ON THE THEORY OF ELEMENTARY PARTICLES

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CONTENTS.

GENERAL INTRODUCTION [1].

The main aim of the present work can be formulated as to be an attempt to overcome, within the framework of quantum mechanics, the well known difficulty of the present day theories of elementary particles, viz. their yielding infinite self-energies, if a model for these particles is adopted which corresponds with a point model in the classical domain.

Within the framework of classical concepts, tentative solutions for these difficulties, already inherent in the classical picture, (cf. e.g. Chap. II, § 1) have been put forward by various authors. However, not only is there no stringent argument why classical theory should provide the appropriate starting point for the introduction of new features necessary to remove these inconsistencies, but moreover it must be emphasized that one cannot consider a classical theory, in which the singularities in the self-energies have been eliminated by some procedure or other, as to be necessarily connected, on correspondence arguments, with a from the self-energy point of view consistent picture of elementary particles. Indeed, by the process itself of subjecting such a divergence-free theory to quantization new divergences are again introduced. This circumstance is essentially due to the fact that the very concept "one elementary particle in a for the rest empty space" needs in quantum theory a revision as compared with classical theory and it follows in fact directly from WEISSKOPF's calculations [2] (surveyed in Chap. II, § 2) of the electromagnetic quantum field self-energy of the electron that there is no correspondence at all between this quantity and its classical counterpart.

The method which is followed in this paper in trying to obtain finite self-energies consists in assuming the elementary particles to be the sources of sets of fields in such a way that the various infinite contributions to the self-energy to which these fields give rise cancel each other so as to make the total outcome finite. In chapter II this idea is worked out for the electron, while in chapter III the nucleons are envisaged from this point of view. As already stated, we base our investigations on quantum field theory. In particular the wave equation of the electron and the nucleons are taken to be DIRAC's equation for spin $\frac{1}{2}$ particles and it may be directly stated that only then it appears possible, on the present method, to obtain finite results if a treatment is given on the lines of the so-called hole theory, i.e. if it is assumed that in vacuum all negative energy states of the electron, the proton as well as the neutron are filled up in accordance with the exclusion principle.

Within the classical scheme analogous ways of attack have been proposed by STÜCKELBERG [3] and BOPP [4] who have envisaged classical theories of the electron where this particle is taken to be the source of a

short range field besides the electromagnetic field (Chap. II, § 1). As was to be expected, there is again — as in the purely electromagnetic theory — no correspondence at all between the classical results of these authors and those obtained, on similar assumptions, in the quantum theory of the electron, (Chap. II, § 9).

In chapter I the more technical aspects of the problem on hand are considered and the results obtained by others are criticized (cf. loc. cit. § 5). The reader who is interested in the general outline of the present theory, rather than in computational details, may directly turn to the second and following chapters, where the main results of chapter I will be found summarized where necessary.

Chapter II is devoted to the electron. The final conclusion (loc. cit. § 8) is that, at any rate in first approximation with respect to an expansion in powers of the fine structure constant, a consistent hole theory of the electron, yielding a finite self-energy, is obtained if the electron is assumed to create a short range scalar field, termed f-field. This field appears to play a fundamental rôle in the theory of nucleons, too: indeed the f-field appears to be inseparable from the electromagnetic field (the equations for which are not modified by the presence of the f-field, though) and thus the proton, too, is a source of the f-field. In particular it is shown, (Chap. III § 4) how on these lines the mass difference of proton and neutron becomes amenable to interpretation. Further physical implications of the f-field hypothesis, amongst which the indetectability of the f-field quanta themselves on account of their extremely short life time and a deviation from COULOMB's law in the hydrogen atom are the most characteristic features, are discussed in chapter IV.

Chapter III deals with the self-energy of nucleons and the consequences which the present ideas may have on the theory of nuclear forces. It is shown that, at any rate in first approximation, the self-energy difficulties due to the coupling of the nucleons with the nuclear force fields can be overcome by the introduction of a neutral short range scalar field, the "F-field" (not to be confused with the f-field mentioned above) with which the nucleons are strongly coupled and which has a range considerably shorter than the customary meson field range.

For the electrons as well as the nucleons we arrive in this paper at the conclusion that, rather than to consider the mass of these particles to be entirely due to their field energy — as has often been assumed, especially within the framework of classical considerations — the latter quantity (divided by c^2) is, within the present scheme, a small perturbation compared to the particle mass which should thus mainly be of other ("non-field") origin.

Thus in the present dualistic theory a quantity like the so-called classical electron radius r_0, defined by

$$r_0 = \frac{e^2}{mc^2}$$

(e and m being the electronic charge and mass) loses its original interpretation which, in fact, just is intimately connected with the unitary point of view (mass entirely due to field energy).

It is true that the present results are based on an analysis of only the first approximation with respect to a development in powers of dimensionless parameters $(\text{charge})^2/hc$, where "charge" stands for the constants describing the coupling between the particles concerned and the fields they create. It would not seem unreasonable to expect, however, that a theory which in this first approximation is consistent might in itself provide an appropriate starting point for the discussion of the higher approximations which, not only as regards the self-energy question, but moreover with respect to all problems concerning the interaction between elementary particles, presents us with inconsistencies.

The self-energy problem of the mesons has been discussed elsewhere [5]; for the sake of completeness, the main conclusion there arrived at may be quoted here, viz. that the present method of eliminating infinities in the self-energies by means of compensation cannot be applied to the meson and that it would seem that our concept of the meson itself as an elementary point particle might need a revision in order to attain a consistent theory of the meson.

The present work was virtually completed in the summer of 1944 but war circumstances prevented its earlier publication. I take the opportunity to convey my heartfelt thanks to all those, without whose friendship and hospitality the carrying out of this work would have been impracticable.

Finally I wish to express my deep gratitude to Prof. L. ROSENFELD and Prof. H. A. KRAMERS for their kind interest in this work and for valuable discussions.

Amsterdam, 1942—1945.

Notes added in Proof: 1. The t-field has been chosen to be of the scalar type for reasons of simplicity. Further implications of the theory may not only show whether the t-field hypothesis can be upheld but also whether the choice of a scalar field is the most adequate one; similarly for the F-field. 2. An indication of fundamental differences in the treatment of the self-energy of FERMI–DIRAC as compared with BOSE–EINSTEIN particles may perhaps be seen in the results of SCHIFF, SNYDER and WEINBERG, Phys. Rev. **59**, 307, 315, 1940.

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3. E. C. G. STÜCKELBERG, Nature **144**, 118, 1939; Helv. Phys. Acta **14**, 51, 1941.
4. Cf. Chap. II, reference 2.
5. A. PAIS, Physica, **12**, 81, 1946.

CHAPTER I.

On the self-energy of particles with spin ½.

Summary.

§ 1. Introduction. — § 2. Hamiltonian of the total system; quantization of free meson fields. — § 3. General formalism of the Proca-field. — § 4. Calculation of the f_v-self-energy: *a*) the static, *b*) the spin, *c*) the fluctuation self-energy. — § 5. Calculation of the f_v-self-energy without canonical transformation. — § 6. Self-energies due to other fields. — § 7. On the higher order self-energies. — Appendix: 1. On the calculation of $W^{(1)}(\vec{p}_0)$. 2. Self-energy due to charged fields. 3. On the electromagnetic self-energy. 4. Canonical transformation in the scalar theory.

§ 1. *Introduction.*

As is well known, the interaction between spin ½ particles generally is described by assuming these particles to be the sources of (sets of) fields which are quantized according to the Bose–Einstein scheme. These fields can be characterized by the following properties of the corresponding field quanta:

a. Their mass. This may be either $\neq 0$, in which case we will in the following always speak of "mesons"; or it may be zero, as in the electromagnetic case, (photons).

b. Their charge: one can consider neutral or charged fields.

c. The transformation properties of their wave functions (field potentials). Assuming the spin of the field quanta to be ≤ 1 one can distinguish between scalar (*s*), vector (*v*), pseudovector (*pv*) and pseudoscalar (*ps*) fields.

Yet, the three mentioned attributes of the field being fixed, the interaction is not uniquely determined. In fact, taking due account of invariance conditions, we still can dispose of the form of the interaction operator with regard to its containing derivatives of the meson field wave functions or not *). If the operator is built up by using the field potentials only, we shall speak of *f*-interactions and the constant, (having the dimensions of a charge) determining the strength of the coupling between particle and field shall be denoted by *f*. The invariant interactions containing derivatives

*) We adhere to the customary restrictions of considering no higher than *first* derivatives of the potentials and no derivatives of the wave function of the spin ½ particle in the interaction operator.

only shall be denoted by g-interactions. The corresponding coupling constant here has the dimension of charge (g) times length.

Before continuing the discussion, it may be useful, in order to avoid confusion, to point out that the present use of the symbols f and g, which has appeared to me to be very appropriate to indicate the deep rooted distinctions between the interactions containing either or no derivatives, does not conform with the notations of KEMMER [1]). The following table gives a survey of the various interactions and the notations for the corresponding "charges", according to KEMMER (K.), MØLLER and ROSENFELD [2]), (M. R.) as well as the present paper; $\omega \equiv \sqrt{\varkappa c/4\pi}$.

Type of field	Type of interaction	Charge			Type of interaction	Charge		
		Pr. paper	K.	M. R.		Pr. paper	K.	M. R.
scalar = s	scalar	f_s	$g_a\,\omega$	f_1'	vector	g_s	$f_a\,\varkappa\omega$	f_2'
vector = v	vector	f_v	$g_b\,\omega$	g_1	tensor	g_v	$f_b\,\varkappa\omega$	g_2
pseudovector = pv	pseudovector	f_{pv}	$f_c\,\omega$	g_1'	pseudotensor	g_{pv}	$g_c\,\varkappa\omega$	g_2'
pseudoscalar = ps	pseudoscalar	f_{ps}	$f_d\,\omega$	f_1	pseudovector	g_{ps}	$g_d\,\varkappa\omega$	f_2

The self-energy $W(\vec{p})$ of the particle, ($\vec{p}$ denoting its momentum), i.e. the energy that must be attributed to it due to its interaction with the field concerned, can be developed in an infinite series with respect to the parameter $(\text{charge})^2/\hbar c$:

$$W(\vec{p}) = \sum_{n=1}^{\infty} W^{(n)}(\vec{p}) \quad \ldots\ldots\ldots \quad (1)$$

The development (1) applies equally to a "one particle theory", where it is assumed that all particle levels of negative energy are empty, as to "hole theory" where the vacuum is considered as a state in which all these levels are occupied in accordance with the exclusion principle.

The aim of the present chapter is twofold:

First, to derive the explicit expression for the first order self-energy $W^{(1)}(\vec{p})$ for the various types of fields, on one particle as well as on hole theory. For the electromagnetic self-energy of the electron this problem has been discussed by WEISSKOPF [3, 4]). Furthermore, KEMMER [1]) investigated, on hole theory, $W^{(1)}(\vec{p})$ for the case of nucleons interacting with meson fields. Unfortunately, however, his results are not correct, a point to which we will find opportunity to return in the course of the discussion, (§ 5);

Secondly, to discuss some general properties of the self-energy of any order on similar lines as was done by WEISSKOPF [4]) for the electromagnetic case.

In connection with the electromagnetic problem it has been remarked by WEISSKOPF, cf. loc. cit. [4]), p. 81, that the direct calculation of $W^{(n)}(\vec{p})$, $\vec{p} \neq 0$ by means of the representation of this quantity as an integral over

momentum space is ambiguous, essentially because, in this case, the problem is not spherically symmetrical with respect to that space. However, the total self-energy generally can be written as (cf. loc. cit. [4]) p. 80)

$$W(\vec{p}) = \langle \int S_{..} F^{..} \, dv \rangle_{\vec{p}}, \quad \ldots \ldots \quad (2)$$

where $S_{..}$ and $F^{..}$ denote the tensor components of a source function and of a function of the field variables respectively, in such a way that the tensor product of $S_{..}$ and $F^{..}$ is a scalar density; $\langle \; \rangle_{\vec{p}}$ indicates that the expectancy value of the integral should be taken for a situation in which one particle in the state of momentum $\vec{p}$ is present. Hence, calling W the value of $W(\vec{p})$ for $\vec{p} = 0$, we have

$$W(\vec{p}) = W\sqrt{1-\beta^2}$$

and thus

$$W^{(n)}(\vec{p}) = W^{(n)}\sqrt{1-\beta^2}, \quad (W = \sum_n W^{(n)}), \quad \ldots \quad (1a)$$

because the series in the right member of (1) represents a development with respect to the invariant parameter $(\text{charge})^2/\hbar c$. Thus, once one has defined the way in which the momentum space integral $W^{(n)}$ is to be calculated, $W^{(n)}(\vec{p})$ follows unambiguously from (1a). Now the only natural way to compute $W^{(n)}$ is [4]) to sum first over all contributions of one spherical shell concentric around the origin and then over all shells. Defining $W^{(n)}$ in this way, $W^{(n)}(\vec{p})$ is then fixed by (1a), or, in other words, the contributions of the various momentum space regions to $W^{(n)}(\vec{p})$ should be taken together in such a way as to yield (1a). It should be noted that the computation of $W^{(n)}(\vec{p})$ by means of the same prescription as adopted for $W^{(n)}$ would not lead to the connection (1a). An example of this is given in the Appendix, note 1.

