

STOCHASTIC PDE's WITH FUNCTION-VALUED SOLUTIONS

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Abstract

The paper provides necessary and sufficient conditions under which stochastic heat and wave equations on \mathbb{R}^d have function-valued solutions. The results extend, to all dimensions d and to all spatially homogeneous perturbations, recent characterizations by Dalang and Frangos [2]. The paper proposes a natural framework for a study of nonlinear stochastic equations. It is based on the harmonic analysis technique and on the stochastic integration theory in functional spaces. Generalizations to the d -dimensional torus and to nonlinear equations are discussed as well.

1 Introduction

The paper is concerned with the stochastic heat and wave equations:

$$\begin{cases} \frac{\partial u}{\partial t}(t, \theta) = \Delta u(t, \theta) + \frac{\partial W_\Gamma}{\partial t}(t, \theta), & t > 0, \quad \theta \in \mathbb{R}^d, \\ u(0, \theta) = 0, & \theta \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

and

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, \theta) = \Delta u(t, \theta) + \frac{\partial W_\Gamma}{\partial t}(t, \theta), & t > 0, \quad \theta \in \mathbb{R}^d, \\ u(0, \theta) = 0, \quad \frac{\partial u}{\partial t}(0, \theta) = 0, & \theta \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

where W_Γ is a spatially homogeneous Wiener process with the space correlation Γ . The correlation Γ can be any positive definite distribution. It defines the covariance operator of the Wiener process by the formula $Q\varphi = \Gamma * \varphi$, $\varphi \in S(\mathbb{R}^d)$.

It is well-known, see [25], that if $\frac{\partial W_\Gamma}{\partial t}$ is a space-time white noise, or equivalently if $\Gamma = \delta_{\{0\}}$, then the equations (1.1), (1.2) have function-valued solutions if and only if the space dimension $d = 1$. It is therefore of interest to find out in dimensions $d \geq 1$ for what space-correlated noise, equations (1.1) and (1.2), have function-valued solutions. This problem has been recently investigated, for the stochastic wave equation, by Dalang and Frangos [2], see also Mueller [18], when $d = 2$. Let $W_\Gamma(t, \theta)$, $t \geq 0, \theta \in \mathbb{R}^2$, be a Wiener process with a space correlation function Γ :

$$\mathbb{E} W_\Gamma(t, \theta) W_\Gamma(s, \eta) = t \wedge s \Gamma(\theta - \eta), \quad \theta, \eta \in \mathbb{R}^2,$$

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where $\Gamma(\theta) = f(|\theta|)$, $\theta \in \mathbb{R}^2$, and f is a non-negative function, continuous except at 0. It has been shown in [2] that the stochastic wave equation (1.2) has a function-valued solution if and only if

$$\int_{|\theta| \leq 1} f(|\theta|) \ln \frac{1}{|\theta|} d\theta < +\infty. \quad (1.3)$$

The proof in [2] is based on an explicit representation of the fundamental solution of the deterministic wave equation in dimension $d = 2$ and can not be extended to higher dimensions.

In the present note we treat the general case of *arbitrary dimension d and of arbitrary spatially homogeneous noise* for both the *stochastic heat and wave equations*. Spatially homogeneous noise processes were introduced by Holley and Stroock [11] and Dawson and Salehi [5] in connection with particle systems, see also Nobel [20], Da Prato and Zabczyk [4] and Peszat and Zabczyk [21], [22] for more recent investigations. We consider also equations (1.1) and (1.2) on the d -dimensional torus T^d . It is interesting to note that for both stochastic heat and wave equations, (1.1) and (1.2) on \mathbb{R}^d and on T^d , the *necessary and sufficient conditions are exactly the same*, despite the different nature of the equations. Also the fact that our conditions are related to *Newtonian potentials* makes the appearance of the logarithmic function in the Dalang and Frangos paper [2] more natural. The obtained characterizations form a natural framework in which *nonlinear heat and wave equations* can be studied. Similar results can be formulated for linear parts of Navier-Stokes equations and other equations of fluid dynamics. Techniques developed in the paper apply also to equations (1.1) and (1.2) with Δ replaced by fractional Laplacien $-(-\Delta)^\alpha$, $\alpha \in (0, 2]$. However, those generalizations are not studied here.

To formulate our main theorems let us recall, see [9], that positive definite, tempered distributions Γ are precisely Fourier transforms of tempered measures μ . The measure μ will be called the *spectral measure* of Γ and of the process W_Γ .

Theorem 1.1. *Let Γ be a positive definite, tempered distribution on \mathbb{R}^d , with the spectral measure μ . Then the equations (1.1) and (1.2) have function-valued solutions if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda) < +\infty. \quad (1.4)$$

Theorem 1.2. *Assume that Γ is not only a positive definite distribution but also a non-negative measure. The equations (1.1) and (1.2) have function-valued solutions:*

- i) *for all Γ if $d = 1$;*
- ii) *for exactly those Γ for which $\int_{|\theta| \leq 1} \ln |\theta| \Gamma(d\theta) < +\infty$ if $d = 2$;*
- iii) *for exactly those Γ for which $\int_{|\theta| \leq 1} \frac{1}{|\theta|^{d-2}} \Gamma(d\theta) < +\infty$ if $d \geq 3$.*

Note that condition (1.3) is a special case of ii).

Similar theorems hold for stochastic heat and wave equations on the d -dimensional torus, see Theorem 5.1 and Theorem 5.3 in §5.

The paper is organized as follows. Preliminaries and the formulation of the problem will be given in section 2. Section 3 contains proofs of the results for the case of \mathbb{R}^d . Applications are discussed in section 4. Extensions to the d -dimensional torus are contained in section 5. We finish the paper with two conjectures and some partial answers in section 6.

The present paper is an updated version of the preprint [14].

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2 Preliminaries

2.1 Heat and wave semigroups

Let $S_c(\mathbb{R}^d)$ denote the space of all infinitely differentiable functions ψ on \mathbb{R}^d taking complex values, for which the seminorms

$$\|\psi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta \psi(x)|$$

are finite. The adjoint space $S'_c(\mathbb{R}^d)$ is then the space of tempered distributions. By $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ we denote the spaces of real functions from $S_c(\mathbb{R}^d)$ and the space of real functionals on $S(\mathbb{R}^d)$.

