = X^\Gamma$ with log-Laplace transition functional* (3.18). *If, in addition, there is a $\delta > 0$ and for each $N \geqslant 1$ a constant $c_N \geqslant 1$ such that*

$$E_{r,x} \int_r^t \phi_p^2(w_s) \, A_w^\Gamma(\mathrm{d}s) \leqslant c_N \, |t-r|^\delta \, \phi_p(x), \qquad 0 \leqslant r \leqslant t \leqslant N, \quad x \in \mathbb{R}^d, \tag{3.23}$$

then X has $\mathcal{M}_p(\mathbb{R}^d)$-valued continuous trajectories [provided it starts in $\mathcal{M}_p(\mathbb{R}^d)$].

The process $X = X^\Gamma$ introduced in the previous theorem is called the *catalytic super-Brownian motion* in $\mathbb{R}^d$ with catalyst Γ. Let $\mathbf{P}_{r,\mu} = \mathbf{P}_{r,\mu}^\Gamma$ denote the law of X when it is started at time r with the measure μ.

Remark 3.7 (strong lifting). The collision local time $L_{[\Gamma,X]}$ which appears in the martingale problem $\mathbf{MP}_{\mathbb{R}^d}(\Gamma)$ in the beginning of this subsection is a continuous additive functional of the Markov process $X = X^\Gamma$, and in fact it is the "strong lifting" of the continuous additive functional $A_w^\Gamma(\mathrm{d}s)$ of Brownian motion w. More precisely, $L_{[\Gamma,X]}$ is a measure-valued continuous additive functional of X such that

$$\mathbf{E}_{r,\mu} L_{[\Gamma,X]}\left((r,t) \times \mathbb{R}^d\right) = \int_{\mathbb{R}^d} \int \int_r^t A_w^\Gamma(\mathrm{d}s) \, P_{r,x}(\mathrm{d}w) \, \mu(\mathrm{d}x) \tag{3.24}$$

(cf. [31, Theorem 6.2.3]). ◊

Example 3.8. Here we collect a couple of special cases of catalysts Γ which meet all requirements posed in Theorem 3.6.

(i) (single point catalyst in $\mathbb{R}$) ([15]) Consider the continuum space and mass analog of Example 2.1 (ii). Let $d = 1$, and $\Gamma_t \equiv \delta_0$, which gives $\int_0^t A_w^\Gamma(\mathrm{d}s) = \ell_t(0)$ with $\ell_t(0)$ denoting the Brownian local time at 0. In ([39]) it is established that

$$L_{[\Gamma,X]}(\mathrm{d}s \times \mathbb{R}^d) \;=\; L_{[\Gamma,X]}\left(\mathrm{d}s \times \{0\}\right), \tag{3.25}$$

the super-local time at the catalyst, has the same law as the total occupation measure of a continuous super-$\frac{1}{2}$-stable subordinator on $\mathbb{R}_+$ (starting with an appropriate initial measure related to X_0). Moreover, this super-local time is almost surely a singular measure (see also [22]).

If the critical branching is replaced by a supercritical one, it would then be possible to obtain results on the total reactant by using the previous result together with a Girsanov transformation.

(ii) (frozen Lévy random catalysts in $\mathbb{R}$) ([25]) Let $\Gamma_t \equiv \Gamma_0$, where Γ_0 is the (infinitely divisible) random measure on $\mathbb{R}$ defined via its log-Laplace functional:

$$-\log \mathbb{E}\exp\langle \Gamma_0, -\varphi\rangle \;=\; \int_{\mathbb{R}}\int_0^\infty \left(1 - \mathrm{e}^{-\lambda\varphi(x)}\right) \nu(d\lambda)\,\mathrm{d}x, \qquad \varphi \geqslant 0. \tag{3.26}$$

Here the Lévy measure ν of Γ_0 is assumed to satisfy $\int_0^\infty \min(\lambda, 1)\,\nu(d\lambda) < \infty$. Note that Γ_0 is a pure point measure, say $\Gamma_0 = \sum_j a_j\,\delta_{y^j}$.

Note also that the stable random measures of index $\gamma' \in (0,1)$ are special cases, where

$$\nu(\mathrm{d}\lambda) \;=\; c\,\lambda^{-1-\gamma'}\,\mathrm{d}\lambda, \tag{3.27}$$

and with atoms y_j dense in $\mathbb{R}$.

(iii) (fluctuating Lévy random catalyst in $\mathbb{R}$) ([13, 24, 14]) Let $\Gamma_0 = \sum_j a_j\delta_{y^j}$ be a stable random measure as in the end of (ii), and set

$$\Gamma_t \;:=\; \sum_j a_j\,\delta_{w_t^j} \tag{3.28}$$

where the w^j, given Γ_0, are independent Brownian motions starting from y^j.