In applying (2) to hole theory, it may be recalled that the expectancy value of any operator O depending on source functions S and field functions F, where now S and F refer to the total distributions of the completely occupied negative energy levels plus the one particle in the state of positive energy, should be replaced by the expectancy value of

$$O(S - S_{\text{vac}}, F - F_{\text{vac}}) - O_{\text{vac}}, \quad \ldots \ldots \quad (3)$$

where S_{vac} is the same source for the vacuum, while F_{vac} represents the field generated by the vacuum distribution; O_{vac} is the expectancy value for this distribution itself. The subtraction of O_{vac} represents a suitable fixation of the zero point of energy, ensuring the symmetry of the theory with respect to particle and anti-particle.

It will be clear that (1a) holds irrespective of whether one has to do with one particle or hole theory. Thus the establishment of (1a) considerably simplifies the computational work we shall have to perform, as it allows us to confine our attention to the case that the particle momentum is zero. Another simplification is obtained by remarking that we can confine us to explicit calculations for neutral interactions only:

Let us consider an arbitrary neutral f- or g-interaction with coupling constants f and g respectively and the corresponding charged interaction involving the same meson Compton wave length $\varkappa^{-1}$; we prime those quantities that refer to charged interaction, thus the coupling constants in the latter case are f', g'. If $f = f'$, $g = g'$, the probability for (virtual) emission and absorption of field quanta is the same for the neutral and the charged interaction. The relative magnitude of $W^{(n)}$ and $W^{(n)'}$ will then only depend on the number of intermediate states in both cases. As, in first approximation, for a proton only the positive, for a neutron only the negative mesons come into play, we have

$$W_f^{(1)} = W_f^{(1)'}, \quad W_g^{(1)} = W_g^{(1)'}$$

if

$$|f| = |f'|, \quad |g| = |g'|;$$

the absolute values of the constants here occur because $W^{(1)}$ is proportional to the square of the charge. If both neutral and charged interaction are present, the latter relations represent the condition for the theory to be "symmetrical". In a symmetrical meson theory, the first approximation to the self-energy is therefore twice that of the neutral theory, while in an unsymmetrical theory, $W^{(1)}$ will be the same function of f, g, $\varkappa$ as $W^{(1)'}$ is of f', g', $\varkappa'$.

In order not to complicate unnecessarily the formalism, we will therefore confine ourselves in the sections dealing with the first order calculations to neutral fields; a simple example of the calculation for charged fields, exhibiting all essential differences with the neutral case is given in the Appendix, note 2, where also the connection between $W^{(n)}$ and $W^{(n)'}$ for $n > 1$ is briefly discussed.

The properties of the electromagnetic first order self-energy have been examined by WEISSKOPF [3, 4]) by gauging vector and scalar potential in such a way that the operator of interaction between particle and field is separated into two parts: one, involving the Coulomb potential, yielding the static interactions and, correspondingly, the "static" self-energy, and one depending on the transverse vector potential $\vec{A}_\perp$, which gives rise to a "dynamic" self-energy which has no classical counterpart. The latter is defined as the expectancy value $\sim e^2$ of

$$-\tfrac{1}{2}\int \vec{s}\,\vec{A}_\perp \, dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\vec{s}$ is the current density. Both $\vec{s}$ and $\vec{A}_\perp$ can be developed in a power series in e, the electric charge:

$$\vec{s} = \sum_{n=1}^{\infty} \vec{s}^{(n)} \quad \vec{A}_\perp = \sum_{n=0}^{\infty} \vec{A}_\perp^{(n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

the term with the superscript (n) being proportional to e^n. The first term in the development of $\vec{A}_\perp$ is independent of e and denotes a superposition of plane waves which always may be added to the solution of the inhomogeneous wave equation for $\vec{A}_\perp$. Therefore, in first approximation (4) may be replaced by

$$-\tfrac{1}{2}\int \vec{s}^{(1)}\,\vec{A}_\perp^{(1)}\,dv - \tfrac{1}{2}\int \vec{s}^{(2)}\,\vec{A}_\perp^{(0)}\,dv \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

The first term denotes the interaction of the unperturbed "proper" current with the proper vector potential (i.e. the vector potential due to the unperturbed current distribution). It has been shown by WEISSKOPF, (cf. loc. cit. [4]), section III) that its contribution to the self-energy (for a particle at rest) can be interpreted as to be due to the magnetic moment of the electron; it is therefore called "spin" energy. The second term accounts for the interaction of the electron with the fluctuating electromagnetic "zero field"; its contribution is called fluctuation self-energy. On hole theory, the subtractions (3) should be performed.

The discussion of $W^{(1)}$ for other fields can of course be given on the same lines as WEISSKOPF's treatment: by means of a canonical transformation, the interaction operator can be divided into a static and a dynamic part. The latter can, by means of developments similar to (1*b*), be written as a sum of a "spin" and a "fluctuation" term, like (4*a*). In order to show clearly the characteristics of the mass of the field quanta being different from zero such a treatment will be given for the f_p-case, or, as it is often called, the Proca-field [5]), which will be shown to involve the electromagnetic field as the limiting case that $\varkappa$, the inverse field range, is zero; this is not quite trivial, because of the well known differences in methods of quantization of Proca- and Maxwell-field *). In particular it will be shown that the divergent part of $W^{(1)}$, as obtained by developing the integrand of the momentum space integral representing $W^{(1)}$ in the region of large p, is independent of $\varkappa$. In the course of this treatment simple general formulae will be obtained by means of which the self-energy can be computed for any coupling.

However, it will appear that the distinction between static and dynamic terms, which underlies most investigations on interaction problems, is not essential as concerns the self-energy, and that the calculations are even

*) Cf. e.g. L. J. F. BROER and A. PAIS, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 190 (1945).

considerably simplified if no canonical transformation is performed, especially in the cases of g-interactions and of charged fields in general: due to the non-commutability of the spin and of the "isotopic" variables such a transformation would here yield a dynamic operator in the form of an infinite series which would make it extremely cumbersome, if feasible, to compute the dynamic self-energy. In order to compare the methods with and without the separation into static and dynamic parts, we shall apply the latter to compute again the self-energy due to the Proca-field. It will be most convenient to consider $W^{(1)}$ as the sum of two terms: one corresponding with the sum of static and spin energy, which is different for one-particle and hole theory; the other, the fluctuation energy which is the same in both cases. Having thus dealt in detail with the f_v-interaction, we will have collected all essential formulae for a simple and straightforward calculation of all f- and g-self-energies.

In the next section, some well-known formulae will be collected concerning the Hamiltonian of the total system and the quantization of free meson fields. § 3 deals with the general formalism of the Proca-field and with the canonical transformation which separates the static and dynamic parts of the interaction. In § 4 the self-energy due to this field is calculated and it is shown that it is legitimate to consider the electromagnetic self-energy as the special case of the former for $\varkappa = 0$. In § 5 the same quantity is again computed, but now without the use of a canonical transformation, while § 6 consists of a survey of the other self-energies and their divergence properties. Finally, § 7 is devoted to the establishment, on hole theory, of some general properties of the self-energy of arbitrary order. In particular it is shown that all f-self-energies $W_f^{(n)}$ diverge at most logarithmically for any n.

§ 2. *Hamiltonian of the total system; quantization of free meson fields.* Generally, the Hamiltonian can be written as

$$\left.\begin{aligned} &H_{\text{total}} = H_M + H + H_{\text{fields}}, \\ \text{with}\quad &H_M = \int \psi^\dagger \left(\frac{\hbar}{i} \vec{a} \vec{\nabla} + \varrho_3 M \right) \psi \, dv \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

The velocity of light is put equal to 1. M is the mass of the particle*); H depends on the particle variables as well as on those of the fields, H_{fields} on the latter only. In view of applications to hole theory, we take ψ and $\psi^\dagger$ to be q-numbers, satisfying the commutation relations

$$\psi_r(\vec{x}, t)\, \psi^\dagger_{r'}(\vec{x}', t) + \psi^\dagger_{r'}(\vec{x}', t)\, \psi_r(\vec{x}, t) = \delta_{rr'}\, \delta(\vec{x} - \vec{x}'), \quad r, r' = 1, \ldots, 4 \quad (6)$$

all other pairs anti-commuting. Introducing a system of one particle wave

*) In applying (5) to nucleons, we take the same M for proton and neutron.

functions φ_q, where the symbolic index q denotes momentum as well as spin and positive or negative energy state, ψ can be expanded in a sum which is in the same sense symbolic:

with
$$\left.\begin{aligned} \psi &= \Sigma a_q \varphi_q \\ a_q a_{q'}^\dagger + a_{q'}^\dagger a_q &= \delta_{qq'} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

while all other pairs anti-commute. The diagonal elements of products of two or four a's are

$$a_q^\dagger a_q = N_q, \qquad a_q a_q^\dagger = 1 - N_q, \quad \ldots\ldots\ldots \quad (8)$$

$$a_q^\dagger a_q a_r^\dagger a_r = N_q N_r, \qquad a_q^\dagger a_r a_r^\dagger a_q = N_q (1 - N_r) \quad \ldots \quad (9)$$

N_q is the occupation number (0 or 1) of the q-th state. The φ_q may be developed similarly to (3); in zeroth approximation they are the free particle wave functions $\varphi_q^{(0)}$:

$$\varphi_q^{(0)} = u_q(\vec{p})\, e^{\frac{i}{\hbar}(\vec{p}\,\vec{x} - E_q t)} \quad \ldots\ldots\ldots \quad (10)$$

which are supposed to be normalized in a big cube of volume 1 *).

Next we consider the field equations in free space, (no particles present). In the vector, (or, which here amounts to the same, the pseudo-vector) case they are, (a dot denotes differentiation with respect to t)

$$\vec{F} = -\operatorname{grad} V - \dot{\vec{U}} \quad \ldots\ldots\ldots \quad (11a)$$

$$\vec{G} = \operatorname{rot} U, \quad \ldots\ldots\ldots \quad (11b)$$

$$\operatorname{div} \vec{F} + \varkappa^2 V = 0 \quad \ldots\ldots\ldots \quad (11c)$$

$$\operatorname{rot} \vec{G} + \varkappa^2 \vec{U} = \dot{\vec{F}} \quad \ldots\ldots\ldots \quad (11d)$$

As a consequence of (11, c, d):

$$\operatorname{div} \vec{U} + \dot{V} = 0 \quad \ldots\ldots\ldots \quad (12)$$

The equations (11a, d) may be derived from the following Hamiltonian

$$H_{\text{fields}} = \tfrac{1}{2} \int \left\{ \vec{F}^2 + \vec{G}^2 + \varkappa^2 (\vec{U}^2 + V^2) \right\} dv, \quad \ldots \quad (13)$$

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*) The actual systems which are considered in this paper consist, in zeroth approximation, of free particles and of free meson fields. Thus the wave functions of the total system are in this approximation given by (10) multiplied with an infinite product of δ-functions denoting the stationarity of the occupation of the meson states.

in which G and V must be considered as "derived" variables [6]), i.e. as to be defined by (11b, c) in terms of the canonical variables $\vec{U}$ and $\vec{F}$ which satisfy the following commutation rules

$$[U_l(\vec{x}, t), F_k(\vec{x}', t)]_- = \frac{\hbar}{i}\delta_{lk}\,\delta(\vec{x}-\vec{x}') \quad . \quad . \quad . \quad . \quad . \quad (14)$$

U may be written as a superposition of plane waves

$$\left.\begin{aligned} \vec{U} = \sum_{j=0,1,2}\sum_{\vec{k}} \vec{\varepsilon}_{jk}\,[U^+(j\vec{k})\,e^{i(\vec{k}\vec{x}-\nu t)} + U^-(j,\vec{k})\,e^{-i(\vec{k}\vec{x}-\nu t)}], \\ \nu = \sqrt{\varkappa^2+k^2}. \end{aligned}\right\} \quad . \quad (15)$$

The wave vectors k have to satisfy periodicity conditions on the boundary of the cube in which the whole system is supposed to be enclosed. For the polarization vectors $\vec{\varepsilon}_{jk}$ the following relations hold

$$\vec{\varepsilon}_{0k} = \vec{k}^0 \equiv \frac{\vec{k}}{k}, \quad k = |\vec{k}|, \quad \vec{\varepsilon}_{jk}\vec{\varepsilon}_{j'k} = \delta_{jj'} \quad . \quad . \quad (15a)$$

Thus $j = 0\,(1, 2)$ denotes the longitudinal (transverse) waves. The quantization yields by standard methods as representation for the Fourier amplitudes

$$\left.\begin{aligned} U^+(j\vec{k})_{n\to n+1} = U^-(j,\vec{k})_{n+1\to n} = \sqrt{\frac{\hbar(n+1)}{2\nu}},\ j = 1, 2 \\ U^+(0,\vec{k})_{n\to n+1} = U^-(0,\vec{k})_{n+1\to n} = \frac{\nu}{\varkappa}\sqrt{\frac{\hbar(n+1)}{2\nu}}, \end{aligned}\right\} \quad . \quad (16)$$

all other matrix elements vanishing. The notation $n \to m$ denotes a transition in which the number of mesons with wave vector $\vec{k}$ and state of polarization j changes from n to m.