For $\psi \in S_c(\mathbb{R}^d)$ define $\psi_{(s)}(x) = \overline{\psi(-x)}$, $x \in \mathbb{R}^d$. By $S_{(s)}(\mathbb{R}^d)$ and $S'_{(s)}(\mathbb{R}^d)$ we denote the spaces of $\psi \in S_c(\mathbb{R}^d)$ such that $\psi(x) = \psi_{(s)}(x)$ and the space of all $\xi \in S'_c(\mathbb{R}^d)$ such that $(\xi, \psi) = (\xi, \psi_{(s)})$ for all $\psi \in S(\mathbb{R}^d)$.

If \mathcal{F} is the Fourier transform on $S_c(\mathbb{R}^d)$:

$$\mathcal{F}(\psi)(\lambda) = \int_{\mathbb{R}^d} e^{i\langle x, \lambda \rangle} \psi(x) dx, \quad \lambda \in \mathbb{R}^d, \quad \psi \in S_c(\mathbb{R}^d),$$

then its inverse \mathcal{F}^{-1} is given by the formula

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, \lambda \rangle} \psi(\lambda) d\lambda, \quad \lambda \in \mathbb{R}^d, \quad \psi \in S_c(\mathbb{R}^d).$$

We use the same notation for the Fourier transforms acting on $S'_c(\mathbb{R}^d)$.

Note that the operators \mathcal{F} and \mathcal{F}^{-1} transform $S'(\mathbb{R}^d)$ onto $S'_{(s)}(\mathbb{R}^d)$ and $S'_{(s)}(\mathbb{R}^d)$ onto $S'(\mathbb{R}^d)$, respectively. The Fourier transforms of $\varphi \in S_c(\mathbb{R}^d)$ and $\xi \in S'_c(\mathbb{R}^d)$ will be denoted by $\hat{\varphi}$ and $\hat{\xi}$.

Consider first the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad t \geq 0, \quad u(0) = \xi, \quad (2.1)$$

where $\xi \in S'_c(\mathbb{R}^d)$. If \hat{u} denotes the Fourier transform of u then

$$\frac{\partial \hat{u}}{\partial t} = -|\lambda|^2 \hat{u} \quad \text{and} \quad \hat{u}(0) = \hat{\xi},$$

and therefore

$$\hat{u}(t) = e^{-|\lambda|^2 t} \hat{\xi}.$$

Consequently, for arbitrary $\xi \in S'_c(\mathbb{R}^d)$, equation (2.1) has a unique solution in $S'_c(\mathbb{R}^d)$ and the solution is given by the formula

$$u(t, x) = p(t) * \xi(x) = (\xi, p(t, x - \cdot)) \quad (2.2)$$

where $\hat{p}(t)(\lambda) = e^{-t|\lambda|^2}$, $p(t, x) = \frac{1}{\sqrt{(4\pi t)^d}} e^{\frac{-|x|^2}{4t}}$, $t > 0$, $\lambda, x \in \mathbb{R}^d$.

The family

$$S(t)\xi = p(t) * \xi, \quad t \geq 0, \quad \xi \in S'_c(\mathbb{R}^d) \quad (2.3)$$

forms a semigroup of operators, continuous in the topology of $S'_c(\mathbb{R}^d)$. The formula (2.2) shows that the semigroup $S(t)$, $t \geq 0$ has a smoothing property: for all $\xi \in S'_c(\mathbb{R}^d)$, $S(t)\xi$ is represented by a C^∞ function.

Similarly, for the wave equation,

$$\begin{aligned} \frac{\partial u}{\partial t} &= v, & u(0) &= \xi \\ \frac{\partial v}{\partial t} &= \Delta u, & v(0) &= \zeta, \end{aligned}$$

one gets, passing again to the Fourier transforms \hat{u} and \hat{v} , that:

$$\frac{\partial \hat{u}}{\partial t} = \hat{v}, \quad \frac{\partial \hat{v}}{\partial t} = -|\lambda|^2 \hat{u}.$$

By direct computation we have

$$\frac{d}{dt} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \cos(|\lambda|t), & \frac{\sin(|\lambda|t)}{|\lambda|} \\ -|\lambda| \sin(|\lambda|t), & \cos(|\lambda|t) \end{pmatrix} \begin{pmatrix} \hat{\xi} \\ \hat{\zeta} \end{pmatrix}.$$

Therefore,

$$\hat{u}(t) = [\cos(|\lambda|t)]\hat{u}(0) + \left[\frac{\sin(|\lambda|t)}{|\lambda|} \right] \hat{v}(0), \quad (2.4)$$

$$\hat{v}(t) = -[|\lambda| \sin(|\lambda|t)]\hat{u}(0) + [\cos(|\lambda|t)]\hat{v}(0). \quad (2.5)$$

Note that for each $t \in \mathbb{R}^1$ functions $\cos(|\lambda|t)$, $\frac{\sin(|\lambda|t)}{|\lambda|}$ and $|\lambda| \sin(|\lambda|t)$, $\lambda \in \mathbb{R}^d$, are smooth and polynomially bounded together with all their partial derivatives. Therefore the formulae (2.4), (2.5) define distributions belonging to $S'_c(\mathbb{R}^d)$.

Let $p_{1,1}(t), p_{1,2}(t), p_{2,1}(t), p_{2,2}(t), t \in \mathbb{R}^1$, be elements from $S'(\mathbb{R}^d)$ such that,

$$\begin{aligned} \cos(|\lambda|t) &= \mathcal{F}(p_{1,1}(t))(\lambda), & \frac{\sin(|\lambda|t)}{|\lambda|} &= \mathcal{F}(p_{1,2}(t))(\lambda) \\ -|\lambda|\sin(|\lambda|t) &= \mathcal{F}(p_{2,1}(t))(\lambda), & \cos(|\lambda|t) &= \mathcal{F}(p_{2,2}(t))(\lambda), \quad \lambda \in \mathbb{R}^d. \end{aligned}$$

Then

$$\begin{aligned} u(t) &= p_{1,1}(t) * u(0) + p_{1,2}(t) * v(0), \\ v(t) &= p_{2,1}(t) * u(0) + p_{2,2}(t) * v(0), \quad t \in \mathbb{R}. \end{aligned}$$

As for the heat equation, explicit formulae for the distributions $p_{i,j}(t)$, $i, j = 1, 2$ are known, see [17, pp. 280–282]. In particular, they have bounded supports.

We shall use the following notation

$$R(t)\xi = p_{1,2}(t) * \xi, \quad t \geq 0, \xi \in S'_c(\mathbb{R}^d). \quad (2.6)$$

For all $\xi \in S'_c(\mathbb{R}^d)$, $R(t)\xi$ is a continuous function of time. For each $t > 0$, the linear transformation $R(t)$ acts continuously on $S'_c(\mathbb{R}^d)$.