(iv) (super-Brownian catalyst) ([17]) Let Γ be a SBM in $\mathbb{R}^d$, $d \leqslant 3$, with Γ_0 the Lebesgue measure. Then for almost all Γ the continuous super-Brownian reactant $X = X^\Gamma$ with super-Brownian catalyst Γ exists. ◊

4 Structural properties of catalytic SBMs

In this section we survey results on the structural properties of catalytic SBMs on $\mathbb{R}^d$.

4.1 Analytic and probabilistic questions

First we raise some questions in terms of the catalytic SBM $X = X^{\Gamma}$.

(a) (absolutely continous states) When are the measures X_t absolutely continuous?

(b) (long-term global extinction) Does the process $X = X^{\Gamma}$ suffer long-term global extinction, that is,

$$\|X_t\| := X_t(\mathbb{R}^d) \to 0 \quad \text{in probability as } t \uparrow \infty, \tag{4.1}$$

for all finite initial measures X_0?

(c) (long-term local extinction) Let X_0 be the Lebesgue measure. Does the process suffer *weak* long-term local extinction, that is, $\lim_{t\uparrow\infty} X_t(B) = 0$ in probability, for all bounded open $B \subset \mathbb{R}^d$? Or does it even suffer *strong* long-term local extinction, that is, the latter convergence statement holds a.s.?

(d) (compact support property) Does X have a compact global support $\mathrm{gsupp} X$ [recall (3.14)], provided that X_0 has compact support?

All these questions can be related to questions in terms of the log–Laplace catalytic reaction-diffusion equation

$$-\frac{\partial u}{\partial r} = \frac{1}{2}\Delta u - \Gamma_r(\mathrm{d}x)u^2, \quad 0 \leqslant r < t, \quad u(t-, \cdot \,|\, t) = \varphi(\cdot) \geqslant 0. \tag{4.2}$$

(a') (fundamental solutions) Does (4.2) have fundamental solutions? That is, can $\varphi = \theta\delta_x$ be taken as terminal condition, for sufficiently many $x \in \mathbb{R}^d$?

(b') (long-term sup-norm extinction) Does $\|u(0, \cdot\,|t)\|_\infty \to 0$ as $t \uparrow \infty$ when $u(t, \cdot\,|t) \equiv 1$?

(c') (long-term L^1–norm extinction) Does $\|u(0, \cdot\,|t)\|_1 \to 0$ as $t \uparrow \infty$ when $u(t, \cdot\,|t) \not\equiv 0$ has compact support?

(d') (elliptic boundary value problem) In the case $\Gamma_t \equiv \Gamma_0$, does for each compact subset $K \subset \mathbb{R}^d$ with non-empty interior K° the non-linear elliptic boundary value problem

$$\frac{1}{2}\Delta v = \Gamma_0 v^2 \text{ in } K^\circ, \quad v(x) \to \infty \text{ as } x \to \partial K, \tag{4.3}$$

have a solution?

In the subsequent subsections we will describe a number of typical results which address questions of this kind.

4.2 Absolutely continuous states

It is proved in ([16, Theorems 2.5.2, 2.6.2]) that, for each fixed $t > 0$, absolute continuity of the measure X_t is implied by the existence of fundamental solutions for Lebesgue-almost all $x \in \mathbb{R}^d$. [That is, (a') implies (a).] (See also [48].)

Example 4.1 (absolutely continuous states). Let $t > 0$.

(i) (constant medium) In the classical case $\Gamma_t(\mathrm{d}x) \equiv \gamma\,\mathrm{d}x$, fundamental solutions exist for all x if and only if $d = 1$. In other words, SBM in $\mathbb{R}^d$ has absolutely continuous states if and only if $d = 1$. (Otherwise the states are singular.)

(ii) (weighted hyperplanes catalyst) ([16, Example 4.4.4]) Let $d > 1$ and $\Gamma_t^{(d)} \equiv \Gamma_0 \times \ell^{(d-1)}$, where Γ_0 is the γ'-stable random measure on $\mathbb{R}$ of Example 3.8 (ii), and $\ell^{(d-1)}$ is the Lebesgue measure on $\mathbb{R}^{d-1}$. Then the states X_t are absolutely continuous if and only if $\gamma' < \frac{1}{2d-1}$. ◊

On the other hand, for the super-Brownian reactant $X = X^\Gamma$ with a super-Brownian catalyst Γ of Example 3.8 (iv) in dimensions $d = 2, 3$, the measures X_t, $t > 0$, are absolutely continuous, and the densities form a very smooth field. This can be shown directly in an L^2–approach, see [37]. Similarly, for this model the existence of the collision local time $L_{[\Gamma,X]}$ in dimensions $d \leqslant 3$, and the absolute continuity of the marginal measures $L_{[\Gamma,X]}\left([0,t] \times \mathrm{d}x\right)$, $t > 0$, in $d = 2$ can be shown by L^2–methods, see [29].