In the (pseudo)scalar theory the field equations are

$$\vec{\Gamma} = \text{grad}\,\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

$$\Lambda = -\dot{\Omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17b)$$

$$\text{div}\,\vec{\Gamma} + \dot{\Lambda} = \varkappa^2\,\Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17c)$$

The Hamiltonian is

$$H_{\text{fields}} = \tfrac{1}{2}\int(\vec{\Gamma}^2 + \Lambda^2 + \varkappa^2\,\Omega^2)\,dv, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

with $\vec{\Gamma}$ as derived variable defined by (17a). We have the commutation rule

$$[\Omega(\vec{x}, t), \Lambda(\vec{x}', t)]_- = \frac{\hbar}{i}\,\delta(\vec{x}-\vec{x}').$$

On developing Ω:

$$\Omega = \sum_{\vec{k}} [\Omega^+ (\vec{k})\, e^{i(\vec{k}\,\vec{x}-\nu t)} + \Omega^- (\vec{k})\, e^{-i(\vec{k}\,\vec{x}-\nu t)}],$$

quantization gives

$$\Omega^+ (\vec{k})_{n\to n+1} = \Omega^- (\vec{k})_{n+1\to n} = \sqrt{\frac{\hbar (n+1)}{2\nu}} \quad . \quad . \quad . \quad . \quad (19)$$

§ 3. *General formalism of the Proca-field.*

The field equations are given by (11a, b), while (11c, d) are replaced by *)

$$\operatorname{div} \vec{F} + \varkappa^2 V = N \quad , \quad N = f \psi^\dagger \psi, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\operatorname{rot} \vec{G} + \varkappa^2 \vec{U} = \vec{M} + \dot{\vec{F}} \quad , \quad M = f \psi^\dagger \vec{\alpha}\, \psi \quad . \quad . \quad . \quad . \quad (21)$$

(20) must again be considered as defining V in terms of F *and* N. The Hamiltonian is given by (5) with (13) and

$$H = -\int \vec{M}\, \vec{U}\, dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

while (14), here holds too.

In order to obtain the desired separation into static and dynamic interaction the case need be considered in which all velocity dependence is neglected. This situation is described by the static equations

$$\vec{F}_0 = -\operatorname{grad} V_0, \quad \operatorname{div} \vec{F}_0 + \varkappa^2 V_0 = N, \quad \vec{U}_0 = 0. \quad . \quad . \quad (23)$$

The corresponding static interaction can be separated from all other terms of (5) by means of a canonical transformation. If $\widetilde{X}$ is a function of the "old" variables, from now on indicated by $\widetilde{U}$, $\widetilde{F}$, ... and X the same function of the "new" ones: U, F, ... the unitary transformation effecting this separation is

$$\widetilde{X} = S^{-1} X S, \quad S = \exp \frac{i}{\hbar} K, \quad K = \int \vec{F}_0\, \vec{U}\, dv = \int \vec{F}_0\, \vec{U}_{||}\, dv, \quad . \quad (24)$$

where $\vec{U}_{||}$ is the longitudinal part of $\vec{U}$. Using (23) we get

$$K = -f \sum_r \int\int dv'\, dv''\, \psi_r^\dagger (\vec{x}'')\, \psi_r (\vec{x}'')\, \vec{U}_{||} (\vec{x}')\, \vec{\nabla}'\, \chi (|\vec{x}' - \vec{x}''|),$$

where

$$\chi (r) = \frac{e^{-\varkappa r}}{4\pi r}$$

*) In §§ 3—5 we write for simplicity f instead of f_v. Throughout this paper all charges are expressed in Heaviside units.

is the Green function of the field, satisfying

$$\Delta \chi(r) - \varkappa^2 \chi(r) = -\delta(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We now transform the old Hamiltonian $\widetilde{H}$, given (in terms of the old variables) by (5), (13) and (22), and will thereto use the commutation rules (6) and (14) which now have to be interpreted as holding between the new variables which indeed are again canonically conjugated. First consider H_M:

$$\widetilde{H}_M = H_M + [K, \int \psi^\dagger \vec{\alpha} \vec{\nabla} \psi \, dv]_- .$$

On the right hand side the first two terms have been written down of the development of H in an infinite series; it can easily be seen, however, that the higher terms disappear. With the help of

$$[\psi^\dagger_r(\vec{x}) \psi_r(\vec{x}), \quad \psi^\dagger_{r'}(\vec{x}') \psi_{r''}(\vec{x}')]_- =$$
$$= \{\psi^\dagger_r(\vec{x}) \psi_{r''}(\vec{x}') \delta_{rr'} - \psi^\dagger_{r'}(\vec{x}') \psi_r(\vec{x}) \delta_{rr''}\} \delta(\vec{x} - \vec{x}')$$

it is found by means of some partial integrations and taking into account (25) that

$$\widetilde{H}_M = H_M + \int \vec{M} \vec{U}_{||} \, dv - \varkappa^2 \int \vec{M}(\vec{x}') \vec{U}_{||}(\vec{x}) \chi(|\vec{x} - \vec{x}'|) \, dv \, dv' .$$

The transformation of (13) is performed with the help of

$$\widetilde{\vec{U}} = \vec{U}, \quad \widetilde{\vec{F}} = \vec{F} + \vec{F}_0, \quad \widetilde{\vec{G}} = \vec{G} = \operatorname{rot} \vec{U} = \operatorname{rot} \vec{U}_\perp ,$$
$$\widetilde{V} = V - \varkappa^{-2} \operatorname{div} \vec{F}_0 = V_0 - \varkappa^{-2} \operatorname{div} \vec{F}.$$

$\vec{U}_\perp$ is the transverse part of $\vec{U}$. Thus

$$\widetilde{H}_{\text{fields}} = H_{\text{fields}} + V$$

where

$$V = \tfrac{1}{2} \int\int N(\vec{x}) N(\vec{x}') \chi(|\vec{x} - \vec{x}'|) \, dv \, dv' \quad . \quad . \quad . \quad . \quad (26)$$

is the static interaction. As $\widetilde{H} = H$, the total transformed Hamiltonian finally becomes

$$\widetilde{H}_{\text{total}} = H_m + H_{\text{fields}} + V + W,$$

where

$$W = -\int \vec{M} \vec{U}_\perp \, dv - \varkappa^2 \int\int dv \, dv' \, \vec{M}(\vec{x}) \vec{U}_{||}(\vec{x}') \chi(|\vec{x} - \vec{x}'|) \quad (27)$$

is the operator of "dynamic" interaction. Taking $\varkappa = 0$, (26) and (27) yield the well known corresponding expressions for the electromagnetic field.

From (26) and (27) the following expressions for the static, spin and fluctuation energy are obtained, cf. (1*b*), (4*a*) or loc. cit. [4]) eq. (21) *)

$$W^{(1)}_{\text{stat}} = \tfrac{1}{2}\overline{\int\int dv\, dv'\, N(\vec{x})\, N(\vec{x}')\, \chi(|\vec{x}-\vec{x}'|)},$$

$$W^{(1)}_{\text{spin}} = -\tfrac{1}{2}\overline{\int \vec{M}^{(1)}\, \vec{U}^{(1)}_{\perp}\, dv} - \frac{\varkappa^2}{2}\overline{\int\int dv\, dv'\, \vec{M}^{(1)}(\vec{x})\, \vec{U}^{(1)}_{\parallel}(\vec{x}')\, \chi(|\vec{x}-\vec{x}'|)},$$

$$W^{(1)}_{\text{fluct}} = -\tfrac{1}{2}\overline{\int \vec{M}^{(2)}\, \vec{U}^{(0)}_{\perp}\, dv} - \frac{\varkappa^2}{2}\overline{\int\int dv\, dv'\, \vec{M}^{(2)}(\vec{x})\, \vec{U}^{(0)}_{\parallel}(x')\, \chi(|x-x'|)}, \quad (28)$$

where the overlining denotes that the expectancy value should be taken for the state in which one particle is present in the lowest positive energy level. In the case of hole theory, the subtractions should be performed in accordance with (3).

§ 4. *Calculation of the f_v-self-energy.*

a) The static self-energy.

The operator is

$$\tfrac{1}{2}\int\int dv\, dv' \{N(\vec{x}) - N_{\text{vac}}(\vec{x})\}\{N(\vec{x}') - N_{\text{vac}}(\vec{x}')\}\chi(|\vec{x}-\vec{x}'|) - \\ - \tfrac{1}{2}\int\int dv\, dv' \{N(\vec{x})\, N(\vec{x}')\}_{\text{vac}}\chi(|\vec{x}-\vec{x}'|).$$

Using (7), (9) and (10), one finds for the diagonal element, if there is one particle present in the positive level $\vec{p}_0$

$$W^{(1)}_{\text{stat}} = \frac{f^2\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}\Big) \frac{\{u^{\dagger}_{q_0}(\vec{p}_0)\, u_q(\vec{p})\}\{u^{\dagger}_q(\vec{p})\, u_{q_0}(\vec{p}_0)\}}{\mu^2 + |\vec{p}-\vec{p}_0|^2}, \quad (29)$$

where

$$\mu = \hbar\varkappa$$

is the meson mass, (having the dimension of a momentum). The plus (minus) sign holds for the one-particle (hole) theory. The summation $q = 1, 2$, $(q = 3, 4)$ means integration over all values of $\vec{p}$ as well as summation over both spin states corresponding with each $\vec{p}$ and with the

*) In the expression for $W^{(1)}_{\text{stat}}$, N has been written for simplicity instead of $N^{(1)}$.

positive (negative) energy sign, while S denotes the averaging over the two possible spin states q_0 corresponding with $\vec{p}_0$ (and positive energy). Generally, $\{u_q^\dagger X u_r\}$, where X operates on the spin variables is the inner product of $u_q^\dagger$ and Xu_r for a fixed spin and sign of energy of the states q and r. The spin summations can be performed in the usual way and one gets for $p_0 = 0$

$$W_{\text{stat}}^{(1)} \begin{cases} = \dfrac{f^2}{16\pi^3\hbar}\int \dfrac{d\vec{p}}{p^2+\mu^2} & \text{(one particle theory)} \quad (30) \\ = \dfrac{f^2 M}{16\pi^3\hbar}\int \dfrac{d\vec{p}}{P(p^2+\mu^2)} & \text{(hole theory)} \quad (31) \end{cases}$$

with

$$P = +\sqrt{p^2+M^2}.$$

Developing the integrals gives, (*f.t.* = finite terms)

$$W_{\text{stat}}^{(1)} \begin{cases} = \dfrac{f^2}{4\pi^2\hbar}\int dp + \text{f.t. (one-p. th.)} \quad (30a) \\ = \dfrac{f^2 M}{4\pi^2\hbar}\int \dfrac{dp}{p} + \text{f.t. (hole th.)} \quad (31a) \end{cases}$$

Thus the divergence on the one-particle theory is of the same type as that of the classical f_v-self-energy of a point particle, *viz.* linear.

The corresponding expressions for the electromagnetic field are found from (30) and (31) by putting $\mu = 0$. Such a proceeding is obviously legitimate as it is identical with the statement that the Yukawa potential goes over into the Coulomb potential for $\mu = 0$.

b) The spin self-energy.

We have to compute the average value of

$$-\tfrac{1}{2}\int dv\,[(\vec{M}^{(1)}-\vec{M}_{\text{vac}}^{(1)})(\vec{U}_{\perp}^{(1)}-\vec{U}_{\perp,\text{vac}}^{(1)}) - (\vec{M}^{(1)}\vec{U}_{\perp}^{(1)})_{\text{vac}}]$$

$$-\frac{\varkappa^2}{2}\int\int dv\,dv'\,[\{M^{(1)}(\vec{x})-M_{\text{vac}}^{(1)}(\vec{x})\}\{\vec{U}_{||}^{(1)}(\vec{x}')-\vec{U}_{||\,\text{vac}}^{(1)}(\vec{x}')\} -$$

$$-\{\vec{M}^{(1)}(\vec{x})\,\vec{U}_{||}^{(1)}(\vec{x}')\}_{\text{vac}}]\,\chi(|\vec{x}-\vec{x}'|).$$

The terms in the first line denote the transverse, those of the second the longitudinal spin energy. $\vec{U}^{(1)}$ is the solution of *)

$$\Box\,U^{(1)} - \varkappa^2 U^{(1)} = -\vec{M}^{(1)}$$

*) It should be noted that this equation is the same in the new as in the old variables, cf. § 3.

and may therefore be written as

$$\vec{U}^{(1)} = \int d\vec{k}\, e^{i\vec{k}\vec{x}} \int_{-\infty}^{t} d\tau \frac{\sin \nu (t-\tau)}{\nu} \int d\vec{\xi}\, \vec{M}^{(1)}(\vec{\xi}, \tau)\, e^{-i\vec{k}\vec{\xi}}. \quad . \quad (32)$$

In $\vec{M}^{(1)}$ we may insert (7) with (10), as it is the "unperturbed" current, ($\sim f$); the dependence on the a_q is the same as for the static energy; again using (9) we get for the spin energy

$$-\frac{f^2\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum'_{q=3,4}\Big) \frac{F}{-2\hbar\nu}\left[\frac{1}{E_0 - E - \hbar\nu} + \frac{1}{E - E_0 - \hbar\nu}\right], \quad (33a)$$

$$F \equiv \{u^{\dagger}_{q_0}(\vec{p}_0)\vec{\alpha}_{\perp} u_q(\vec{p})\}\{u^{\dagger}_{q}(\vec{p})\vec{\alpha}_{\perp} u_{q_0}(\vec{p}_0)\} + \\ + \frac{\mu^2}{\mu^2 + |\vec{p} - \vec{p}_0|^2}\{u^{\dagger}_{q_0}(\vec{p}_0)\vec{\alpha}_{\parallel} u_q(\vec{p})\}\{u^{\dagger}_{q}(\vec{p})\vec{\alpha}_{\parallel} u_{q_0}(\vec{p}_0)\}$$

