Preface

Recently great progress has been made in the field of dynamical systems. In January 1995, the Royal Netherlands Academy of Arts and Sciences enabled us to organize a meeting aimed at a wide audience consisting of mathematicians, physicists, biologists and economists. This collection of articles comprises the contributions of most of the speakers.

We felt that several new developments in dynamical systems are important or will become so in the near future. We decided to select some areas which are close to applications and related to noise, randomness and spatial structures. Broadly speaking our aim was to centre the meeting around three topics.

(i) the effect of noise on data generated by dynamical systems and testing whether these dynamical systems adequately model "reality";

(ii) spatial structures which can be generated by dynamical systems and which act on a network of coupled systems (coupled lattice maps);

(iii) random differential equations and applications to biology.

Noise and chaotic dynamics

One effect of adding noise to a system with chaotic dynamics is that it can drastically change its attractor. One reason for this is that noise is amplified in a system which is not very hyperbolic: this becomes especially important when some components of the basin of an attractor are extremely small. Of course, one is often interested in the underlying deterministic system of a dynamical system f which has noise. To formalize this, one could define the *deterministic approximation* $x \mapsto f^*(x)$ of a "noisy system" f to be the conditional expected value $E(f|x)$. However, $(f^2)^*$ need not be equal to $(f^*)^2$. As Takens shows in his paper, this implies that it is in some ways meaningless to ask whether a system with noise is really chaotic. (Because $(f^2)^*$ can be chaotic even when f^* is not.)

One approach to systems with noise is estimating correlation integrals. Given some numerical data, one can try to estimate some of these numbers. In Keller and Sporer's article linear regression estimators are discussed for the correlation dimension, entropy and detection of noise. These estimators are applied to data related to the Hénon map.

In the article of Cheng and Tong, delay coordinates from Takens' embedding theorem are discussed in the context of stochastic dynamical systems. More specifically, assume that one has a stochastic dynamical system of the form

$$X_t = F(X_{t-1}, \ldots, X_{t-d_0}) + \epsilon_t$$

where the condition expectation of ϵ_t (given $X_{t-1}, \ldots, X_{t-d_0}$) is zero. Estimates for suitable choices of the lag d_0 and the required sample size are discussed.

Coupled lattice maps

Recently, many numerical studies and some theoretical results have been obtained on *Coupled Lattice Maps* abbreviated frequently as CLM's. These are systems which are meant to model spatial structures where the state of a site is determined dynamically by the previous state at that site and that of its neighbours. Such models have a wide range of applications to physics (crystals), biology (nervous systems, population dynamics), economics (interaction of different markets), reaction-diffusion equations (see Section 2 of the paper of Losson and Mackey) and so on. Numerically, one can observe the formation of waves, patterns, synchronization in which coupling plays an important role. In one class of models the state $x_{n+1}(i)$ at time $n+1$ at site i is determined by the states at the neighbouring sites $i-1, i, i+1 \in \mathbb{Z}$ at time n. An example of such a one-dimensional nearest site model is

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{2}\left[f(x_n(i+1)) + f(x_n(i-1))\right]$$

where $x_n(i)$ is the state of the system at site $i \in \mathbb{Z}$ at time $n \in \mathbb{N}$. If $\epsilon = 0$ then each site i has a time dynamics which is completely uncoupled from those at other sites. When $\epsilon \in (0,1)$ then the dynamics at distinct sites will interact. Of course, the non-linearity of the map f also plays an important role. Similar models can also be constructed when $i \in \mathbb{Z}$ is replaced by $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ (the two-dimensional case) in which case the interaction at site (i,j) could be with its nearest neighbours. In many situations one observes random patterns which are "frozen", or "defects" which zigzag in space. Sometimes, also certain regular patterns suddenly break up and one obtains spatiotemporal chaos.

In the paper of Losson and Mackey a survey is given on CLM's. In particular, the effect of adding stochastic perturbations onto a CLM system is discussed. In fact, even without stochastic perturbations a CLM can behave "ergodically". One way of describing these ergodic properties is discussed in Keller's paper: it is shown that if the map f is sufficiently expanding (and itself has a good invariant probability measure) and the coupling is sufficiently small then one has a good invariant probability measure. In his paper, Mackay shows that for a rather general class of CLM's (with coupling parameter ϵ) solutions of the system for $\epsilon = 0$ persist when $\epsilon > 0$. In fact, since the proof of this uses an implicit function theorem, Mackay is able to show that these results even apply for rather large ϵ. This explains why one can have "spatially local" dynamics.

Instead of coupled lattice maps, Mallet-Paret studies coupled lattice differential equations (LDE's). The class of systems Mallet-Paret discusses in his paper, are simplified versions of systems which seem to be able to identify patterns in digitized images (Cellular Neural Networks, studied experimentally by Chua, Hasler and others). Using methods from bifurcation theory, Mallet-Paret shows that - depending on parameters in the model - all kinds of stripe or check pattern solutions exist and discusses the stability of such solutions. In addition, he discusses travelling wave solutions and also defines and describes systems with spatial chaos.

Random differential equations and applications

Many applications are modelled by the telegraph equation

$$u_{tt} + 2\mu u_t = \gamma^2 u_{xx}.$$

This equation is in some senses an interpolation between the wave equation $u_{tt} = \gamma^2 u_{xx}$ (taking $|\mu|$ small) and a diffusion equation $u_t = (\gamma^2/2\mu)$ (taking μ large). Hadeler's article gives an overview of the many applications of the telegraph equation and its connections to random walks. For example, it is shown that the telegraph equation is equivalent to the system

$$u_t^+ + \gamma u_x^+ = \mu(u^- - u^+),$$

$$u_t^- - \gamma u_x^- = \mu(u^+ - u^-)$$

where $u_\pm$ are the densities of a particle performing a correlated random walk on the real line with speed $\pm\gamma$ (i.e., $u^\pm(t,x) \geq 0$ and $\int_{-\infty}^{\infty} u^+(t,x), u^-(t,x))dx = 1$).

Instead of adding noise to a differential equation (as is done in stochastic differential equations), one can also add a term which comes from a chaotic flow. This point of view is considered in Johnson's article. He considers differential equations of the form

$$x' = f(T_t(y), x), \quad y \in Y, x \in \mathbb{R}^n$$

where Y is some topological space with probability measure μ which is ergodic w.r.t. the flow $T_t: Y \to Y$ and $f: Y \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous in the second variable. The notion of exponential dichotomy and its connection with the theorem of Oseledec is discussed. Furthermore, some examples of bifurcations of such systems are considered.

Finally, in an article of Metz e.a. stochastic processes are suggested which model long-term biological evolution.

Clearly, there are many new exciting developments in dynamical systems. Hopefully these proceedings give a good impression of some of these.

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