4.3 Global extinction/persistence

It follows from the log-Laplace functional relationship (3.18) that the questions (b) and (b') in Subsection 4.1 are equivalent. By a martingale argument, one then gets even an almost sure convergence $\|X_t\| \to 0$ as $t \uparrow \infty$.

Example 4.2 (long-term global extinction). Assume that $X_0 \in \mathcal{M}_{\mathrm{f}}(\mathbb{R})\backslash\{0\}$.

(i) (single point catalyst in $\mathbb{R}$) ([39]) Let $d = 1$ and $\Gamma_t \equiv \delta_0 = X_0$. Then

$$0 \;<\; \|X_t\| = \int_0^t \sqrt{\frac{2}{\pi(t-s)}}\, L_{[\Gamma,X]}\left(\mathrm{d}s \times \{0\}\right) \underset{t\uparrow\infty}{\longrightarrow} 0 \;\text{ a.s.} \tag{4.4}$$

(Recall that $L_{[\Gamma,X]}\left(\mathrm{d}s \times \{0\}\right)$ is the super-Brownian local time of X at 0. Consequently, there is no finite time global extinction but long-term global extinction (ultimate global extinction).

(ii) (uniform disk catalyst in $\mathbb{R}^2$) In contrast, in the case $d = 2$ with

$$\Gamma_t(\mathrm{d}x) \;\equiv\; \mathbf{1}_{\{|x|\leqslant 1\}}(x)\,\mathrm{d}x, \tag{4.5}$$

long-term global extinction does not hold:

$$\mathbf{P}\Big(\|X_t\| \underset{t\uparrow\infty}{\longrightarrow} 0\Big) \;<\; 1. \tag{4.6}$$

(To see this, use a Feynman-Kac formula as in (4.10) below.)

(iii) (hyperbolic catalyst) Let $d = 1$ and

$$\Gamma_t(\mathrm{d}x) \equiv |x|^{-2}\,\mathrm{d}x. \tag{4.7}$$

Then all super-Brownian "particles" die before they reach the catalytic center 0. This is made precise in [40]. ◊

Now we turn to our main model of a catalytic SBM.

Theorem 4.3 (catalytic SBM). *Consider the super-Brownian catalyst* Γ *starting from Lebesgue measure, as in Example* 3.8 (iv), *and start off* $X = X^\Gamma$ *with a finite measure* $\mu \neq 0$.

(i) (global persistence in $d = 1$) ([17]) *Let* $d = 1$ *and fix* $s \geqslant 0$ *as well as* $\mu \in \mathcal{M}_f \setminus \{0\}$. *Then for almost all* Γ,

$$m = m^\Gamma := \lim_{t \uparrow \infty} \|X_t\| \text{ exists } \mathbf{P}_{s,\mu}\text{-a.s.}, \tag{4.8}$$

where the limiting mass m^Γ *has full expectation* $\|\mu\|$ *(global persistence) and positive (finite) variance. Moreover,* m^Γ *has log-Laplace functional*

$$-\log \mathbf{P}_{s,\mu} \exp[-\theta m^\Gamma] = \langle \mu, u_\theta(s, \cdot)\rangle \tag{4.9}$$

with u_θ *satisfying the function space integral equation*

$$u_\theta(s,x) = \theta P_{s,x} \exp\left[-\int_s^\infty u_\theta(r, w_r)\, A_w^\Gamma(\mathrm{d}r)\right], \qquad s \geqslant 0, \quad x \in \mathbb{R}. \tag{4.10}$$

(ii) (long-term global extinction in $d = 2$) ([38]) *If* $d = 2$, *then* $\|X_t\| \to 0$ *almost surely as* $t \uparrow \infty$.

(iii) (ultimate global extinction in $d = 3$) ([38]) *If* $d = 3$, *then*

$$0 < \|X_t\| \to 0 \quad \text{a.s. as } t \uparrow \infty. \tag{4.11}$$

Remark 4.4 (open problem). It is not known whether finite time survival as in Theorem 4.3 (iii) holds also in dimension 2. (It would follow from a positive answer to the open problem in Remark 4.11 below.) ◊

Remark 4.5 (finite time extinction). [23] Opposed to Theorem 4.3 (iii), finite time global extinction can be shown by probabilistic methods in the case of the following catalysts in dimension one: deterministic power laws, i.i.d. uniform, and frozen stable catalysts as in Example 3.8 (ii). Interesting counterexamples are also constructed there. (For a survey on these results, see [41].) ◊

Remark 4.6 (models with additional birth term). [32] Analogously to the one-dimensional single point catalytic model of Example 3.8 (i), an additional single point source can be introduced in (4.2) instead, leading to a SBM with an exponential growth of the mean mass. ◊

4.4 Local extinction/persistence

Here we assume that $X = X^\Gamma$ starts from the Lebesgue measure. It follows from the Laplace functional representation that the weak long-term local extinction question in (c) is equivalent with (c').