This may also be written as

$$-\frac{f^2\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum'_{q=3,4}\Big) \frac{F}{\mu^2 + |\vec{p} - \vec{p}_0|^2 - (E - E_0)^2}. \quad . \quad . \quad (33b)$$

$\vec{\alpha}_{\perp}$ is the component of $\vec{\alpha}$ perpendicular to $\vec{p} - \vec{p}_0$, $\vec{\alpha}_{\parallel}$ is parallel to this vector. The first term of F corresponds with the transverse, the second with the longitudinal energy. The terms in (33a) in square brackets represent the characteristic energy denominators. As is well known, those states q must be excluded from the summations for which the corresponding energy denominators are equal to zero. We now show that for one particular value of E the second denominator may vanish, i.e. $E = E_0 - \hbar\nu$. As this E will be seen to be < 0, we have marked the summation $\sum'_{q=3,4}$ with a prime to indicate that this value should be omitted. Taking for simplicity $p_0 = 0$, so $E_0 = M$, we get indeed that $E = M - \hbar\nu$ if

$$E < 0$$

$$p = \sqrt{\frac{\mu^4}{4M^2} - \mu^2} \equiv p' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

provided

$$\mu \geqslant 2M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34a)$$

Consequently if (34a) is satisfied, which means that the mass of the meson is at least twice that of the particle, the momentum space integrals concerned should be understood in the following way

$$\int d\vec{p} = 4\pi \lim_{\eta \to 0}\left[\int_0^{p'-\eta} p^2\, dp + \int_{p'+\eta}^{\infty} p^2\, dp\right].$$

If on the other hand $\mu < 2M$, the integration simply means

$$\int d\vec{p} = 4\pi \int_0^\infty p^2\, dp.$$

In this paper, we shall, for shortness, always write $\int d\vec{p}$. With

$$X = p^2 + \mu^2 - \frac{\mu^4}{4M^2}$$

the separate expressions for the transverse and the longitudinal spin energy are found to be, if $\vec{p}_0 = 0$,

$$W^{(1)}_{\text{spin, transv.}} = \begin{cases} \dfrac{f^2}{32\pi^3\hbar}\left(\dfrac{\mu}{M}\right)^2 \int \dfrac{d\vec{p}}{X} = \dfrac{f^2}{8\pi^2\hbar}\left(\dfrac{\mu}{M}\right)^2 \int dp + \text{f. t. (one p. th.)} & (35a) \\ -\dfrac{f^2}{16\pi^3 M\hbar}\int \dfrac{d\vec{p}}{X}(p^2 + \tfrac{1}{2}\mu^2) = -\dfrac{f^2}{4\pi^2 M\hbar}\left[\int p\, dp - \right. \\ \left. -\dfrac{M^2}{2}\left\{1 + \dfrac{\mu^2}{M^2} - \dfrac{1}{2}\dfrac{\mu^4}{M^4}\right\}\int \dfrac{dp}{p}\right] + \text{f. t. (hole th.)} \ldots & (36a) \end{cases}$$

$$W^{(1)}_{\text{spin, long.}} = \begin{cases} \dfrac{f^2}{64\pi^3\hbar}\dfrac{\mu^4}{M^2}\int \dfrac{d\vec{p}}{X(p^2+\mu^2)} = \text{finite (one p. th.)} \ldots\ldots\ldots & (35b) \\ -\dfrac{f^2}{32\pi^3 M\hbar}\cdot\mu^2 \int \dfrac{d\vec{p}(p^2+\frac{1}{2}\mu^2)}{X(p^2+\mu^2)} = -\dfrac{f^2\mu^2}{8\pi^2 M\hbar}\int \dfrac{dp}{p} + \text{f. t. (hole th.)} & (36b) \end{cases}$$

It is easily seen that, if (34a) is satisfied, which means that X becomes zero for the p-value (34), there occurs no additional contribution due to the singularity of the integrand for this p; thus, in particular, the divergent part of integrals like (35) and (36) is independent of (34a) being fulfilled or not.

Again, as in all cases where quantization is not explicitly involved, there is no difficulty in performing the transition $\mu = 0$, whch gives the electromagnetic spin energy. In this case the longitudinal contribution vanishes, of course, for both one particle and hole theory, while in the former the transverse part also becomes zero, in accordance with the result of WEISSKOPF *).

It may be remarked here, and it will be useful to note this for further purposes, that (33a) can also be written as

$$W^{(1)}_{\text{spin}} = \tfrac{1}{2} S\Big(\sum_{q=1,2} \pm \sum'_{q=3,4}\Big) \sum_{j=0,1,2} \left\{ \left[\frac{(q_0, 0 \mid W_j \mid q, \vec{k})\,(q, \vec{k} \mid W_j \mid q_0, 0)}{E_0 - E - \hbar\nu} + \right.\right. \\ \left.\left. + \frac{(q_0, -\vec{k} \mid W_j \mid q, 0)\,(q, 0 \mid W_j \mid q_0, -\vec{k})}{E - E_0 - \hbar\nu} \right] \right\} \quad (37)$$

*) Cf. loc. cit. [4]) eq. (19) and (23).

where

$$\vec{hk} = \vec{p}_0 - \vec{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and where the matrix elements ($|W_j|$) refer to the interaction between particle and quantized meson field:

$$\left.\begin{aligned} j=1,2 \quad (q_0, 0 \,|\, W_j \,|\, q, \vec{k}) &= -f\sqrt{\frac{\hbar}{2\nu}}\,\{u_0^\dagger \vec{a}\vec{\varepsilon}_{jk} u_q\}, \\ (q_0, 0 \,|\, W_0 \,|\, q, \vec{k}) &= -f\frac{\varkappa}{\nu}\sqrt{\frac{\hbar}{2\nu}}\,\{u_0^\dagger \vec{a}\vec{k}^0 u_q\}, \text{ etc.} \end{aligned}\right\} \quad (39)$$

Cf. especially (16) and (27).

An interesting result is obtained by taking together (29) and (33*b*). By means of the operator identities

$$\left.\begin{aligned} \vec{a}_\perp \vec{a}_\perp &= \vec{a}\vec{a} - \frac{(\vec{a}, \vec{p}-\vec{p}_0)(\vec{a}, \vec{p}-\vec{p}_0)}{|\vec{p}-\vec{p}_0|^2}, \\ (\vec{a}, \vec{p}-\vec{p}_0) &= E-E_0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

it follows that

$$\left.\begin{aligned} &W^{(1)}_{\text{stat}} + W^{(1)}_{\text{spin}} = \\ &= \frac{f^2\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum'_{q=3,4}\Big) \frac{\{u_0^\dagger u_q\}\{u_q^\dagger u_0\} - \{u_0^\dagger \vec{a} u_q\}\{u_q^\dagger \vec{a} u_0\}}{\mu^2 + |\vec{p}-\vec{p}_0|^2 - (E-E_0)^2} \end{aligned}\right\} . \quad (40a)$$

The invariant expression behind the summation signs is the generalization to the case $\mu \neq 0$ of the invariant electromagnetic matrix elements of MØLLER [7]). We will therefore denote (40*a*) by $W^{(1)}_{\text{inv}}$:

$$W^{(1)}_{\text{inv}} = W^{(1)}_{\text{stat}} + W^{(1)}_{\text{spin}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40b)$$

Similar invariant matrix elements, with other numerators of course, will be found to occur for any *f*- or *g*-interaction; this enables one to write down directly the expression for $W^{(1)}_{\text{inv}}$, or, more generally, the matrix elements of interaction, for all KEMMER cases.

c) The fluctuation self-energy.

It is well known that the lowest (positive) energy state of a "free" particle which produces a field is defined as follows: the particle has zero momentum, while all occupation numbers of the quantum states of the field have zero expectancy value.

Notwithstanding the latter condition, the particle can, by virtual emissions and reabsorptions of field quanta, interact with the zero field. From this interaction originates the fluctuation self-energy of the particle.

The average value which we have to consider is given by (28), in which, in de first place, $\vec{M}^{(2)}$ should be replaced by $\vec{M}^{(2)}-\vec{M}_{\text{vac}}$. It can easily be seen, however, that the terms $\vec{M}_{\text{vac}}\,\vec{U}^{(0)}$ yield no contribution. Thus $W^{(1)}_{\text{fluc}}$ is given by the difference of the expectancy values of (28) for the situations "vac + q_0" and "vac".

$\vec{M}^{(2)}$ is the current due to the field $\vec{U}^{(0)}$ which is considered as a time dependent perturbation. We have

$$\vec{M}^{(2)} = f\,\psi^{(0)\dagger}\,\vec{\alpha}\,\psi^{(1)} + f\,\psi^{(1)\dagger}\,\vec{\alpha}\,\psi^{(0)}$$

$$\psi^{(0)} = \Sigma\, a_q\, \varphi^{(0)}_q \quad , \quad \psi^{(1)} = \Sigma\, a_q\, \varphi^{(1)}_q,$$

where $\varphi^{(0)}_q$ is given by (10) and $\varphi^{(1)}_q$ is the next term in the development, similar to (3), of φ_q in a power series in f. Thus $\varphi^{(1)}_q(\vec{x}, t)$ is the additional part of the one particle wave function due to the occurrence of the perturbation term (cf. (27))

$$H^{(1)} = -f\,\vec{\alpha}\,\vec{U}^{(0)}_{\perp} - \varkappa^2 f\,\vec{\alpha}\int d\vec{\xi}\,\chi(|\vec{x}-\vec{\xi}|)\,\vec{U}^{(0)}_{\parallel}(\xi)$$

in the one particle Hamiltonian. Therefore

$$\varphi^{(1)}_q(\vec{x}, t) = -\frac{i}{\hbar}\sum_r \varphi^{(0)}_r \int_0^t d\tau \int d\vec{\xi}\,\varphi^{(0)\dagger}_r(\vec{\xi}, \tau)\,H^{(1)}(\vec{\xi}, \tau)\,\varphi^{(0)}_q(\vec{\xi}, \tau).$$

After some calculations one finds, using (9) and (16)

$$\left.\begin{aligned} \vec{M}^{(2)} &= f^2 \sum'_{q,r} N_q \sum_{k,j} \{u^\dagger_q\,\vec{\alpha}\,u_r\} \left\{ u^\dagger_r\,\vec{\alpha}\left[\frac{\vec{Q}(j,\vec{k})}{E_r - E_q - \hbar\nu} + \frac{\vec{Q}(j,-\vec{k})}{E_r - E_q + \hbar\nu}\right] u_q \right\} e^{-i\nu t}, \\ \vec{Q}(j,\vec{k}) &= \frac{\varkappa^2 + k^2(1-\delta_{j,0})}{\varkappa^2 + k^2}\, U^+(j,\vec{k})\,\vec{\varepsilon}_{jk}\,e^{i\vec{k}\vec{x}} \end{aligned}\right\} \quad (41)$$

where some terms containing $U^-(j, k)$ which, on account of the choice of the sequence of factors in (28) do not contribute and some that lead to terms in (28) with zero time average have been omitted. $\vec{k}$ is given by (38). The prime affixed to the summation $\sum'_{q,r}$ indicates that terms which would make zero one of the energy denominators in (41) have to be excluded from the summation. It is obvious that the results will be the same on one particle and hole theory. From (41) and (28) we infer that

$$\left.\begin{aligned} W^{(1)}_{\text{fluct}} = \frac{f^2\hbar^2}{2}\, S \sum'_{q=1,\ldots,4} \frac{1}{\text{inv}} \Bigg[&\sum_{j=1,2} \{u^\dagger_{q_0}\,\vec{\alpha}\,\vec{\varepsilon}_{jk}\,u_q\}\{u^\dagger_q\,\vec{\alpha}\,\vec{\varepsilon}_{jk}\,u_{q_0}\} + \\ &+ \frac{\varkappa^2}{\varkappa^2 + k^2}\{u^\dagger_{q_0}\,\vec{\alpha}\,\vec{k}^0\,u_q\}\{u^\dagger_q\,\vec{\alpha}\,\vec{k}^0\,u_{q_0}\}\Bigg] \cdot \frac{E - E_0}{\hbar\nu} \end{aligned}\right\} . \quad (42)$$

where

$$\text{inv} = \mu^2 + |\vec{p} - \vec{p}_0|^2 - (E - E_0)^2. \quad . \quad . \quad . \quad . \quad . \quad (42a)$$

is the invariant denominator we already met in (33b) and (40a).