2.2 Spatially homogeneous Wiener process

Let Γ be a positive definite, tempered distribution. By W_Γ we denote an $S'(\mathbb{R}^d)$ -valued Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}(W(t), \varphi)(W(s), \psi) = t \wedge s (\Gamma, \varphi * \psi_{(s)}),$$

where $\psi_{(s)}(x) = \overline{\psi(-x)}$, $x \in \mathbb{R}^d$, see [21]. It is well-known that this way one can describe all space homogeneous $S'(\mathbb{R}^d)$ -valued Wiener processes, see e.g. [21].

The crucial role for stochastic integration with respect to W_Γ is played by the Hilbert space $S'_\Gamma \subset S'(\mathbb{R}^d)$ consisting of all distributions $\xi \in S'(\mathbb{R}^d)$ for which there exists a constant C such that,

$$|(\xi, \psi)| \leq C \sqrt{(\Gamma, \psi * \psi_{(s)})}, \quad \psi \in S.$$

The norm in S'_Γ is given by the formula:

$$|\xi|_{S'_\Gamma} = \sup_{\psi \in S} \frac{|(\xi, \psi)|}{\sqrt{(\Gamma, \psi * \psi_{(s)})}}.$$

The space S'_Γ is called the *kernel* of W_Γ , see [21].

Let H be a Hilbert space continuously embedded into $S'(\mathbb{R}^d)$ and let $L_{HS}(S'_\Gamma, H)$ be the space of Hilbert-Schmidt operators from S'_Γ into H . Assume that $\Psi(t)$, $t \in [0, T]$, is a predictable $L_{HS}(S'_\Gamma, H)$ -valued process such that

$$\mathbb{E} \left(\int_0^T \|\Psi(s)\|_{L_{HS}(S'_\Gamma, H)}^2 ds \right) < +\infty.$$

Then the stochastic integral

$$\int_0^T \Psi(s) dW_\Gamma(s),$$

can be defined in a standard way, see [12], [3], [21]. It is an H -valued random variable such that

$$\mathbb{E} \left(\int_0^T \Psi(s) dW_\Gamma(s) \right) = 0,$$

and

$$\mathbb{E} \left| \int_0^T \Psi(s) dW_\Gamma(s) \right|_H^2 = \mathbb{E} \left(\int_0^T \|\Psi(s)\|_{L_{HS}(S'_\Gamma, H)}^2 ds \right).$$

The definition of the stochastic integral can be extended to all predictable Ψ such that

$$\mathbb{P} \left(\int_0^T \|\Psi(s)\|_{L_{HS}(S'_\Gamma, H)}^2 ds < +\infty \right) = 1. \quad (2.7)$$

This is, in a sense, the largest class of integrands. One says that the *stochastic integral* (2.2) *exists in* H if the integrability condition (2.7) holds. Finally if $\Psi(t, \omega), t \geq 0, \omega \in \Omega$ are linear operators on $S'(\mathbb{R}^d)$, not necessarily everywhere defined, and for some Hilbert space H , continuously embedded into $S'(\mathbb{R}^d)$, the process Ψ is H stochastically integrable, than one says that the process Ψ is $S'(\mathbb{R}^d)$ *stochastically integrable* or that the *stochastic integral* (2.2) *exists in* $S'(\mathbb{R}^d)$, see [12].

We will need a characterization of the space S'_Γ from [21, Proposition 1.2]. In the proposition below $L^2_{(s)}(\mathbb{R}^d, \mu)$ denotes the subspace of $L^2(\mathbb{R}^d, \mu; \mathbb{C})$ consisting of all functions u such that $u_{(s)} = u$, see §2.1.

Proposition 2.1. *A distribution ξ belongs to S'_Γ if and only if $\xi = \widehat{u\mu}$ for some $u \in L^2_{(s)}(\mathbb{R}^d, \mu)$. Moreover, if $\xi = \widehat{u\mu}$ and $\eta = \widehat{v\mu}$, then*

$$\langle \xi, \eta \rangle_{S'_\Gamma} = \langle u, v \rangle_{L^2_{(s)}(\mathbb{R}^d, \mu)}.$$

2.3 Questions

The solution X to the *stochastic heat equation* and the solution Y to the *stochastic wave equation* is understood in the evolution form,

$$X(t) = \int_0^t S(t-s) dW_\Gamma(ds), \quad t \geq 0, \quad (2.8)$$

$$Y(t) = \int_0^t R(t-s) dW_\Gamma(ds), \quad t \geq 0, \quad (2.9)$$

where $S(\cdot)$ is given by (2.3) and $R(\cdot)$ by (2.6). It is not difficult to show, see [15], that the stochastic integrals (2.8), (2.9), exist in $S'(\mathbb{R}^d)$.

We say that the *solutions* X and Y are *function valued* if the integrals (2.8), (2.9) exist in $L^2_\rho(\mathbb{R}^d) = L^2(\mathbb{R}^d, \rho(x)dx)$ for some positive, continuous, integrable weight ρ . We will see that characterizations of function-valued solutions are independent of the choice of the weight ρ .

To motivate the definition of the function valued solutions assume that an $S'(\mathbb{R}^d)$ valued process $X(t), t \geq 0$, can be identified with function valued process $Z(t, x), t \geq 0, x \in \mathbb{R}^d$, in the sense that $X(t)$, (which depends also on ω), can be represented as a function $Z(t, \cdot)$. As X is a Gaussian and spatially homogeneous process we can expect that for each $t \geq 0$, random variables $Z(t, x), x \in \mathbb{R}^d$, are Gaussian with identical probabilistic laws. In particular, for all positive, integrable functions $\rho(x), x \in \mathbb{R}^d$,

$$\mathbb{E} \left(\int_{\mathbb{R}^d} Z^2(t, x) \rho(x) dx \right) = \int_{\mathbb{R}^d} (\mathbb{E} Z^2(t, x)) \rho(x) dx = \left(\int_{\mathbb{R}^d} \rho(x) dx \right) \mathbb{E}(Z^2(t, 0)) < +\infty.$$

Consequently, $\mathbb{P}(Z(t, \cdot) \in L^2_\rho(\mathbb{R}^d)) = 1$.

The main questions of the present paper are formulated as follows.

Question 1. *Under what conditions on Γ the stochastic integrals $X(t), t \geq 0$, exist in $L^2_\rho(\mathbb{R}^d)$ for some (any) positive integrable weight ρ ?*

Question 2. *Under what conditions on Γ the stochastic integrals $Y(t), t \geq 0$, exist in $L^2_\rho(\mathbb{R}^d)$ for some (any) positive integrable weight ρ ?*

Answers to these questions have been formulated in the Introduction as Theorem 1.1 and Theorem 1.2. The case of d -dimensional torus T^d is treated in §5.