Example 4.7 (extinction/persistence). Recall that $X_0(\mathrm{d}x) = \mathrm{d}x$.

(i) (classical SBM) It is well-known that classical SBM suffers weak long-term local extinction in dimensions 1 and 2 but converges to a steady state with full intensity (local persistence) in dimensions $d \geqslant 3$ (cf. [3, 4]).

The question of strong local extinction is more difficult. In dimension one, results from [46] can be used to show that

$$\mathbf{P}\left(X_t(B(0,r)) = 0 \text{ for all } t > T\right) \underset{T\uparrow\infty}{\longrightarrow} 1, \tag{4.12}$$

where $B(0,r)$ denotes a ball of radius r centered at the origin, implying strong long-term local extinction.

(ii) (single point catalyst) ([15]) In the single point-catalytic model of Example 3.8 (i), weak long-term local extinction holds true. ◊

Theorem 4.8 (local persistence under a super-Brownian catalyst in $\mathbb{R}^d$). *Consider the super-Brownian catalyst of Example* 3.8 (iv), *starting from Lebesgue measure.*

(i) ($d = 1$) ([17]) *For almost all Γ-realizations, X_t converges in probability as $t \uparrow \infty$ to the Lebesgue measure.*

(ii) ($d = 2$) ([33, 37]) *For almost all Γ-realizations, X_t converges in law as $t \uparrow \infty$ to a random multiple of Lebesgue measure of full expectation.*

(iii) ($d = 3$, **infinite biodiversity**) *Consider the super-Brownian catalyst Γ in its equilibrium state. Then X_t converges in law as $t \uparrow \infty$ to a random limit X_∞ of full expectation* ([18]).

For almost all Γ, each non-empty open ball in $\mathbb{R}^3$ is hit by infinitely many clusters in the cluster representation of the infinitely divisible random measure X_∞ ([38]).

4.5 Compact support property

In the classical case of a constant medium $\Gamma_t(\mathrm{d}x) \equiv \gamma\,\mathrm{d}x$, the compact support property as formulated in (d) of Subsection 4.1 can be shown with the help of the elliptic boundary value problem as in (d'); see [46]. Therefore the ordinary super-Brownian motion in $\mathbb{R}^d$ has the compact support property. This type of relation between the compact support property and an elliptic boundary value problem is used also in the following example.

Example 4.9 (frozen random Lévy catalyst). [25] Let $d = 1$ and $\Gamma_t \equiv \Gamma_0$ as in Example 3.8 (ii).

(i) (stable catalyst) If Γ_0 is a stable random catalyst as in the end of Example 3.8 (ii), then $X = X^\Gamma$ has the compact support property.

(ii) (rarefied random Lévy catalyst) Let Γ_0 have Lévy measure

$$\nu(d\lambda) = \frac{1}{\lambda} \mathbf{1}(0 < \lambda \leqslant 1)\, d\lambda. \tag{4.13}$$

Then the global support gsupp$X$ is a.s. non-compact, provided that $X_0 \neq 0$. ◊

The previous example (ii) shows a case where the compact support property is lost if the catalyst is too rarefied. We mention still another example in this direction.

Example 4.10 (instantaneous propagation of reactant matter). ([38])
Consider the super-Brownian reactant $X = X^\Gamma$ in $\mathbb{R}^3$ with a super-Brownian catalyst Γ of Example 3.8 (iv). Then for all $t > 0$ and almost all Γ,

$$\mathbf{P}\left\{X_t(B) > 0 \ \forall \text{ non-empty open balls } B \subset \mathbb{R}^3 \;\middle|\; X_t \neq 0\right\} = 1. \tag{4.14}$$

Combined with the absolute continuity of X_t, the equivalence of X_t with the Lebesgue measure follows, provided that $X_t \neq 0$. ◊

Remark 4.11 (open problem). It is not known whether the instantaneous propagation of reactant matter property according to Example 4.10 also holds in dimension $d = 2$. ◊

5 Mutually catalytic branching

In the catalytic branching models of Sections 3 and 4, the catalyst was always an autonomous random process Γ which catalyzes the reactant $X = X^\Gamma$. In this last part of the paper we return to the mutually catalytic model in $\mathbb{Z}^d$ introduced in Subsection 2.4, and its continuum space analog.