Furthermore, it is easily seen that, using the same notations as in (37)—(39)

$$W^{(1)}_{\text{fluct}} = \tfrac{1}{2} S \sum_q{}^{\nu} \sum_j \left[\frac{(q_0, 0 \mid W_j \mid q, \vec{k})\,(q, \vec{k} \mid W_j \mid q_0, 0)}{E_0 - E - \hbar\nu} + \right.$$
$$\left. + \frac{(q_0, -\vec{k} \mid W_j \mid q, 0)\,(q, 0 \mid W_j \mid q_0, -\vec{k})}{E_0 - E + \hbar\nu} \right] . \quad (43)$$

We now turn to the discussion of (42) for $p_0 = 0$. If (34a) is satisfied, the p-value (34) should again be excluded from the integration in the way indicated in § 4b. The first term in the numerator of (42) corresponds with the transverse, the second with the longitudinal fluctuation energy. The results are

$$W^{(1)}_{\text{fluct, trans.}} = \frac{f^2}{16\pi^3 M\hbar} \int \frac{d\vec{p}}{X} \sqrt{p^2 + \mu^2} = \frac{f^2}{4\pi^2 M\hbar} \left[\int p\,dp - \left(\frac{\mu^2}{2} - \frac{\mu^4}{4M^2} \right) \int \frac{dp}{p} \right] + \text{f.t.}$$
$$W^{(1)}_{\text{fluct, long.}} = \frac{f^2}{32\pi^3 M\hbar} \int \frac{d\vec{p}}{X} \cdot \frac{\mu^2}{\sqrt{p^2 + \mu^2}} = \frac{f^2 \mu^2}{8\pi^2 M\hbar} \int \frac{dp}{p} + f.t. \quad (44)$$

However, the question has to be considered whether it is justified to identify the expressions obtained by means of (41)—(43) with the fluctuation energy due to the electromagnetic field if $\varkappa = 0$. As is well known, the quantization of this field cannot be performed on the same lines as for the Proca-field essentially because the relation (12), which for $\varkappa \neq 0$ is a consequence of the field equations (11c, d), should here be considered as an accessory condition in the sense that its left member, operating on the occupation number functional of the electromagnetic field should yield zero in all cases actually realized in nature. The different footing on which the Maxwell- and Proca-quantization are based is clearly expressed by the fact that, whereas the matrix elements of the transverse vector potential smoothly go over into those of the electromagnetic vector potential if $\varkappa = 0$, those of the longitudinal part become infinite for $\varkappa = 0$, cf. (16). Thus it is evident that the transverse fluctuation energy for $\varkappa \neq 0$ yields the electromagnetic spin energy by simply putting $\varkappa = 0$. But, notwithstanding the singular behaviour of $U^{\pm}(o, \vec{k})$, the same transition may be performed for the longitudinal term, as the operator of longitudinal dynamic interaction stands proportional to $\varkappa^2$, cf. (27), which makes its matrix elements $\sim \varkappa$, as we have seen in (39). Thus we may state more generally:

All final results concerning particle interaction or dispersion, in which neutral vector mesons are involved, the interaction of which with the particles is described by a four vector source in the meson field equations, continuously go over into the corresponding results for the electromagnetic field by putting $\varkappa = 0$, if we start from the suitably transformed Hamiltonian of the Proca-field.

Thus the electromagnetic fluctuation energy is obtained from (44) by putting $\varkappa = 0$. The longitudinal part then vanishes and the total result is *)

$$W^{(1)}_{\text{fluct}} = \frac{e^2}{16\,\pi^3\,M\,\hbar}\int\frac{\vec{dp}}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (44a)$$

Collecting (30a), (31a), (34), (35) and (44) we obtain for the divergent part of the total first order self-energy

$$\frac{f^2}{4\,\pi^2\,M\,\hbar}\left[\int p\,dp + M\left(1+\frac{\mu^2}{2\,M^2}\right)\int dp + \frac{\mu^4}{4\,M^2}\int\frac{dp}{p}\right], \text{(one-p. th.)}$$

$$\frac{3\,f^2\,M}{8\,\pi^2\,\hbar}\int\frac{dp}{p}, \text{(hole th.)}$$

the latter being independent of $\varkappa$, which means that the divergent term of the Proca- and the electromagnetic self-energy are identical.

Finally, it follows from (37) and (43) that the total dynamic self-energy may be written as

$$S\sum_q{}'\sum_j \frac{(q_0,0\mid W_j\mid q,\vec{k})\,(q,\vec{k}\mid W_j\mid q_0,0)}{E_0-E-\hbar\nu}, \text{(one p.-th.)}$$

$$S\sum_j\left[\sum_{q=1,2}\frac{(q_0,0\mid W_j\mid q,\vec{k})\,(q,\vec{k}\mid W_j\mid q_0,0)}{E_0-E-\hbar\nu} - \right.$$

$$\left. - \sum_{q=3,4}{}'\frac{(q_0,-\vec{k}\mid W_j\mid q,0)\,(q,0\mid W_j\mid q_0,-\vec{k})}{E-E_0-\hbar\nu}\right], \text{(hole th.)}$$

a result which we would have obtained directly, had we treated the problem on hand from the start by means of the general formulae of second order perturbation theory. With regard to the case $\varkappa = 0$ the same

*) WEISSKOPF's expression for this quantity, cf. loc. cit. [4]), p. 81, is too large by a factor 2. The expression for W_{sp} should be, in the notation of loc. cit. (e is expressed in ordinary units)

$$W_{sp} = -\frac{e^2}{2\,\pi\,m\,c\,\hbar}\lim_{(P=\infty)}\left[P\,P_0 - m^2\,c^2\,lg\,\frac{P+P_0}{m\,c}\right],$$

cf. loc. cit. eq. (20) and (23). Therefore eq. (26) loc. cit. is again correct. The result (44a) is also obtained by HEITLER, *Quantum Theory of Radiation*, Oxford Clarendon Press, 1936, p. 183 eq. (23).

considerations apply to the total dynamic self-energy, calculated by perturbation theory, as those just made in our above treatment of the fluctuation energy.

§ 5. *Calculation of f_v-self-energy without canonical transformation.*

From the developments of the previous section, it appears that the separation of the self-energy into static and dynamic terms presents no advantages. It e.g. separates the static from the spin energy, but we have seen that it is more natural to take these terms together. We shall now show that the calculation becomes much simpler by not performing a canonical transformation at all. Thus we have now to consider the expectancy value of

$$H = \tfrac{1}{2}\int (N V - \vec{M}\,\vec{U})\, dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

From the point of view of quantization, it is more convenient to express H in terms of the canonical field variables. As has already been remarked, V is a derived variable, defined by (20). Therefore we write

$$H = \tfrac{1}{2}\int (\varkappa^{-2} N \operatorname{div} \vec{F} - \vec{M}\,\vec{U})\, dv + \frac{\varkappa^{-2}}{2}\int N^2\, dv. \quad . \quad . \quad . \quad (46)$$

Developing all quantities similarly to (1*b*) we get

$$\left.\begin{aligned} H = \tfrac{1}{2}\int (\varkappa^{-2} N^{(1)} \operatorname{div} \vec{F}^{(1)} - \vec{M}^{(1)}\,\vec{U}^{(1)})\, dv + \frac{\varkappa^{-2}}{2}\int N^{(1)2}\, dv \\ + \tfrac{1}{2}\int (\varkappa^{-2} N^{(2)} \operatorname{div} \vec{F}^{(0)} - \vec{M}^{(2)}\,\vec{U}^{(0)})\, dv. \end{aligned}\right\} \quad . \quad (47)$$

The terms of the first line will yield $W^{(1)}_{\text{inv}}$, those of the second $W^{(1)}_{\text{fluct}}$. The last term of the first line gives, by means of first order perturbation theory

$$\frac{f^2}{2\varkappa^2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}\Big) \{u^\dagger_{q_0} u_q\}\{u^\dagger_q u_{q_0}\}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

To obtain the other terms of the first line, which will be called $W^{(1)'}_{\text{inv}}$, a formula similar to (37) may be used. Putting

$$H' = \int (\varkappa^{-2} N \operatorname{div} \vec{F} - \vec{M}\,\vec{U})\, dv, \quad . \quad . \quad . \quad . \quad . \quad (45a)$$

we thus get, using (38)

$$\left.\begin{aligned} W^{(1)'}_{\text{inv}} = \tfrac{1}{2} S\Big(\sum_{q=1,2} \pm \sum'_{q=3,4}\Big) \sum_j \Bigg[\frac{(q_0, 0 \,|\, H'_j \,|\, q, \vec{k})\,(q, \vec{k} \,|\, H'_j \,|\, q_0, 0)}{E_0 - E - \hbar\nu} + \\ + \frac{(q, 0 \,|\, H'_j \,|\, q_0, -\vec{k})\,(q_0, -\vec{k} \,|\, H'_j \,|\, q, 0)}{E - E_0 - \hbar\nu} \Bigg] \end{aligned}\right\} \quad (49)$$

where (cf. (15a))

$$(q_0, 0\,|\,H'_0|\,q, \vec{k}) = \frac{f}{\varkappa}\sqrt{\frac{\hbar}{2\nu}}\left[k\{u^\dagger_{q_0} u_q\} - \nu\{u^\dagger_{q_0} \vec{\vec{a}}\,\vec{\vec{k}}^0 u_q\}\right],$$

$$i = 1, 2\colon (q_0, 0\,|\,H'_j|\,q, \vec{k}) = f\sqrt{\frac{\hbar}{2\nu}}\,\{u^\dagger_{q_0} \vec{\vec{a}}\,\varepsilon_{jk} u_q\};\ \text{etc.}$$

Therefore, remembering that H' is Hermitian, the expression between square brackets in (49) may be written, using (42a), as

$$\left.\begin{aligned}\frac{f^2\hbar}{2\varkappa^2\nu}\Bigg[&\frac{\{u^\dagger_{q_0}(k - \nu\vec{\vec{a}}\,\vec{\vec{k}}^0) u_q\}\{u^\dagger_q(k - \nu\vec{\vec{a}}\,\vec{\vec{k}}^0) u_{q_0}\}}{E_0 - E - \hbar\nu} + \\ &+ \frac{\{u^\dagger_{q_0}(k + \nu\vec{\vec{a}}\,\vec{\vec{k}}^0) u_q\}\{u^\dagger_q(k + \nu\vec{\vec{a}}\,\vec{\vec{k}}^0) u_{q_0}\}}{E - E_0 - \hbar\nu}\Bigg] \\ &- \frac{f^2\hbar^2}{2}\sum_{j=1,2}\frac{\{u^\dagger_{q_0}\vec{\vec{a}}\,\varepsilon_{jk} u_q\}\{u^\dagger_q \vec{\vec{a}}\,\varepsilon_{jk} u_{q_0}\}}{\text{inv}}\end{aligned}\right\} \quad (50)$$

The difference in sign of $\nu\vec{\vec{a}}\,\vec{\vec{k}}^0$ in the first and second term of (50) should be noted. Using (40) and (40b), (49) can be written as

$$W^{(1)'}_{\text{inv}} = W^{(1)}_{\text{inv}} - \frac{f^2}{2\varkappa^2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}\Big)\{u^\dagger_{q_0} u_q\}\{u^\dagger_q u_{q_0}\}.$$

The latter term is cancelled by (48), and so the first line of (47) indeed gives $W^{(1)}_{\text{inv}}$. This result can also be obtained by starting from (45), developing as in (47) and integrating the equations for V and U in a way similar to (32).

Similar to (43), the terms of the second line of (47) can be written as

$$\left.\begin{aligned}W^{(1)}_{\text{fluct}} = \tfrac{1}{2} S \sum_q \sum_j \Bigg[&\frac{(q_0, 0|H'_j|q, \vec{k})(q, \vec{k}|H'_j|q_0, 0)}{E_0 - E - \hbar\nu} + \\ &+ \frac{(q_0, -\vec{k}|H'_j|q, 0)(q, 0|H'_j|q_0, -\vec{k})}{E_0 - E + \hbar\nu}\Bigg]\end{aligned}\right\}. \quad (51)$$

After some calculations, one again finds (44) on putting $p_0 = 0$. However, one does *not* get (42), in which p_0 has not yet been put equal to zero, but this only means that, in computing the fluctuation energy for a moving particle, starting either from (42) or (51), the contributions of the various regions of momentum space should be taken together in a different way, so as to yield the uniquely determined relation (1a); cf. further § 1 *).

*) The same phenomenon we have also found for other fields, cf. e.g. the Appendix, notes 3 and 4.

Thus, as it should be, the present method yields the same result as that described in the previous section. From the considerations of § 4 we know already that the electromagnetic field self-energy is obtained from that of the Proca-field by putting $\varkappa = 0$. However, contrary to the calculation of § 4, it is by no means justified (although we know it indirectly to be true) to say, that the results of calculations which do not involve a canonical transformation yield the corresponding electromagnetic ones if $\varkappa = 0$: whereas it was seen in § 4 that the matrix elements of the interaction operator involving longitudinal "free" mesons are proportional to $\varkappa$, this is not the case in this section, where the distinction between "free" and "bound" mesons is not made, but where the whole particle-field interaction is treated from the start as a perturbation. Thus it is necessary to consider separately the electro-magnetic self-energy as computed without a canonical transformation. This is done in the Appendix, note 3, by quantizing the field with the help of FERMI's method [10]).

From (47), (49) and (51) the following expressions can be obtained for $W^{(1)}$, noting that H' is hermitian

$$\text{Hole th.:}\quad W^{(1)} = \frac{\varkappa^{-2}}{2}\overline{\int N^{(1)2}\,dv} + \left.\begin{aligned} &+ S\sum_j\left[\sum_{q=1,2}\frac{(q_0,0|H'_j|q,\vec{k})(q,\vec{k}|H'_j|q_0,0)}{E_0-E-\hbar\nu} - \sum_{q=3,4}'\frac{(q_0,-\vec{k}|H'_j|q,0)(q,0|H'_j|q_0,-\vec{k})}{E-E_0-\hbar\nu}\right]\\ &\end{aligned}\right\}. \tag{52}$$

$$\text{One-p. th.:}\quad W^{(1)} = \frac{\varkappa^{-2}}{2}\overline{\int N^{(1)2}\,dv} + \left.\begin{aligned} &+ S\sum_j\sum_{q=1,\ldots,4}'\frac{(q_0,0|H'_j|q,\vec{k})(q,\vec{k}|H'_j|q_0,0)}{E_0-E-\hbar\nu}\\ &\end{aligned}\right\}. \tag{53}$$

We have now obtained sufficiently general formulae to be able to compute straightforwardly all self-energies due to fields of all types. To this purpose one can either use (52) and (53), (the N^2-term having to be replaced by corresponding other terms, of course), or retain the division in W_{inv} and W_{fluct}. We shall here follow the latter procedure and thus will use (49) and (51).