3 Proofs of Theorem 1.1 and Theorem 1.2

3.1 Proof of Theorem 1.1.

(a) Stochastic heat equation

Let us recall that we denote by S'_Γ the kernel of the Wiener process W_Γ and that

$$S(t)\xi = p(t) * \xi, \quad t \geq 0.$$

It follows from §2.2 that the stochastic integral

$$\int_0^t S(t-s) dW_\Gamma(s), \quad t > 0,$$

takes values in $L^2_\rho(\mathbb{R}^d)$ if and only if

$$\int_0^t \|S(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma < +\infty.$$

Let $\{u_k\}$ be an orthonormal basis in $L^2_{(s)}(\mathbb{R}^d, \mu)$. Then by Proposition 1.2 of [21], $e_k = \widehat{u_k \mu}$, $k \in \mathbb{N}$, is an orthonormal basis in S'_Γ .

Thus we have

$$\|S(\sigma)\|_{L_{HS}(S'_\Gamma, L^2_\rho)}^2 = \sum_{k=1}^{\infty} |S(\sigma)\widehat{u_k\mu}|_{L^2_\rho}^2 = \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} |p(\sigma) * \widehat{u_k\mu}(x)|^2 \rho(x) dx, \quad \sigma > 0.$$

However, $p(\sigma) \in S(\mathbb{R}^d)$ and therefore

$$p(\sigma) * \widehat{u_k\mu}(x) = (p(\sigma, x - \cdot), \widehat{u_k\mu}) = (u_k\mu, \hat{p}(\sigma, x - \cdot)).$$

The last identity follows from the definition of the Fourier transform of the distribution $u_k\mu$. However,

$$\hat{p}(\sigma, x - \cdot)(\lambda) = e^{i\langle x, \lambda \rangle} e^{-\sigma|\lambda|^2}$$

and therefore

$$(u_k\mu, \hat{p}(\sigma, x - \cdot)) = (u_k\mu, e^{i\langle x, \cdot \rangle} e^{-\sigma|\cdot|^2}).$$

Consequently

$$\begin{aligned} \|S(\sigma)\|_{L_{HS}(S'_\Gamma, L^2_\rho)}^2 &= \sum_k \int_{\mathbb{R}^d} \left| (u_k\mu, e^{i\langle x, \cdot \rangle} e^{-\sigma|\cdot|^2}) \right|^2 \rho(x) dx \\ &= \sum_k \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} u_k(\lambda) e^{i\langle x, \lambda \rangle} e^{-\sigma|\lambda|^2} \mu(d\lambda) \right|^2 \rho(x) dx \\ &= \int_{\mathbb{R}^d} \left[\sum_k \left| \langle u_k, e^{-i\langle x, \cdot \rangle} e^{-\sigma|\cdot|^2} \rangle_{L^2_{(s)}(\mathbb{R}^d, \mu)} \right|^2 \right] \rho(x) dx. \end{aligned}$$

By the Parseval identity in $L^2_{(s)}(\mathbb{R}^d, \mu)$,

$$\sum_k \left| \langle u_k, e^{-i\langle x, \cdot \rangle} e^{-\sigma|\cdot|^2} \rangle_{L^2_{(s)}(\mathbb{R}^d, \mu)} \right|^2 = \int_{\mathbb{R}^d} \left| e^{-i\langle x, \lambda \rangle} e^{-\sigma|\lambda|^2} \right|^2 \mu(d\lambda) = \int_{\mathbb{R}^d} e^{-2\sigma|\lambda|^2} \mu(d\lambda).$$

Finally,

$$\begin{aligned} \int_0^t \|S(\sigma)\|_{L_{HS}(S'_\Gamma, L^2_\rho)}^2 d\sigma &= \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \int_0^t \int_{\mathbb{R}^d} e^{-2\sigma|\lambda|^2} \mu(d\lambda) d\sigma \\ &= \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \frac{1}{2} \int_{\mathbb{R}^d} \frac{1 - e^{-2t|\lambda|^2}}{|\lambda|^2} \mu(d\lambda). \end{aligned}$$

Therefore

$$\int_0^t \|S(\sigma)\|_{L_{HS}(S'_\Gamma, L^2_\rho)}^2 d\sigma < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda) < +\infty.$$

(b) Stochastic wave equation

Let us recall that $R(\sigma)\xi = p_{1,2}(\sigma) * \xi$, $\sigma \geq 0$, $\xi \in S'_c(\mathbb{R}^d)$, see (2.6). The process $Y(t)$, $t \geq 0$, is well defined as an $L^2_\rho(\mathbb{R}^d)$ -valued process if and only if

$$\int_0^t \|R(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma < +\infty.$$

But

$$\|R(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \sum_k \int_{\mathbb{R}^d} |p_{1,2}(\sigma) * \widehat{u_k \mu}(x)|^2 \rho(x) dx.$$

However

$$p_{1,2}(\sigma) * \widehat{u_k \mu}(x) = (p_{1,2}(\sigma, x - \cdot), \widehat{u_k \mu}) = (\hat{p}_{1,2}(\sigma)(x - \cdot), u_k \mu). \quad (3.1)$$

To justify the identity (3.1) we need the following lemma, see [8].

Lemma 3.1. *Let ξ and η be distributions with bounded supports. Then the convolution $\xi * \hat{\eta}$ exists and is a function of class C^∞ . Moreover,*

$$\xi * \hat{\eta}(x) = (\hat{\xi}(x - \cdot), \eta), \quad x \in \mathbb{R}^d.$$

Note that the distribution $p_{1,2}$ has bounded support and one can assume also that functions u_k , $k \in \mathbb{N}$, have bounded supports as well.