5.1 Weak uniqueness via duality

First of all, we reformulate the stochastic equation (2.9) as a *martingale problem* for the pair $\mathbf{X} = (X^1, X^2)$ of processes:

MP$_{\mathbb{Z}^d}$ For $j = 1, 2$ and test functions $\varphi^j : \mathbb{Z}^d \to \mathbb{R}$ of compact support,

$$M_t^j(\varphi^j) := \left\langle X_t^j, \varphi^j \right\rangle - \langle \mu^j, \varphi^j \rangle - \int_0^t \left\langle X_s^j, \frac{1}{2}\Delta\varphi^j \right\rangle ds, \qquad t \geqslant 0, \tag{5.1}$$

are continuous square integrable martingales with continuous square function

$$\left\langle\!\left\langle M_\cdot^j(\varphi^j), M_\cdot^k(\varphi^k) \right\rangle\!\right\rangle_t = \delta_{j,k}\, \gamma \left\langle L_{\mathbf{X}}(t), \varphi^j \varphi^k \right\rangle, \tag{5.2}$$

where

$$\langle L_{\mathbf{X}}(t), \varphi^j \rangle := \int_0^t \sum_{y \in \mathbb{Z}^d} \varphi^j(y)\, X_s^1(y)\, X_s^2(y)\, \mathrm{d}s, \qquad t \geqslant 0. \tag{5.3}$$

Here $\delta_{j,k}$ denotes the Kronecker symbol. Consequently, $L_{\mathbf{X}}(t)$ is the *collision local time* by time t, in analogy with Definition 3.3.

Theorem 5.1 (mutually catalytic branching on $\mathbb{Z}^d$). ([26]) *There is a continuous non-degenerate Markov process $\mathbf{X} = (X^1, X^2)$ on $\mathbb{Z}^d$ with states in an appropriate space of tempered measures which satisfies the martingale problem $\mathbf{MP}_{\mathbb{Z}^d}$. The law of $\mathbf{X}$ is unique.*

Existence of solutions to the system (2.9) of stochastic differential equations, that is, to the martingale problem $\mathbf{MP}_{\mathbb{Z}^d}$, can be obtained by standard methods. However, the proof of uniqueness requires a duality argument which goes back to [50]. It turns out that an independent copy $\widetilde{\mathbf{X}} = (\widetilde{X}^1, \widetilde{X}^2)$ of the same system provides a duality relationship ([26]), which we next want to describe.

By a simple coordinate transformation, we introduce a new pair $\mathbf{Y} = (Y^1, Y^2)$ of measure-valued processes,

$$Y^1 := X^1 + X^2, \quad Y^2 := X^1 - X^2, \tag{5.4}$$

and we denote by $\widetilde{\mathbf{Y}} = (\widetilde{Y}^1, \widetilde{Y}^2)$ an independent copy of $\mathbf{Y}$. Moreover, we introduce the self-duality function

$$\Phi(x, y) := \mathrm{e}^{-x+iy}, \qquad x, y \in \mathbb{R}, \tag{5.5}$$

(with i the imaginary unit). According to our convention (2.2), the measures $\widetilde{Y}_t^j$ on $\mathbb{Z}^d$ are identified with their density functions. Then by Itô's formula it can be checked that

$$\begin{aligned} &\Phi\left(\langle Y_t^1, \widetilde{Y}_0^1\rangle, \langle Y_t^2, \widetilde{Y}_0^2\rangle\right) - \Phi\left(\langle Y_0^1, \widetilde{Y}_0^1\rangle, \langle Y_0^2, \widetilde{Y}_0^2\rangle\right) \qquad (5.6)\\ &= \int_0^t \left[-\left\langle Y_s^1, \tfrac{1}{2}\Delta \widetilde{Y}_0^1\right\rangle + i\left\langle Y_s^2, \tfrac{1}{2}\Delta \widetilde{Y}_0^2\right\rangle + \frac{\gamma}{4}\left\langle (Y_s^1)^2 - (Y_s^2)^2, (\widetilde{Y}_0^1)^2 - (\widetilde{Y}_0^2)^2\right\rangle\right] \\ &\qquad \times \Phi\left(\langle Y_s^1, \widetilde{Y}_0^1\rangle, \langle Y_s^2, \widetilde{Y}_0^2\rangle\right) \mathrm{d}s \;+\; M_t, \qquad t \geqslant 0, \end{aligned}$$

where M is a martingale. A similar formula holds by interchanging the role of $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$. By standard arguments (cf. [34, Section 4.4]), this then implies the following *self-duality* relation:

$$\mathbf{E}\Phi\left(\langle Y_t^1, \widetilde{Y}_0^1\rangle, \langle Y_t^2, \widetilde{Y}_0^2\rangle\right) = \mathbf{E}\Phi\left(\langle Y_0^1, \widetilde{Y}_t^1\rangle, \langle Y_0^2, \widetilde{Y}_t^2\rangle\right). \tag{5.7}$$

Based on this, existence proves uniqueness in the martingale problem for $\mathbf{Y} = (Y^1, Y^2)$ which is related to $\mathbf{MP}_{\mathbb{Z}^d}$ via the transformation (5.4).