With the help of (52) and (53), the present results can be compared with those of KEMMER's [1]). This author divides the interaction operator, (referring to charged fields, but this is immaterial to the issue), into two parts, H_1 and H_2. In the case of f_v-interaction, f_v corresponding with KEMMER's g_b, the term $\varkappa^{-2}\int N^2\,dv$ corresponds with H_2, the other part of (45), viz. H', cf. (45a), with H_1. It may be remarked that, from the point of view of the self-energy, the separate consideration of H_1 and H_2 is disadvantageous, as the integrands of H_1 and H_2 are no scalars, so that (1a) does not apply to the expectancy value of H_1 and H_2 but only to that of $H_1 + H_2$. It may be repeated that the separation into H_1 and H_2 must be considered as no more than a convenient method for expressing the interactions in terms of canonical variables; from the correspondence point of view it

is the operator $H_1 + H_2$ which should be taken as representative for the interaction *).

However, even taking together the corresponding contributions of loc. cit. [2]) eq. (72) and (77), there still are many discrepancies with the present results. The origin of the disagreement is to be found in the terms of (52) in square brackets which differ from loc. cit. eq. (71) in that the first and second one contain $+\vec{k}$ and $-\vec{k}$, respectively, while KEMMER has taken $\vec{k}$ in both cases. The importance of the difference in relative sign has already been pointed out in connection with (50). From a closer examination of (52), it follows that the difference in sign affects those (and only those) terms which are "cross terms", i.e. contain products of two matrix elements which involve different source functions, in the cited case the terms containing both N and $\vec{M}_{11}$. Now all such cross terms have erroneously been omitted in the cited paper. There are only two cases in which these do not occur, viz. for f_s- and f_{ps}-interactions. In these cases our results (cf. § 6) agree with those of KEMMER's, apart from a factor two by which his expressions are too large.

§ 6. Self-energies due to other fields.

We will now briefly survey all f- and g-self-energies, treating simultaneously the s- with their dual ps-cases and likewise v with pv. First, expressions like (37) and (42) will be derived for $W^{(1)}_{\text{inv}}$ and $W^{(1)}_{\text{fluct}}$; their representations, for $p_0 = 0$, as integrals over momentum space will then be collected in table 1. Where no confusion is possible, we shall write $\{X, Y\}$ instead of $\{u^\dagger_{q_0} X u_q\}\{u^\dagger_q Y u_{q_0}\}$. Further, the following notations are used

$$W^{(1)}_{\text{inv}} = \frac{\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}{}'\Big) m_{\text{inv}} \qquad W^{(1)'}_{\text{inv}} = \frac{\hbar^2}{2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}{}'\Big) m'_{\text{inv}}$$

$$W^{(1)}_{\text{fluct}} = \frac{\hbar^2}{2} S \sum_q{}' m_{\text{fluct}}.$$

1. f_s- and f_{ps}-self-energy.

The field equations are given by (17a) and (17b), while, instead of (17c),

$$\operatorname{div} \vec{\Gamma} + \dot{A} = \varkappa^2 \Omega - \psi^\dagger R \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

with

$$\text{s: } R = f_s \varrho_3, \quad \text{ps: } R = f_{ps} \varrho_2.$$

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*) FRÖHLICH, HEITLER and KEMMER [9]) take H^b_1 to describe the f_v-interaction, cf. loc. cit. eq. (30b), whereas MØLLER and ROSENFELD [2]) take $H^b_1 + H^b_2$, cf. loc. cit. eq. (7). Apart from the invariance argument, quoted above, which is of no interest if only *static* interactions are considered, it should be noted, that also from the point of view of nuclear interaction the second choice seems preferable, as this, contrary to the first one, (cf. loc. cit. [9]) p. 164) does not lead to terms in the static interaction potential of the δ-type, the occurrence of which is incompatible with a finite binding energy of the deuteron [10]).

The Hamiltonian is given by (5) with (18) and

$$H = -\int \psi^\dagger R \psi \,.\, \Omega \, dv.$$

Thus, $W^{(1)}$ is defined as the expectancy value $\sim f^2$ of $-\frac{1}{2}\int \psi^\dagger R \psi \,.\, \Omega \, dv$. Using (19), the matrix elements of H are found to be

$$(q_0, 0 \,|\, H \,|\, q, \vec{k}) = \sqrt{\frac{\hbar}{2\nu}} \{u^\dagger_{q_0} R \, u_q\}, \text{ etc.}$$

Hence, by means of (49) and (51), (there are no terms like the last of (46))

$$m_{\text{inv}} = -\frac{\{R, R\}}{\text{inv}} \qquad m_{\text{fluct}} = \frac{\{R, R\}}{\text{inv}} \cdot \frac{E - E_0}{\hbar \nu}. \quad . \; . \; (55)$$

In view of the important rôle the f_s-interaction will play in subsequent chapters, its treatment by means of a canonical transformation will be discussed in the Appendix (note 4). It should be noted that the f_{ps}-self-energy is of purely non-static origin.

2. g_s- and g_{ps}-self-energy.

The field equations are, besides (17c),

$$\vec{\Gamma} = \text{grad}\, \Omega + \psi^\dagger \vec{P} \psi \quad , \quad \Lambda = -\dot{\Omega} + \psi^\dagger Q \psi, \; . \; . \; . \; (56)$$

with

$$\text{s}: \; \vec{P} = \frac{g_s}{\varkappa_s} \varrho_1 \vec{\sigma} \quad , \quad Q = \frac{g_s}{\varkappa_s},$$

$$\text{ps}: \; \vec{P} = \frac{g_{ps}}{\varkappa_{ps}} \vec{\sigma} \quad , \quad Q = \frac{g_{ps}}{\varkappa_{ps}} \varrho_1.$$

The interaction term in the Hamiltonian (5) is $-\int \psi^\dagger Q \psi \,.\, \Lambda \, dv$. The operator which must be considered to obtain the self-energy is

$$H = \tfrac{1}{2}\int [\psi^\dagger \vec{P} \psi \,.\, \vec{\Gamma} - \psi^\dagger Q \psi \,.\, \Lambda] \, dv =$$

$$= \tfrac{1}{2}\int [\psi^\dagger \vec{P} \psi \,.\, \text{grad}\, \Omega - \psi^\dagger Q \psi \,.\, \Lambda] \, dv + \tfrac{1}{2}\int \psi^\dagger \vec{P} \psi \,.\, \psi^\dagger \vec{P} \psi \, dv,$$

where, similar to (46), H is expressed in the canonical variables Ω and Λ. The last term of H yields

$$\tfrac{1}{2} S \Big(\sum_{q=1,2} \pm \sum_{q=3,4} \Big) \{\vec{P}, \vec{P}\}. \; . \; . \; . \; . \; . \; . \; . \; (57)$$

Denoting the other part of H by H', we have, with the help of (38)

$$(q_0, 0 \mid H' \mid q, \vec{k}) = i \sqrt{\frac{\hbar}{2\nu}} \{u^{\dagger}_{q_0} (\vec{P}\vec{k} - Q\nu)\, u_q\}, \text{ etc.}$$

thus

$$m'_{\text{inv}} = -\frac{\{\vec{P}\vec{k}, \vec{P}\vec{k}\} - 2\{\vec{P}\vec{k}, Q\varepsilon\} + \nu^2 \{Q, Q\}}{\text{inv}}$$

with

$$\varepsilon = \frac{E_0 - E}{\hbar}.$$

Therefore, by adding (57):

$$m_{\text{inv}} = -\frac{\{\vec{P}\vec{k}, \vec{P}\vec{k}\} - 2\{\vec{P}\vec{k}, Q\varepsilon\} + \varepsilon^2 \{Q, Q\}}{\text{inv}} + \{\vec{P}, \vec{P}\} - \{Q, Q\}.$$

Its invariance is evident. In the same way we find from (51)

$$2\nu\, m_{\text{fluct}} = \frac{\{\vec{P}\vec{k} - Q\nu, \vec{P}\vec{k} - Q\nu\}}{E_0 - E - \hbar\nu} + \frac{\{\vec{P}\vec{k} + Q\nu, \vec{P}\vec{k} + Q\nu\}}{E_0 - E + \hbar\nu}.$$

3. The $f_s g_s$- and $f_{ps} g_{ps}$-self-energy.

Considering the case that the f- and g-sources are simultaneously present, i.e. that the field equations are given by (54) and (56), there occur cross-terms proportional to fg *). It is easily seen that for these

$$m_{\text{inv}} = \frac{-i\{R, \vec{P}\vec{k} - Q\varepsilon\} + \text{conj.}}{\text{inv}}$$

while

$$2\nu\, m_{\text{fluct}} = \frac{i\{R, \vec{P}\vec{k} - Q\nu\} + \text{conj.}}{E_0 - E - \hbar\nu} + \frac{i\{R, \vec{P}\vec{k} + Q\nu\} + \text{conj.}}{E_0 - E + \hbar\nu}.$$

4. The "direct" g_s- and g_{ps}-self-energy.

It is well known [1]) that the addition to the Hamiltonian of a term

$$\frac{n}{2} \int S_{..}(g)\, S^{..}(g)\, d\nu \quad \ldots\ldots\ldots \quad (58)$$

where the integrand is the scalar product of (the tensor components of) g-source functions $S_{..}(g)$ and n a numerical coefficient, has no effect on

*) In loc. cit. [1]) terms of this type have not been taken into account.

the equations of the meson field *). It leads to a direct interaction, (i.e. not brought through the intermediary of fields), between particles, the spatial dependence of which is described by the highly singular δ-function. We will now compute the self-energy to which such terms give rise and which will be called direct self-energy. This quantity obviously satisfies (1a). In the g_s and g_{ps}-case we have to consider the operator

$$\frac{n}{2}\int dv\,(\psi^\dagger \vec{P}\psi\,.\,\psi^\dagger \vec{P}\psi - \psi^\dagger Q\psi\,.\,\psi^\dagger Q\psi).$$

By first order perturbation theory its expectancy value is found to be

$$\frac{n}{2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}\Big)[\{\vec{P},\vec{P}\} - \{Q,Q\}] \quad . \quad . \quad . \quad . \quad (58a)$$

5. The f_v- and f_{pv}-self-energy.

The field equations are (11a, b) and (20), (21). In the latter two we write $\psi^\dagger \vec{M}\psi$, $\psi^\dagger N\psi$ instead of $\vec{M}$, N. We have

$$\text{v: } \vec{M} = f_v \varrho_1 \vec{\sigma}, \quad N = f_v; \quad \text{pv: } \vec{M} = f_{pv}\vec{\sigma}, \quad N = f_{pv}\varrho_1.$$

The f_v-case has extensively been dealt with in the foregoing. The corresponding formulae for the pseudovector theory are obtained by replacing $\varrho_1\vec{\sigma}$ by $\vec{\sigma}$ and "1" by ϱ_1.

6. The g_v- and g_{pv}-self-energy.

The field equations are (11a, b), while (11c, d) are replaced by

$$\left.\begin{aligned} \vec{F} &= -\dot{\vec{U}} - \operatorname{grad} V + \psi^\dagger \vec{T}\psi, \\ G &= \operatorname{rot}\vec{U} + \psi^\dagger \vec{S}\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\text{v: } \vec{T} = -\frac{g_v}{\varkappa_v}\varrho_2\vec{\sigma} \qquad \vec{S} = \frac{g_v}{\varkappa_v}\varrho_3\vec{\sigma}$$

$$\text{pv: } \vec{T} = \frac{g_{pv}}{\varkappa_{pv}}\varrho_3\vec{\sigma} \qquad \vec{S} = \frac{g_{pv}}{\varkappa_{pv}}\varrho_2\vec{\sigma}.$$

---

*) Such terms should not be confused with the terms H_2 of KEMMER's Hamiltonian, cf. loc. cit.[1]) eq. (50), as the latter arise from the separation of the invariant operator of nucleon-field interaction into a part depending on the *canonical* field variables (H_1) and a part depending on the source functions only (H_2).

One may also introduce terms of direct f-interaction in the Hamiltonian. As for later purposes only the direct g-self-energies are needed, however, direct f-self-energies have not been considered here.