But

$$\hat{p}_{1,2}(\sigma)(x - \cdot)(\lambda) = e^{i\langle x, \lambda \rangle} \frac{\sin(|\lambda| \sigma)}{|\lambda|}.$$

Therefore

$$\begin{aligned} \|R(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} &= \sum_k \int_{\mathbb{R}^d} \left| \left(u_k \mu, e^{i\langle x, \cdot \rangle} \frac{\sin(|\cdot| \sigma)}{|\cdot|} \right) \right|^2 \rho(x) dx \\ &= \sum_k \int_{\mathbb{R}^d} \left| \langle u_k, e^{i\langle x, \cdot \rangle} \frac{\sin(|\cdot| \sigma)}{|\cdot|} \rangle_{L^2_{(\cdot)}(\mathbb{R}^d, \mu)} \right|^2 \rho(x) dx. \end{aligned}$$

Again, by the Parseval identity,

$$\|R(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \int_{\mathbb{R}^d} \frac{(\sin(|\lambda| \sigma))^2}{|\lambda|^2} \mu(d\lambda).$$

Consequently,

$$\int_0^t \|R(\sigma)\|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma = \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \int_{\mathbb{R}^d} \left[\int_0^t \frac{(\sin(|\lambda| \sigma))^2}{|\lambda|^2} d\sigma \right] \mu(d\lambda).$$

By an elementary argument one shows now that the integral is finite, for all $t > 0$, if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda) < +\infty.$$

This completes the proof of Theorem 1.1. ■

3.2 Proof of Theorem 1.2.

Let

$$G_d(x) = \int_0^{+\infty} e^{-t} p(t, x) dt, \quad x \in \mathbb{R}^d,$$

where

$$p(t, x) = \frac{1}{\sqrt{(4\pi t)^d}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^d.$$

Thus G_d is the resolvent kernel of the d -dimensional Wiener process. It is easy to see that

$$G_d(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, \lambda \rangle} \frac{1}{1 + |\lambda|^2} d\lambda, \quad x \in \mathbb{R}^d.$$

The following properties of G_d are well-known, see [16], [10], [8]:

Proposition 3.2. *One has that:*

$$G_1(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}^1; \quad G_3(x) = \frac{1}{4\pi|x|} e^{-|x|}, \quad x \in \mathbb{R}^3,$$

and, in general, for $d \geq 2$,

$$G_d(x) = (2\pi)^{-\frac{d}{2}} \frac{1}{|x|^{\frac{d-2}{2}}} K_{\frac{d-2}{2}}(|x|),$$

where K_γ , $\gamma \geq 0$, denotes the modified Bessel function of the third order.

We will need also a characterization of the behaviour of G_d near 0 and near ∞ , see [10, Proposition 7.2.1].

Proposition 3.3. *The function G_d has the following properties:*

- (a) for $d \geq 1$, for $|x|$ bounded away from a neighbourhood of zero and for a constant $c > 0$

$$G_d(x) \leq \frac{c}{|x|^{\frac{d-1}{2}}} e^{-|x|};$$

- (b) for $d \geq 3$ and for a constant $c > 0$, in a neighbourhood of zero

$$G_d(x) \sim \frac{c}{|x|^{d-2}};$$

- (c) for $d = 2$ and for a constant $c > 0$, in a neighbourhood of zero

$$G_2(x) \sim -c \ln |x|.$$

We will also use the following lemma:

Lemma 3.4. *Assume that the distribution Γ is not only positive definite but it is also a non-negative measure. Then*

$$(\Gamma, G_d) = (2\pi)^d \int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda).$$

Proof. Since $\mu = \mathcal{F}^{-1}(\Gamma)$ and $e^{-t|\cdot|^2} \frac{1}{1+|\cdot|^2} \in S(\mathbb{R}^d)$, by the definition of the Fourier transform of a distribution,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{e^{-t|\lambda|^2}}{1+|\lambda|^2} \mu(d\lambda) &= \left(\frac{e^{-t|\cdot|^2}}{1+|\cdot|^2}, \mu \right) = \left(\frac{e^{-t|\cdot|^2}}{1+|\cdot|^2}, \mathcal{F}^{-1}(\Gamma) \right) = \\ &= \left(\mathcal{F}^{-1} \left(\frac{e^{-t|\cdot|^2}}{1+|\cdot|^2} \right), \Gamma \right) = \frac{1}{(2\pi)^d} (p(t) * G_d, \Gamma). \end{aligned}$$

Therefore

$$\lim_{t \downarrow 0} \frac{1}{(2\pi)^d} (p(t) * G_d, \Gamma) = \int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda).$$

Moreover

$$p(s) * G_d = \int_0^{+\infty} e^{-t} p(t) * p(s) dt = e^s \int_0^{+\infty} e^{-(t+s)} p(t+s) dt = e^s \int_s^{+\infty} e^{-\sigma} p(\sigma) d\sigma.$$

So

$$e^{-s} p(s) * G_d = \int_s^{+\infty} e^{-\sigma} p(\sigma) d\sigma$$

and then

$$e^{-s} p(s) * G_d \uparrow G_d \quad \text{as } s \downarrow 0.$$

Hence, if Γ is a non-negative distribution on \mathbb{R}^d , then

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) = \lim_{t \downarrow 0} e^{-t} \frac{1}{(2\pi)^d} (p(t) * G_d, \Gamma) = \frac{1}{(2\pi)^d} (G_d, \Gamma).$$

This completes the proof of Lemma 3.4. ■

We pass now to the proof of the theorem. It is well-known that a non-negative measure Γ belongs to $S'(\mathbb{R}^d)$ if and only if for some $r > 0$,

$$\int_{\mathbb{R}^d} \frac{1}{1+|x|^r} \Gamma(dx) < +\infty. \quad (3.2)$$

Moreover, for arbitrary $d \geq 1$,

$$\int_{\mathbb{R}^d} G_d(x) \Gamma(dx) = \int_{|x| \leq 1} G_d(x) \Gamma(dx) + \int_{|x| > 1} G_d(x) \Gamma(dx).$$

But, by Proposition 3.3 (a),

$$\int_{|x| > 1} G_d(x) \Gamma(dx) \leq c \int_{|x| > 1} e^{-|x|} \Gamma(dx),$$

and from (3.2)

$$\int_{|x| > 1} G_d(x) \Gamma(dx) < +\infty.$$

Since the function G_1 is continuous,

$$\int_{|x| \leq 1} G_1(x) \Gamma(dx) < +\infty,$$

and the theorem is true for $d=1$.

If $d=2$ then $\int_{\mathbb{R}^d} G_2(x) \Gamma(dx) < +\infty$ if and only if $\int_{|x| \leq 1} G_2(x) \Gamma(dx) < +\infty$. But $G_2(x) \sim c \ln \frac{1}{|x|}$ for some $c > 0$ in the neighbourhood of 0, so

$$G_2(x)/c \ln \frac{1}{|x|} \rightarrow 1 \text{ as } |x| \rightarrow 0.$$

Therefore, for some $c_1 > 0$, $c_2 > 0$:

$$c_2 \ln \frac{1}{|x|} \leq G_2(x) \leq c_1 \ln \frac{1}{|x|}, \text{ for } |x| \leq 1.$$

Consequently,

$$\int_{\mathbb{R}^d} G_2(x) \Gamma(dx) < +\infty \text{ if and only if } \int_{|x| \leq 1} \ln \frac{1}{|x|} \Gamma(dx) < +\infty.$$

If $d \geq 3$, in the same way,

$$\int_{\mathbb{R}^d} G_d(x) \Gamma(dx) < +\infty \text{ if and only if } \int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}} \Gamma(dx) < +\infty.$$

This completes the proof of Theorem 1.2. ■

4 Applications

We illustrate the main results by several examples. We start with the case of bounded functions Γ .