5.2 Long-time behavior: Finite initial states

Assume now that the mutually catalytic branching process $\mathbf{X} = (X^1, X^2)$ on $\mathbb{Z}^d$ of Theorem 5.1 starts off at time 0 with non-zero finite deterministic measures X_0^1 and X_0^2. Then it is easy to verify that $\|\mathbf{X}_t\| := (\|X_t^1\|, \|X_t^2\|)$ coincides in law with $B(\gamma \|L_{\mathbf{X}}(t)\|)$ (recall the definition (5.3) of the collision local time), where $t \mapsto B(t)$ is a planar Brownian motion starting from $\|\mathbf{X}_0\| \in \mathbb{R}_+^2$. By the martingale convergence theorem,

$$\|\mathbf{X}_t\| \underset{t\uparrow\infty}{\longrightarrow} \text{some } \|\mathbf{X}_\infty\| = (\|X_\infty^1\|, \|X_\infty^2\|) \text{ a.s.}, \tag{5.8}$$

and hence $\mathbf{E}\|\mathbf{X}_\infty\| = B(0) = \|\mathbf{X}_0\|$. This already gives part (i) in the following theorem.

Theorem 5.2 (finite measure-valued mutually catalytic branching in $\mathbb{Z}^d$). *Assume* $0 < \|X_0^1\| \cdot \|X_0^2\| < \infty$.

(i) (global persistent convergence) ([26]) $\|\mathbf{X}_t\|$ *converges a.s. as* $t \uparrow \infty$ *to some* $\|\mathbf{X}_\infty\|$ *with full expectation.*

(ii) (global coexistence of types in $d \geqslant 3$) ([26]) *In dimensions* $d \geqslant 3$,

$$\mathbf{E}\left\|L_{\mathbf{X}}(\infty)\right\| < \infty \quad \textit{and} \quad \mathbf{P}\left(\|X_\infty^1\| \cdot \|X_\infty^2\| > 0\right) > 0. \tag{5.9}$$

(iii) (global segregation of types in $d \leqslant 2$) *In dimensions* $d \leqslant 2$, *the limit* $\|\mathbf{X}_\infty\|$ *coincides in law with* $B(\tau)$, *where*

$$\tau := \inf\{t : B^1(t)B^2(t) = 0\} \tag{5.10}$$

is the almost surely finite exit time from $\mathbb{R}_+^2$ *of the planar Brownian motion* $B = (B^1, B^2)$ *starting from* $\|\mathbf{X}_0\|$. *In particular,*

$$\|X_\infty^1\| \cdot \|X_\infty^2\| = 0 \quad \textit{a.s.} \tag{5.11}$$

([26]). *However* ([49]), *for certain initial states* $\mathbf{X}_0$, *finite time survival holds:*

$$\|X_t^1\| \cdot \|X_t^2\| > 0, \qquad t > 0, \quad \textit{a.s.}, \tag{5.12}$$

whereas for certain other cases, for each fixed $T > 0$ *and* $\varepsilon \in (0,1)$,

$$\mathbf{P}\left(\|X_t^1\| \cdot \|X_t^2\| = 0 \text{ for } t \geqslant T\right) \geqslant 1 - \varepsilon, \tag{5.13}$$

provided that one of X_0^1 *or* X_0^2 *is sufficiently small (that is, finite time global segregation occurs with large probability).*

The long-term global extinction of one type in statement (iii) is proved by showing that

$$\left\|L_{\mathbf{X}}(t)\right\| \underset{t\uparrow\infty}{\longrightarrow} \infty \text{ on } \{\|X_\infty^1\| \cdot \|X_\infty^2\| > 0\}. \tag{5.14}$$

Summarizing, in the finite measure case, a global segregation of types occurs in low dimensions, whereas coexistence of types occurs in high dimensions.