The operator which yields the self-energy is

$$H = \tfrac{1}{2}\int (\psi^\dagger \vec{S}\,\psi \,.\, \vec{G} - \psi^\dagger \vec{T}\,\psi \,.\, \vec{F})\, dv =$$

$$= \tfrac{1}{2}\int (\psi^\dagger \vec{S}\,\psi \,.\, \mathrm{rot}\, \vec{U} - \psi^\dagger \vec{T}\,\psi \,.\, \vec{F})\, dv + \tfrac{1}{2}\int \psi^\dagger \vec{S}\,\psi \,.\, \psi^\dagger \vec{S}\,\psi\, dv,$$

G being a derived variable. The last term gives

$$\tfrac{1}{2} S\Big(\sum_{q=1,2} \pm \sum_{q=3,4}\Big)\{\vec{S}, \vec{S}\}.$$

Calling H'_j the part of H, depending on the canonical variables U and F, which corresponds with transitions involving mesons with state of polarization j, we have *)

$$(q_0, 0 \,|\, H_0' \,|\, q, \vec{k}) = -i\varkappa \sqrt{\frac{\hbar}{2\nu}}\, \{u^\dagger_{q_0} \vec{T}\, \vec{k}^0\, u_q\},$$

$$(q_0, 0 \,|\, H_j' \,|\, q, \vec{k}) = i \sqrt{\frac{\hbar}{2\nu}}\, \{u^\dagger_{q_0} [\vec{S}(\vec{k} \wedge \vec{\varepsilon}_{jk}) - \nu\, \vec{T}\, \vec{\varepsilon}_{jk}]\, u_q\}; \quad j = 1, 2.$$

Hence, with the help of simple vector calculus,

$$m'_{\mathrm{inv}} = -\frac{1}{\mathrm{inv}} [k^2 \{\vec{S}, \vec{S}\} - \{\vec{k}\vec{S}, \vec{k}\vec{S}\} -$$

$$- \{\vec{k}\vec{T}, \vec{k}\vec{T}\} + \nu^2 \{\vec{T}, \vec{T}\} + 2\varepsilon \vec{k} \{\vec{S} \wedge \vec{T}\}],$$

Therefore

$$m_{\mathrm{inv}} = -\frac{1}{\mathrm{inv}} [k^2 \{\vec{S}, \vec{S}\} - \{\vec{k}\vec{S}, \vec{k}, \vec{S}\} - \{\vec{k}\vec{T}, \vec{k}\vec{T}\} +$$

$$+ \varepsilon^2 \{\vec{T}, \vec{T}\} + 2\varepsilon \vec{k} \{\vec{S} \wedge \vec{T}\}] + \{\vec{S}, \vec{S}\} - \{\vec{T}, \vec{T}\}.$$

That the expression between square brackets is indeed a scalar can be seen by writing $S_{ik} = (\vec{S}, \vec{T})$, where S_{ik} is a (pseudo)tensor, and

$$\vec{k}, -\varepsilon = \Pi_i,$$

Π_i being a four vector. This expression can then be written as

$$S_i^{\cdot j} S^{ik}\, \Pi_j\, \Pi_k.$$

For the fluctuation energy we get

$$2\nu\, m_{\mathrm{fluct}} = \frac{k^2 \{\vec{S}, \vec{S}\} - \{\vec{k}\vec{S}, \vec{k}\vec{S}\} - \{\vec{k}\vec{T}, \vec{k}\vec{T}\} + \nu^2 \{\vec{T}, \vec{T}\} + 2\nu \vec{k} \{\vec{S} \wedge \vec{T}\}}{E_0 - E - \hbar\nu} +$$

$$+ \frac{\text{same with } -\nu \text{ instead of } \nu}{E_0 - E + \hbar\nu}.$$

*) $\vec{a} \wedge \vec{b}$ denotes the vector product of $\vec{a}$ and $\vec{b}$.

TABLE 1.

Type	$W^{(1)}$ inv		$W^{(1)}_{\text{fluct}}$
	One-part. theory	Hole theory	
f_s	$-f^2\left(1-\frac{\mu^2}{4M^2}\right)A$	$-\frac{1}{2}f^2C(p^2+2M^2-\frac{1}{2}\mu^2)$	$\frac{1}{2}f^2Dp^2$
g_s	$\left(\frac{g}{\varkappa}\right)^2B$	$-2\left(\frac{g}{\varkappa}\right)^2E$	0
$f_s\,g_s$	0	0	0
g_s; direct	$n\left(\frac{g}{\varkappa}\right)^2B$	$-2n\left(\frac{g}{\varkappa}\right)^2E$	—
f_v	$f^2\left(1+\frac{\mu^2}{2M^2}\right)A$	$-f^2C(p^2-M^2+\mu^2)$	$f^2D(p^2+\frac{3}{2}\mu^2)$
g_v	$\left(\frac{g}{\hbar\varkappa}\right)^2A(p^2+3\mu^2)$	$-\frac{1}{2}\left(\frac{g}{\hbar\varkappa}\right)^2C(10M^2p^2+\mu^2p^2+6M^2\mu^2)+3\left(\frac{g}{\varkappa}\right)^2E$	$\frac{1}{2}g^2D(p^2+3\mu^2)$
$f_v\,g_v$	$-3\frac{fg}{\hbar}A\varkappa$	$3\frac{fg}{\varkappa}.M\hbar C(2p^2+\mu^2)$	$-\frac{3}{2}\frac{fg}{M}.\hbar^2\mu^3D$
g_v; direct	0	$3n\left(\frac{g}{\varkappa}\right)^2E$	—
f_{pv}	$-3f^2\left(1-\frac{\mu^2}{6M^2}\right)A$	$-f^2C(p^2+3M^2-\mu^2)$	$f^2D(p^2-\frac{1}{2}\mu^2)$
g_{pv}	$\left(\frac{g}{\hbar\varkappa}\right)^2Ap^2$	$-\frac{1}{2}\left(\frac{g}{\hbar\varkappa}\right)^2(\mu^2-2M^2)Cp^2-3\left(\frac{g}{\varkappa}\right)^2E$	$\frac{1}{2}g^2Dp^2$
$f_{pv}\,g_{pv}$	0	0	0
g_{pv}; direct	0	$-3n\left(\frac{g}{\varkappa}\right)^2E$	—
f_{ps}	$f^2.\frac{\mu^2}{4M^2}A$	$-\frac{1}{2}f^2C\left(p^2+\frac{\mu^2}{2}\right)$	$\frac{1}{2}f^2D(p^2+\mu^2)$
g_{ps}	$g^2A+\left(\frac{g}{\varkappa}\right)^2B$	$-\left(\frac{g}{\varkappa}\right)^2M^2C(2p^2+\mu^2)+2\left(\frac{g}{\varkappa}\right)^2E$	$\frac{1}{2}g^2.\mu^2D^2$
$f_{ps}\,g_{ps}$	$-\frac{fg}{\hbar}A\varkappa$	$\frac{fg}{\varkappa}.M\hbar C(2p^2+\mu^2)$	$-\frac{1}{2}\frac{fg}{M}.\hbar^2\mu^3D$
g_{ps}; direct	$n\left(\frac{g}{\varkappa}\right)^2B$	$2n\left(\frac{g}{\varkappa}\right)^2E$	—

7. The $f_v g_v$- and $f_{pv} g_{pv}$-self-energy.

These are due to the cross terms in fg which are found if the field equations are (20), (21) and (59). We obtain

$$m_{\text{inv}} = -\frac{i\vec{k}\{\vec{M}\wedge\vec{S}\}-i\varepsilon\{\vec{M}\vec{T}\}+i\vec{k}\{N\vec{T}\}+\text{conj.}}{\text{inv}}$$

$$2\nu m_{\text{fluct}} = \frac{i\vec{k}\{\vec{M}\wedge\vec{S}\}-i\nu\{\vec{M}\vec{T}\}+i\vec{k}\{N\vec{T}\}+\text{conj}}{E_0-E-\hbar\nu} + \\ + \frac{\text{same with } -\nu \text{ instead of } \nu}{E_0-E+\hbar\nu}.$$

8. The direct g_v- and g_{pv}-self-energy.

Similarly to 4. we get

$$\frac{n}{2} S\left(\sum_{q=1,2} \pm \sum_{q=3,4}\right)[\{\vec{S},\vec{S}\}-\{\vec{T},\vec{T}\}]. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

In table 1 the expressions are given for $W^{(1)}_{\text{inv}}$ and $W^{(1)}_{\text{fluct}}$; the following abbrevations have been used, (for X see (35), (36))

$$\frac{1}{16\pi^3\hbar}\int\frac{d\vec{p}}{X}=A,\ \frac{1}{8\pi^3\hbar^3}\int d\vec{p}=B,\ \frac{1}{16\pi^3 M\hbar}\int\frac{d\vec{p}}{PX}=C,$$

$$\frac{1}{16\pi^3 M\hbar}\int\frac{d\vec{p}}{X\sqrt{p^2+\mu^2}}=D,\ \frac{M}{8\pi^3\hbar^3}\int\frac{d\vec{p}}{P}=E.$$

Where one of the integrals $A-E$ stands "multiplied" with a function of p, it is meant that this function should be taken under the integral. To simplify the formulae, we have, except in the first column, omitted the "indices" s, v, pv, ps.

Table 2 gives a survey of the kinds of divergences to which $W^{(1)}$ gives rise; cu = cubic, qu = quadratic, lin = linear, log = logarithmic divergence. In the case of the f_v-self-energy, the divergences indicated between brackets refer to the electromagnetic self-energy.

For later purposes we need the explicit expressions for the coefficients with which, on hole theory, the various divergences stand multiplied; we write

$$W^{(1)}=\frac{c_{\text{qu}}}{\pi^2 M\hbar}\int^{\infty} p\,dp+\frac{3M c_{\text{log}}}{8\pi^2\hbar}\int^{\infty}\frac{dp}{p}+\text{f.t.}$$

and introduce

$$\eta=\frac{M}{\mu}.$$

TABLE 2.

Kinds of divergence of $W^{(1)}$.

	$W^{(1)}_{\text{inv}}$		$W^{(1)}_{\text{fluct}}$		$W^{(1)}_{\text{inv}}$		$W^{(1)}_{\text{fluct}}$
Type	One-p. th.	Hole th.		Type	One-p. th.	Hole th.	
f_s	lin	qu, log	qu, log	f_{pv}	lin	qu, log	qu, log
g_s	cu	qu	—	g_{pv}	cu, lin	qu, log	qu, log
$f_s\, g_s$	—	—	—	$f_{pv}\, g_{pv}$	—	—	—
g_s; dir.	cu	qu	—	g_{pv}; dir.	—	qu	—
f_v	lin (—)	qu, log (id.)	qu, log	f_{ps}	lin	qu, log	qu, log
g_v	cu, lin	qu, log	qu, log	g_{ps}	cu, lin	qu, log	log
$f_v\, g_v$	lin	qu, log	log	$f_{ps}\, g_{ps}$	lin	qu, log	log
g_v; dir.	—	qu	—	g_{ps}; dir.	cu	qu	—

c_{qu} and c_{log} are given in table 3. In the second and third column the indices s, v, pv, ps have again been omitted.

TABLE 3.

Coefficients of divergences of $W^{(1)}$ on hole theory.

Type	c_{qu}	c_{log}
f_s	—	$-\frac{1}{2} f^2$
g_s	$-\ g^2 \eta^2$	$\frac{4}{3} g^2 \eta^2$
$f_s\, g_s$	—	—
g_s; direct	$-\ n g^2 \eta^2$	$\frac{4}{3} n g^2 \eta^2$
f_v	—	f^2, (e^2)
g_v	$\frac{1}{4} g^2 \eta^2$	$-\frac{1}{3} g^2 \eta^2 (1 - \frac{9}{2} \eta^{-2})$
$f_v\, g_v$	$\frac{3}{2} fg\, \eta$	$-2\, fg\, \eta\, (1 + \eta^{-2})$
g_v; direct	$\frac{3}{2} n g^2 \eta^2$	$-2\, n g^2 \eta^2$
f_{pv}	—	$-\frac{5}{3} f^2$
g_{pv}	$-\frac{5}{4} g^2 \eta^2$	$\frac{1}{3} g^2 \eta^2 (5 - \frac{3}{2} \eta^{-2})$
$f_{pv}\, g_{pv}$	—	—
g_{pv}; direct	$-\frac{3}{2} n g^2 \eta^2$	$2\, n g^2 \eta^2$
f_{ps}	—	$\frac{1}{6} f^2$
g_{ps}	$\frac{1}{2} g^2 \eta^2$	$-\frac{2}{3} g^2 \eta^2 (1 - \eta^{-2})$
$f_{ps}\, g_{ps}$	$\frac{1}{2} fg\, \eta$	$-\frac{2}{3} fg\, \eta\, (1 + \eta^{-2})$
g_{ps}; direct	$n g^2 \eta^2$	$-\frac{4}{3} n g^2 \eta^2$

§ 7. *On the higher order self-energies.*

It has been shown by WEISSKOPF [4]) that $W^{(n)}_{\text{elm}}$, the electromagnetic self-energy of arbitrary order, diverges, on hole theory, at most logarithmically. In this section we shall show, by similar reasonings, that to a certain extent, general conclusions as to the divergence properties can be established for any field, again starting from hole theory. As this case will mainly be dealt with in the following, we shall from now on understand by $W^{(n)}$ the *n-th* order self-energy on hole theory. We now first turn to a short recapitulation of WEISSKOPF's argument.

a) The electromagnetic case.

The main point is that

$$W^{(n)}_{\text{elm}} \to 0 \text{ if } M \to 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where M here denotes the rest mass of the electron.