Proposition 4.1. *If the positive definite distribution Γ is a bounded function then the equations (1.1) and (1.2) have function-valued solutions in any dimension d .*

Proof. If the positive definite distribution Γ is a bounded function then Γ is a continuous function and the corresponding spectral measure μ is finite. Since the function $\frac{1}{1+|\lambda|^2}$, $\lambda \in \mathbb{R}^d$, is bounded therefore

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) < +\infty$$

and by Theorem 1.1 the result follows. ■

Stochastic evolution equations with noise of such type have been introduced by Dawson and Salehi [5] with an extra requirement that μ is absolutely continuous with respect to Lebesgue measure on \mathbb{R}^d . In the case of $d=2$ they have appeared in the recent paper by Mueller [18].

Example 4.2. It is well-known that functions $\Gamma(x) = e^{-|x|^\alpha}$, $x \in \mathbb{R}^d$, for $\alpha \in (0, 2]$ are positive definite. In fact, they are Fourier transforms of the so called symmetric stable distributions, see [16] or [6]. Consequently with such covariances Γ the equations (1.1) and (1.2) have function-valued solutions.

We consider now some examples of unbounded covariances Γ .

Proposition 4.3. *For arbitrary $\alpha \in (0, d)$ the function $\Gamma_\alpha(x) = \frac{1}{|x|^\alpha}$, $x \in \mathbb{R}^d$ is a positive definite distribution. Equations (1.1) and (1.2) with the covariance Γ_α have function-valued solutions if and only if $\alpha \in (0, 2 \wedge d)$.*

Proof. It is well-known, see [17], [8] or [16], that Γ_α is the Fourier transform of the function $c_1 \frac{1}{|\lambda|^{d-\alpha}}$, $\lambda \in \mathbb{R}^d$, where c_1 is a positive constant. The condition (1.4) is equivalent to

$$I := \int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \frac{1}{|\lambda|^{d-\alpha}} d\lambda < +\infty.$$

By standard calculation

$$I = c_2 \int_0^{+\infty} \frac{1}{(1 + r^2)} \frac{1}{r^{d-\alpha}} r^{d-1} dr,$$

where c_2 is a constant. One obtains, that $I < +\infty$ if and only if

$$\int_0^1 \frac{1}{r^{1-\alpha}} dr < +\infty \quad \text{and} \quad \int_1^\infty \frac{1}{r^{3-\alpha}} dr < +\infty,$$

or equivalently, $\alpha > 0$ and $\alpha < 2$. Since $\alpha \in (0, d)$, the result follows. ■

Remark 4.4. Note that Proposition 4.3 contains, as a special case, an application from the paper [2, Examples].

We pass now to examples for which Γ are genuine distributions.

Example 4.5. If $\frac{\partial W_\Gamma}{\partial t}$ is the space-time white noise then $\Gamma = \delta_{\{o\}}$ and the corresponding spectral measure μ has a constant density, say $c > 0$. Since

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} d\lambda < +\infty,$$

if and only if $d = 1$, the equations (1.1) and (1.2), perturbed by such noise, have function-valued solutions iff $d = 1$.

Example 4.6. Walsh [25], in his study of particle systems, arrived at the following equation for fluctuations:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, \xi) &= \frac{\partial^2 u}{\partial \xi^2}(t, \xi) + \frac{\partial}{\partial t} \left[\frac{\partial}{\partial \xi} W_{\delta_{\{o\}}}(t, \xi) \right], \\ u(0, \xi) &= 0, \quad t > 0, \quad \xi \in \mathbb{R}^1. \end{aligned} \tag{4.1}$$

It is easy to calculate that the covariance function corresponding to $\frac{\partial}{\partial \xi} W_{\delta_{\{o\}}}(t, \xi)$, $t \geq 0$, is $\Gamma = -\delta''_{\{0\}}$ and the appropriate spectral measure μ has the density $d_\mu(\lambda) = \lambda^2 d\lambda$.

Since

$$\int_{-\infty}^{+\infty} \frac{1}{1 + \lambda^2} \lambda^2 d\lambda = +\infty,$$

the equation (4.1) does not have a function-valued solution. This fact has already been noticed by Walsh [25].

5 Equations on the d -dimensional torus

Many of the previous considerations can be extended from \mathbb{R}^d to stochastic equations on more general groups¹. As an illustration we discuss here the case of d -dimensional torus T^d , for more details we refer to [13]. The d -dimensional torus T^d can be identified with the Cartesian product, $(-\pi, \pi]^d$, regarded as a group with the addition modulo 2π (coordinatewise). We assume that W_Γ is a $D'(T^d)$ -valued Wiener process spatially homogeneous with the space correlation Γ . The distribution Γ can be uniquely expanded into its Fourier series

$$\Gamma(\theta) = \sum_{n \in \mathbb{Z}^d} e^{i\langle n, \theta \rangle} \gamma_n,$$

with the non-negative coefficients such that $\gamma_n = \gamma_{-n}$ and $\sum_{n \in \mathbb{Z}^d} \frac{\gamma_n}{1 + |n|^r} < +\infty$, for some $r > 0$.

Denote $\mathbb{Z}_s^1 = \mathbb{N}$ and, by induction, $\mathbb{Z}_s^{d+1} = (\mathbb{Z}_s^1 \times \mathbb{Z}^d) \cup \{(0, n); n \in \mathbb{Z}_s^d\}$. Then $\mathbb{Z}^d = \mathbb{Z}_s^d \cup (-\mathbb{Z}_s^d) \cup \{0\}$.

The corresponding spatially homogeneous Wiener process $W(t)$, $t \geq 0$ can be represented in the form:

$$W(t, \theta) = \sqrt{\gamma_0} \beta_0(t) + \sum_{n \in \mathbb{Z}_s^d} \sqrt{2\gamma_n} ((\cos \langle n, \theta \rangle) \beta_n^1(t) + (\sin \langle n, \theta \rangle) \beta_n^2(t)),$$

$$\theta \in T^d, \quad t \geq 0, \quad (5.1)$$

where $\beta_0, \beta_n^1, \beta_n^2$, $n \in \mathbb{Z}_s^d$ are independent, real Brownian motions and the convergence is in the sense of $D'(T^d)$.