5.3 Long-time behavior: Uniform initial states

In this subsection we assume that the initial state $\mathbf{X}_0 = (X_0^1, X_0^2)$ is given by uniform measures $\mathbf{c}\ell = (c^1\ell, c^2\ell)$ with intensities $c^1, c^2 > 0$. First we again ask for the long-term behavior without scaling. In order to address the question "How big are the one-type blocks for t large?", we let $r, \beta, \varepsilon > 0$ and define for $A = \{x \in \mathbb{R}^d : |x| \leqslant r\}$,

$$X_t^{j,\varepsilon}(A) := \varepsilon^{\beta d} \sum_{x \in \varepsilon^{-\beta} A} X^j_{\varepsilon^{-1}t}(x), \qquad j = 1, 2, \quad t \geqslant 0. \tag{5.15}$$

Furthermore, we denote by $\ell^{(d)}$ the Lebesgue measure on $\mathbb{R}^d$.

Theorem 5.3 (long-term behavior for uniform initials). *Let* $\mathbf{X}_0 = \mathbf{c}\ell$.

(i) (local persistent convergence) ([26]) $\mathbf{X}_t$ *converges in law as* $t \uparrow \infty$ *to a pair of spatially homogeneous random measures* $\mathbf{X}_\infty = (X_\infty^1, X_\infty^2)$ *having full intensity* $\mathbf{c}$.

(i1) (local coexistence of types in high dimensions) *Assume* $d \geqslant 3$. *Then*

$$X_\infty^1(x)\, X_\infty^2(x) > 0, \quad x \in \mathbb{Z}^d, \quad a.s. \tag{5.16}$$

(i2) (local segregation of types in low dimensions) *If* $d \leqslant 2$, *then*

$$\mathbf{X}_\infty = B(\tau)\, \ell^{(d)}, \tag{5.17}$$

where τ *is the exit time* [*recall* (5.10)] *from* $\mathbb{R}^2_+$ *of planar Brownian motion* B *starting from* $\mathbf{c}$.

(ii) (growth of one-type blocks in $d \leqslant 2$) ([7]) *Let* $d \leqslant 2$, *fix* $t > 0$, *and let* $\varepsilon \downarrow 0$.

(ii1) (supercritical blocks) *If* $\beta > \frac{1}{2}$, *then*

$$\mathbf{X}_t^\varepsilon := \left(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}\right) \to \mathbf{c}\ell^{(d)} \quad \text{in law.} \tag{5.18}$$

(ii2) (subcritical blocks) *If* $d = 1$ *and* $\beta < \frac{1}{2}$, *then*

$$\mathbf{X}_t^\varepsilon \to B(\tau)\, \ell^{(1)} \quad \text{in law.} \tag{5.19}$$

(iii) (oscillations of types in low dimension) ([8]) *Let* $d \leqslant 2$. *Then with probability one,*

$$0 = \liminf_{t\uparrow\infty} X_t^j(x) < \limsup_{t\uparrow\infty} X_t^j(x) = \infty, \qquad j = 1, 2, \quad x \in \mathbb{Z}^d. \tag{5.20}$$

Consequently, in all dimensions there is local persistent convergence. In high dimensions local coexistence of types prevails. In low dimensions, however, locally only one type survives finally. More precisely, in the long run locally there is one predominant type near 0. "Unitype blocks" at time $t = 1/\varepsilon$ are of size $O(1/\sqrt{\varepsilon})$. Moreover, in the case $d = 1$ the origin lies in an even larger single-type block as time increases. Finally, the predominant type near zero oscillates infinitely often as $t \uparrow \infty$.

For the case of more general initial states, we refer to [9].

5.4 Mutually catalytic branching on $\mathbb{R}$

Next we ask for a continuum analog of the mutually catalytic branching model of Theorem 5.1. Formally, instead of (2.9) we obtain a pair of non-linear stochastic partial differential equations:

$$\frac{\partial}{\partial t}X_t^j(x) = \frac{1}{2}\Delta X_t^j(x) + \sqrt{\gamma X_t^2(x)\, X_t^2(x)}\, \dot{W}_t^j(x), \quad t > 0,\ x \in \mathbb{R}^d,\ j = 1, 2, \tag{5.21}$$

where $\dot{W}^1, \dot{W}^2$ are independent (standard) space-time white noises on $\mathbb{R}^d$.

In the one-dimensional case, the stochastic equation (5.21) can be made precise:

Theorem 5.4 (mutually catalytic branching in $\mathbb{R}$). ([26]) *Let $d = 1$.*

(i) (existence and uniqueness in law) *There is a function-valued solution $\mathbf{X} = (X^1, X^2)$ to the stochastic equation* (5.21) *such that*

$$\sup_{s \leqslant t} \mathbf{P}\left(\int_{\mathbb{R}} \left[X_s^1(x)^q + X_s^2(x)^q \right] \mathrm{e}^{-\lambda |x|}\, \mathrm{d}x \right) < \infty \tag{5.22}$$

for some $\lambda, q > 0$. Uniqueness in law holds ([51]).

(ii) (persistent convergence) *In analogy with Theorems* 5.2 (i) *and* 5.3 (i), *persistent convergence holds.*

(iii) (segregation of types) *In analogy with Theorems* 5.2 (iii) *and* 5.3 (i2), *segregation of types occurs.*

For a recent cyclically catalytic model in $\mathbb{R}$ we refer to [42].

5.5 Mutually catalytic branching on $\mathbb{R}^2$

Next we ask for a continuum space mutually catalytic model if $d \geqslant 2$, at least in a measure-valued setting, as in the martingale problem $\mathbf{MP}_{\mathbb{R}^d}(\Gamma)$ of Subsection 3.3.

Based on the (one-way) catalytic results of Section 4, heuristically we expect for a possible limit that

- $X_t^j(\mathrm{d}x) \ll \mathrm{d}x$ on the complement of $\mathrm{supp} X_t^k$, where $j \neq k$, and the symbol $\ll$ denotes absolute continuity,
- $\mathrm{supp} X_t^1 \cap \mathrm{supp} X_t^2$ has zero Lebesgue measure,
- $X_t^1(\mathrm{d}x) \ll \mathrm{d}x$ and $X_t^2(\mathrm{d}x) \ll \mathrm{d}x$ off this interface.

For the rest of the paper we assume that $d = 2$.

Theorem 5.5 (mutually catalytic branching in $\mathbb{R}^2$). ([10]) *Let $\gamma > 0$ be sufficiently small, and restrict attention to a "reasonable" pair $\mathbf{X}_0 = (X_0^1, X_0^2)$ of initial measures on $\mathbb{R}^2$.*

(i) (existence) *There is a pair* $\mathbf{X} = (X^1, X^2)$ *of measure-valued processes with the following property. For* $j = 1, 2$ *and* φ^j *in a nice space of smooth test functions,*

$$M_t^j(\varphi^j) := \left\langle X_t^j, \varphi^j \right\rangle - \langle \mu^j, \varphi^j \rangle - \int_0^t \left\langle X_s^j, \frac{1}{2}\Delta\varphi^j \right\rangle \mathrm{d}s, \qquad t \geqslant 0, \tag{5.23}$$

is a continuous square integrable (zero-mean) martingale with continuous square function

$$\left\langle\!\left\langle M_\cdot^j(\varphi^j), M_\cdot^k(\varphi^k) \right\rangle\!\right\rangle_t = \delta_{j,k}\gamma \int_{[0,t]\times\mathbb{R}^2} \varphi^j(x)\varphi^k(x)\, L_{\mathbf{X}}\left(\mathrm{d}[s,x]\right) \tag{5.24}$$

with $L_{\mathbf{X}}$ *the collision local time between* X^1 *and* X^2 *according to Definition* 3.3.

(ii) (absolutely continuous states) *For* $t > 0$ *fixed,* $\mathbf{X}_t$ *has absolutely continuous states:* $\mathbf{X}_t(\mathrm{d}x) = \mathbf{X}_t(x)\,\mathrm{d}x$.

(iii) (segregation of types) *For* $\ell^{(2)}$*-almost all* x*, the law of* $\mathbf{X}_t(x)$ *coincides with the exit state* $B(\tau)$ *[recall* (5.10)*] of planar Brownian motion starting from the point* $\mathbf{X}_0 * \mathrm{p}_t\,(x)$*. In particular,*

$$\mathbf{Var}\, X_t^j(x) = \infty, \qquad j = 1, 2, \tag{5.25}$$

(provided that $X_0^j \neq 0$, $j = 1, 2$*), and* $X_t^1(x)X_t^2(x) = 0$ *almost surely.*

(iv) (self-similarity) *If* $\mathbf{X}_0 = \mathbf{c}\ell^{(2)}$*, then*

$$t \mapsto \varepsilon^2 \mathbf{X}_{\varepsilon^{-2}t}\left(\varepsilon^{-1}\cdot\right) \tag{5.26}$$

has the same law as $\mathbf{X}$*, for all* $\varepsilon > 0$*. In particular, the law for the densities in* (iii) *is constant in time and space.*

(v) (persistent convergence) *In analogy with Theorems* 5.2 (i) *and* 5.3 (i)*, persistent convergence holds.*

(vi) (segregation of types) *In analogy with Theorems* 5.2 (iii) *and* 5.3 (i2)*, long-term segregation of types occurs.*

Note that the existence claim on the collision local time $L_{\mathbf{X}}$ of the processes X^1 and X^2 is an integral part of the martingale problem in (i).

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