Denote $H = H^0 = L^2(T^d)$, $H^\alpha = H^\alpha(T^d)$ and $H^{-\alpha} = H^{-\alpha}(T^d)$, $\alpha \in \mathbb{R}_+$, the real Sobolev spaces of order α and $-\alpha$, respectively. The norms are expressed in terms of the Fourier coefficients, see [1]

$$\|\xi\|_{H^\alpha} = \left(\sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^\alpha |\xi_n|^2 \right)^{\frac{1}{2}} = \left(|\xi_0|^2 + 2 \sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^\alpha ((\xi_n^1)^2 + (\xi_n^2)^2) \right)^{\frac{1}{2}},$$

¹The case of linear and nonlinear heat equations on compact Lie groups has been recently discussed in [24].

and

$$\|\xi\|_{H^{-\alpha}} = \left(\sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^{-\alpha} |\xi_n|^2 \right)^{\frac{1}{2}} = \left(|\xi_0|^2 + 2 \sum_{n \in \mathbb{Z}_*^d} (1 + |n|^2)^{-\alpha} ((\xi_n^1)^2 + (\xi_n^2)^2) \right)^{\frac{1}{2}},$$

where $\xi_n = \xi_n^1 + i\xi_n^2$, $\xi_n = \bar{\xi}_{-n}$, $n \in \mathbb{Z}^d$.

We have the following result.

Theorem 5.1. *Equations (1.1) and (1.2) on the torus T^d have $H^{\alpha+1}(T^d)$ -valued solution if and only if the Fourier coefficients (γ_n) of the kernel Γ satisfy:*

$$\sum_{n \in \mathbb{Z}^d} \gamma_n (1 + |n|^2)^\alpha < +\infty. \quad (5.2)$$

Remark 5.2. Recently, the problem of existence of solution to stochastic wave equation in $S'(\mathbb{R}^d)$ has been considered by Gaveau [7].

As in the case of \mathbb{R}^d , condition (5.1) can be written in a more explicit way.

Theorem 5.3. *Assume that Γ is not only a positive definite distribution but is also a non-negative measure. Then equations (1.1) and (1.2) have function-valued solutions:*

- i) *for all Γ if $d = 1$;*
- ii) *for exactly those Γ for which $\int_{|\theta| \leq 1} \ln |\theta| \Gamma(d\theta) < +\infty$ if $d = 2$;*
- iii) *for exactly those Γ for which $\int_{|\theta| \leq 1} \frac{1}{|\theta|^{d-2}} \Gamma(d\theta) < +\infty$ if $d \geq 3$.*

The proofs of both theorems are similar to those for \mathbb{R}^d . For details we refer to our preprint [13].

In fact the proof of Theorem 5.1 can be done in a different way by taking into account the expansion (5.1) of the Wiener process W , with respect to the basis 1, $\cos\langle n, \theta \rangle$, $\sin\langle n, \theta \rangle$, $n \in \mathbb{Z}_s^d$, $\theta \in T^d$. We sketch the different proof now.

Equations (1.1) and (1.2) can be solved coordinatewise with the following explicit formulae for the solutions:

$$\begin{aligned} X(t, \theta) &= \sqrt{\gamma_0} \beta_0(t) + \sum_{n \in \mathbb{Z}_s^d} \sqrt{2\gamma_n} \left[\cos\langle n, \theta \rangle \int_0^t e^{-|n|^2(t-s)} d\beta_n^1(s) \right. \\ &\quad \left. + \sin\langle n, \theta \rangle \int_0^t e^{-|n|^2(t-s)} d\beta_n^2(s) \right], \end{aligned} \quad (5.3)$$

$$\begin{aligned} Y(t, \theta) &= \sqrt{\gamma_0} \beta_0(s) ds + \sum_{n \in \mathbb{Z}_s^d} \sqrt{2\gamma_n} \left[\cos\langle n, \theta \rangle \int_0^t \frac{\sin(|n|(t-s))}{|n|} d\beta_n^1(s) \right. \\ &\quad \left. + \sin\langle n, \theta \rangle \int_0^t \frac{\sin(|n|(t-s))}{|n|} d\beta_n^2(s) \right]. \end{aligned} \quad (5.4)$$

It is easy to check that the series defining the processes X and Y converge in $D'(T^d)$ \mathbb{P} -almost surely. It follows from the definition of the space $H^{\alpha+1}$ that $X(t) \in H^{\alpha+1}$, \mathbb{P} -almost surely if and only if,

$$\sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^{\alpha+1} \gamma_n \left[\left(\int_0^t e^{-|n|^2(t-s)} d\beta_n^1(s) \right)^2 + \left(\int_0^t e^{-|n|^2(t-s)} d\beta_n^2(s) \right)^2 \right] < +\infty. \quad (5.5)$$

Since the stochastic integrals in (5.5) are independent Gaussian random variables therefore (5.5) holds \mathbb{P} -almost surely if and only if,

$$\sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^{\alpha+1} \gamma_n \mathbb{E} \left[\left(\int_0^t e^{-|n|^2(t-s)} d\beta_n^1(s) \right)^2 + \left(\int_0^t e^{-|n|^2(t-s)} d\beta_n^2(s) \right)^2 \right] < +\infty. \quad (5.6)$$

Or equivalently, if and only if,

$$\sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^{\alpha+1} \gamma_n \int_0^t e^{-2|n|^2 s} ds < +\infty. \quad (5.7)$$

Similarly, $Y(t) \in H^{\alpha+1}$, \mathbb{P} -almost surely, if and only if,

$$\sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^{\alpha+1} \frac{\gamma_n}{|n|^2} \int_0^t \sin^2(|n|s) ds < +\infty. \quad (5.8)$$

Since, for arbitrary $t > 0$,

$$|n|^2 \int_0^t e^{-2|n|^2 s} ds \rightarrow \frac{1}{2}, \text{ as } |n| \rightarrow +\infty,$$

$$\int_0^t \sin^2(|n|s) ds \rightarrow \frac{t}{\pi} \int_0^\pi \sin^2 \sigma d\sigma, \text{ as } |n| \rightarrow +\infty,$$

therefore inequalities (5.7) and (5.8) hold if and only if,

$$\sum_{n \in \mathbb{Z}_s^d} (1 + |n|^2)^{\alpha+1} \frac{\gamma_n}{|n|^2} < +\infty,$$

as required. ■

Expansions (5.3), (5.4) lead also to more refined results.

Theorem 5.4. *Assume that*

$$\sum_{n \in \mathbb{Z}^d} \frac{\gamma_n}{1 + |n|^\alpha} < +\infty,$$

for some $\alpha \in (0, 2)$. Then solutions $X(t)$, $Y(t)$, $t \geq 0$ are Hölder continuous with respect to $t > 0$ and $\theta \in T^d$ with any exponent smaller than $\frac{1}{2} - \frac{\alpha}{4}$.

The theorem is a consequence of Theorem 5.20 and Theorem 5.22 from [3]. For the case of R^2 and the stochastic wave equation a similar result was obtained in [2].

We finish the section with some applications of Theorems 5.1 and 5.3.

Corollary 5.5. *Assume that $\Gamma \in L^2(T^d)$ and $d = 1, 2, 3$. Then the stochastic heat and wave equations (1.1) and (1.2) have solutions with values in $L^2(T^d)$.*

Corollary 5.6. *Assume that for some $1 \leq p \leq 2$, $\hat{\Gamma} \in l^p(\mathbb{Z}^d)$. If $d < \frac{2p}{p-1}$, then the stochastic heat and wave equations (1.1) and (1.2) have solutions in $L^2(T^d)$.*

6 Conjectures

Taking into account Theorem 1.1 it is natural to expect that the following conjecture is true.

Conjecture 1. *If $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) < +\infty$ and functions $g : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz then nonlinear heat equation*

$$\begin{cases} \frac{\partial u(t, \theta)}{\partial t} = \Delta u(t, \theta) + g(u(t, \theta)) + b(u(t, \theta)) \frac{\partial W_\Gamma}{\partial t}(t, \theta), & t > 0, \quad \theta \in \mathbb{R}^d, \\ u(0, \theta) = u_0(\theta), & \theta \in \mathbb{R}^d, \end{cases} \quad (6.1)$$

and nonlinear wave equation

$$\begin{cases} \frac{\partial^2 u(t, \theta)}{\partial t^2} = \Delta u(t, \theta) + g(u(t, \theta)) + b(u(t, \theta)) \frac{\partial W_\Gamma}{\partial t}(t, \theta), & t > 0, \quad \theta \in \mathbb{R}^d, \\ u(0, \theta) = u_0(\theta), \quad \frac{\partial u}{\partial t}(0, \theta) = v_0(\theta), & \theta \in \mathbb{R}^d, \end{cases} \quad (6.2)$$

have function valued solutions.

At the moment there are only partial confirmations of the conjectures. Namely, the following result concerned with nonlinear heat equation (6.1) is contained in the paper [22]. It strengthens an earlier result from [21].

Theorem 6.1. *Assume that g and b are Lipschitz and that for a positive number γ , the distribution $\Gamma + \gamma$, is a non-negative measure. Nonlinear heat equation (6.1) has a unique Markovian solution in L^2_ρ ,*

- i) for all Γ if $d = 1$;*
- ii) for exactly those Γ for which $\int_{|\theta| \leq 1} \ln |\theta| \Gamma(d\theta) < +\infty$ if $d = 2$;*
- iii) for exactly those Γ for which $\int_{|\theta| \leq 1} \frac{1}{|\theta|^{d-2}} \Gamma(d\theta) < +\infty$ if $d \geq 3$.*

An existence result for nonlinear stochastic wave equations was proved in the paper [2] by Dalang and Frangos and more recently by Millet and Sanz-Sole in [19]. The following result was shown in [2].

Theorem 6.2. *If the correlation Γ is a function of the form $\Gamma(\theta) = f(|\theta|)$, with f positive and continuous outside 0, and if the dimension $d = 2$ and the condition (1.3) is satisfied then the equation (6.2) has a local solution.*

The above theorem has been recently strengthened and extended to dimension $d = 3$, in the paper [22] by Peszat and Zabczyk.

Theorem 6.3. *Assume that g and b are Lipschitz, that for a positive number γ the distribution $\Gamma + \gamma$, is a non-negative measure and that $d \leq 3$. Nonlinear wave equation (6.2) has a unique solution in L^2_ρ ,*

- i) for all Γ if $d = 1$;
- ii) for exactly those Γ for which $\int_{|\theta| \leq 1} \ln |\theta| \Gamma(d\theta) < +\infty$ if $d = 2$;
- iii) for exactly those Γ for which $\int_{|\theta| \leq 1} \frac{1}{|\theta|^{d-2}} \Gamma(d\theta) < +\infty$ if $d \geq 3$.

It is interesting to note that conditions of Theorem 6.1 and Theorem 6.3 imply that the reproducing kernel space S'_Γ consists only of functions, namely, $S'_\Gamma \subset L^2(\mathbb{R}^d) + C_b(\mathbb{R}^d)$, see [21]. This seems to be essential for the definition of the stochastic integral with the integrands being multiplication operators. We therefore pose the following conjecture.

Conjecture 2. *If $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) < +\infty$ then elements of S'_Γ are represented by locally integrable functions.*

The following proposition is a partial confirmation of Conjecture 2.

Proposition 6.4. *If $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) < +\infty$ then $\delta_{\{o\}} \notin S'_\Gamma$.*

Proof. Assume, to the contrary, that for some $u \in L^2_{(s)}(\mathbb{R}^d, \mu)$, $\widehat{u\mu} = \delta_{\{o\}}$.

Then, for a constant $c > 0$, $u(x)\mu(dx) = c dx$ and $u(x) > 0$ for almost all $x \in \mathbb{R}^d$ and measure μ can be identified with its density $\gamma(x) = \frac{c}{u(x)}$, $x \in \mathbb{R}^d$.

We also have

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \gamma(x) dx < +\infty,$$

and

$$\int_{\mathbb{R}^d} u^2(x) \mu(dx) = \int_{\mathbb{R}^d} \frac{c^2}{\gamma^2(x)} \gamma(x) dx = \int_{\mathbb{R}^d} \frac{c^2}{\gamma(x)} dx < +\infty.$$

Consequently

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}} \frac{\gamma(x)}{\sqrt{1+|\lambda|^2}} dx &< +\infty, \\ \int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}} \frac{\sqrt{1+|\lambda|^2}}{\gamma(x)} dx &< +\infty. \end{aligned}$$

Adding both inequalities and taking into account that $a + \frac{1}{a} \geq 2$ for all $a > 0$, one arrives at $\int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}} d\lambda < +\infty$, a contradiction. ■

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