To show this, WEISSKOPF considers the identity

$$\left.\begin{aligned} W^{(n)}(\text{vac}+e_0^+)+W^{(n)}(\text{vac}-e_0^-)-2\,W^{(n)}(\text{vac}) = \\ \{T^{(n)}(e_0^+\to)-T^{(n)}(e_0^-\to)\}-\{T^{(n)}(\to e_0^+)-T^{(n)}(\to e_0^-)\}\end{aligned}\right\} \quad (61)$$

The first term on the left denotes the energy $\sim (e^2/\hbar)^n$ of the system of completely filled vacuum states with an additional electron (of given spin direction) in the lowest positive energy level e_0^+: in the same sense the second term indicates the energy of the vacuum with one electron lacking in the highest negative energy level e_0^-, while $W^{(n)}(\text{vac})$ is the energy $\sim (e^2/\hbar)^n$ of the vacuum distribution itself. Thus the left hand side of (61) is equal to the sum of the self-energy of positive and negative electron at rest $=$ twice the self energy $W^{(n)}_{\text{elm}}$ of the electron on account of the symmetry of the whole theory with respect to the sign of the charge. $T^{(n)}(e_0^\pm \to)$ denotes the sum of the contributions due to all transitions involving $2n-1$ intermediate states in which the electron in the state $e_0^\pm$ takes part, while similarly $T^{(n)}(\to e_0^\pm)$ is the sum of the contributions due to all transitions in which an electron jumps into the (unoccupied) state $e_0^\pm$. In order to show that $W^{(n)}_{\text{elm}} \to 0$ if $M \to 0$ it is, therefore, sufficient to prove that the right hand side of (61) tends to zero under this condition. This amounts to proving that [4])

$$P_+ = P_- \quad \text{if} \quad M=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

where

$$P_\pm = \text{Trace}\, H_1 \left(1+\frac{\vec{\alpha}\vec{p}_1+\varrho_3 M}{E_1}\right) H_2 \left(1+\frac{\vec{\alpha}\vec{p}_2+\varrho_3 M}{E_2}\right) \ldots H_{2n}(1\pm\varrho_3), \quad (63)$$

$$E_l = \pm\sqrt{p_l^2+M^2}.$$

H_i is the operator causing the transition from the $(i-1)$-*th* to the *i-th* state; (the final = initial state is indicated by $2n$). That (62) is true then follows from well known properties of the Dirac matrices.

b) Generalization of (60) to other fields

We next show that

$$W^{(n)} \to 0 \quad \text{if} \quad M \to 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

for any *f*- or *g*-interaction. For this purpose it suffices to prove that (62) here holds too, and this is easily seen to be the case. For all H_i depend in the same way on the Dirac matrices ϱ_i and σ_i and therefore the $\pm \varrho_3$-term in the last bracket of (63) does, in the limit $M = 0$, not contribute to $P_\pm$ as the number of interaction operators H_i occurring in (63) is even. Furthermore, (64) holds whether the interaction concerned is of the charged or of the neutral type. In fact, if we consider the case of charged fields, and to that purpose introduce isotopic matrices τ_i, representative of the charge coordinate of the nucleons, (63) will contain additional factors τ_i compared to the corresponding neutral interactions. This, however, cannot affect the conclusion that the $\pm \varrho_3$-term does not contribute.

Thus (64) has been established for the case that we have to do with either an *f*- or a *g*-interaction. Let us now consider the case that an *f*- and a *g*-interaction of the same type are simultaneously present. This yields, in first order, self-energy terms proportional to fg. But for these it is no longer true that the dependence of H_1 and H_2 on the Dirac matrices is the same, and therefore the reasoning of the preceding paragraph cannot be applied here. We thus have to insert the expressions for the operator H involved into (63) and to verify by explicit calculation whether (62) is satisfied. As to the first order approximation, it appears *) that in the *s*- and *pv*-case $P_+ = P_- = 0$, so that (64) again is true; as we have seen in § 6, 3° and 7°, $W^{(1)}$ $(M \neq 0) = 0$ for these cases. On the other hand $P_+ = -P_-$ for the *v*- and *ps*-interaction.

Finally it should be noted that in higher order there may occur cross terms proportional to $f_i^{2p} f_k^{2q}$, $g_i^{2p} g_k^{2q}$, $f_i^{2p} g_k^{2q}$, $f_i^{2p} f_k^{2q} f_l^{2r}$, ..., *i*, *k* and *l* now referring to different types of fields. There, however, we have again

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*) Take e.g. the scalar case and let e.g. H_1 be the operator corresponding with the spatial part of the g_s-interaction, i.e. $H_1 \sim i\varrho_1(\vec{\sigma}\,\vec{p})$ and let H_2 be the operator of f_s-interaction: $H_2 \sim \varrho_3$. Then, for $M = 0$,

$$P_\pm \backsim \text{Trace}\, i\varrho_1 \vec{\sigma}\vec{p} \left(1 + \frac{\varrho_1 (\vec{\sigma}\,\vec{p})}{E}\right) \varrho_3 (1 \pm \varrho_3) = \pm \frac{ip^2}{E} \quad . \quad . \quad (63)$$

However, there will also be a transition involving the same intermediate state, in which the rôles of H_1 and H_2 are interchanged, i.e. we have to take the sum of the right member of (63) and its conjugated to obtain the total $P_\pm$. Thus $P_+ = P_- = 0$.

$P_+ = P_-$ as the number of operators corresponding with the different kinds of fields is even. Generally we may state that, if the operators $H_1, \ldots, H_{2n}$ occurring in (63) can be grouped in such a way that each group contains an even number of H's depending in the same way on the Dirac matrices, (62) is satisfied. If they cannot be grouped in the manner indicated, the relation between P_+ and P_- has to be found by explicit calculations.

We shall now consider the bearing of (64) on the divergence properties of $W^{(n)}$. First we take

c) The f_v-self-energy.

We recall that all *n–th* order self-energy effects can be described as due to consecutive transitions in which the total system, after passing through $2n-1$ intermediate states returns to the initial state. The divergent results are due to those states having very large momenta, (ultra-violet catastrophe). Considering provisorily the domain in momentum space of the intermediate states to be cut off at a large but finite value P, all self-energies will be finite. We especially choose P to be so large as to be $\gg \hbar\varkappa$; then $\eta = \hbar\varkappa/P$ may be considered as a small dimensionless parameter of the problem and, in the domain of large p, we can develop the integrals representing the contributions to the self-energy of various order in a power series in η *).

Now it may be remarked in the first place that no negative powers of η will occur. Indeed, such terms would, for $\varkappa \to 0$, become infinite, contrary to the fact that the electromagnetic self-energy remains finite for finite P **). The lowest exponent of η which occurs is therefore zero: this term in the development is the only one remaining if $\varkappa = 0$ and is, therefore, according to the previously mentioned results for the electromagnetic field logarithmically divergent.

To discuss the other terms, we note that $W^{(n)}$ is a series of the form

$$W^{(n)} = \sum_{k=0}^{\infty} W_k^{(n)}(P)\,\eta^k$$

in which the coefficient of η^k may be written as a power series in P which, for a particle at rest, generally is ***)

$$W_k^{(n)}(P) = \sum_l c_{kl}^{(n)} \left(\frac{P}{Mc}\right)^l \cdot \left(\lg \frac{P}{Mc}\right)^{t_l} \cdot Mc^2$$

*) The series developments used here and in the following find their justification in the circumstance that the self-energy is an integral over p of a rational function of p, $\hbar\varkappa$, $\sqrt{p^2 + \hbar^2\varkappa^2}$, $\sqrt{p^2 + M^2}$, etc. By developing the integrand in the region of large p, which is the only domain of momentum space that interests us here, and then integrating term by term, the quoted series are obtained.

**) It will be remembered that it was shown in § 6*b* that it is indeed legitimate to consider the electromagnetic self-energy as the limiting case of the f_v-self-energy for $\varkappa \to 0$.

***) It is immaterial to the argument whether the coefficients $c_{kl}^{(n)}$ depend on $lg(\hbar\varkappa/Mc)$, as may happen for $k > 0$.

factors M and c having been introduced to obtain the right dimensions. Now, according to (64) *) $W^{(n)}(P) = 0$ if $M = 0$, for any $\varkappa$, and so $W_k^{(n)}(P) = 0$ if $M = 0$. Consequently the exponent l of P/Mc must be ≤ 0. This implies that all terms with $k > 0$ do not, in the subsequent transition $P \to \infty$, lead to divergent results as they are at most $\sim P^{-k}$ (or $\sim P^{-k} \lg P$). Thus the only divergent term is the same one as occurs in the electromagnetic case. Hence:

> The f_v-self-energy diverges at most logarithmically to any order of approximation. The divergent term is independent of $\varkappa$.

The explicit first order calculations of §§ 4, 5 are in accordance with this general result. The argument is furthermore independent of the charged or neutral character of the interaction **).

d) *f*-fields of other type.

In the first place we remark that, for the corresponding "photon" fields ($\varkappa = 0$), it can be shown in the same way as was done by WEISSKOPF for the electromagnetic field that the corresponding self-energies diverge at most logarithmically in any approximation. Furthermore the f-fields are connected with the corresponding photon fields in the same way as the f_v- with the electromagnetic field **) and this means that the proof of § 7*c* can be given for all f-fields. Therefore:

> All f-self-energies diverge at most logarithmically to any order of approximation. The divergent term is independent of $\varkappa$.

e) *g*-fields.

As already strifed before, the difference between f- and g-interactions is essentially that the dimension of the coupling constants in the latter case is charge (g) times length. For reasons of simplicity it is customary to identify this length with the Compton wave length $1/\varkappa$ of the mesons. However, in discussing the properties of the g-divergences, it will prove convenient not to introduce this identification. In this section we shall therefore put the g-sources proportional to

$$\frac{g}{\varkappa^0}$$

where $\varkappa^0$ is an inverse length fundamentally not identical with $\varkappa$.

We can again develop $W_g^{(n)}$ in a power series in η like we did with $W_f^{(n)}$. However, besides η there occurs a similar dimensionless parameter

$$\eta_0 = \frac{\hbar \varkappa^0}{P}.$$

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*) It should be noted that we always consider the double limiting process $\lim_{P\to\infty} \lim_{M\to 0} W(P)$ and that the order of these limiting processes is never reversed.

**) The legitimacy of the (formal) transition $\varkappa \to 0$ is shown on similar lines in the charged as in the neutral f_v-case.

Now, while we may for g-interactions, too, let η tend to zero, this is impossible for η^0, as η^0 is different from zero whatever the mass of the field quanta is. Still, the development of $W_g^{(n)}$ with respect to η can easily be written down in such a form that the explicit dependence on η^0 is simultaneously taken into account. In fact, while $W_g^{(n)}$ will generally be an intricate function of η, its dependence on η^0 is quite simple:

$$W_g^{(n)} \backsim \left(\frac{g}{\varkappa^0}\right)^{2n}.$$

Therefore, the most general representation of $W_g^{(n)}(P)$ is

$$W_g^{(n)}(P) = (\eta^0)^{-2n} \sum_k W_k^{(n)}(P)\, \eta^k.$$

Following the same reasoning as for f-interactions, it can be shown that no negative powers of η can occur, while the $W_k^{(n)}(P)$ are at most $\sim lg\, P$. Hence the highest divergent term which *may* occur in $W_g^{(n)}(P)$ is

$$\backsim P^{2n} \lg P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Thus the divergences of $W_g^{(n)}$ are generally of higher order than that of $W_f^{(n)}$ which eventually is $\sim lg\, P$, the difference being due to the occurrence of η^0. Moreover, $W_g^{(n)}$ will generally involve divergences of more than one order, as (65) only gives the highest possible divergent term. Consider e.g. $W_g^{(1)}$. According to (65), the highest divergent term which may be found here is $\sim P^2 lg\, P$, but actually the occurrence of such a term is impossible: Indeed $W_g^{(1)}(P)$ can be written as

$$W_g^{(1)}(P) = \left(\frac{g}{\varkappa^0}\right)^2 \int_0^P \Phi(p, \varkappa)\, dp,$$

where Φ is an algebraic function of p. As the highest divergence of $W_g^{(1)}$ corresponds with the integral over the highest order term of the development of Φ with respect to p, a transcendental term $\sim P^2 lg\, P$ cannot exist. This affords an example of the fact that not all divergences compatible with (65) need necessarily occur. However, to decide this, arguments of other kind are apparently necessary. As we have seen in preceding sections $W_g^{(1)}$ involves a quadratic and a logarithmic divergence.

f) fg-self-energy.

Here (64) does not hold and consequently the general dimensional considerations from which conclusions can be drawn as to the kinds of divergences do not apply. The explicit calculations showed $W_{fg}^{(1)}$ to have a *qu* and a *log* divergence (on hole theory) in the *v-* and *ps-*case.

g) Direct self-energy.

We have already seen that in first approximation these yield a *qu* and a *log* divergence. The occurrence of terms of direct interaction also influences the higher order approximations of the self-energy, as for these there may occur "cross terms" which are partly of "field", partly of "direct" origin. As their "field dependence" can be discussed by the methods of § 7, *d*, *e*, it follows that these cross terms will give rise to divergences in $W^{(n)}$ the highest possible order of which is $2n$.

Appendix.

Note 1. *On the calculation of* $W^{(1)}(\vec{p}_0)$.

In § 1 a prescription has been given for computing the p-integral representing the self-energy for the case that the particle is at rest, which is essentially based on the fact that, if $\vec{p}_0 = 0$, the problem is spherically symmetrical in momentum space. We will now show by two examples that, generally, the same prescription cannot be used if $p_0 \neq 0$: the divergent part, on hole theory, has been computed of the f_s- and f_v-self-energy, using (55) and (40), (42) respectively. The results are for the divergent parts

$$\frac{f_s^2 M^2}{4\pi\hbar^2 P_0}\left[-\frac{3}{4}-\frac{\mu^2}{M^2}+\frac{\mu^2}{2}\left(1+\frac{p_0^2}{M^2}\right)\frac{1}{p_0 P_0}\lg\frac{P_0+p_0}{P_0-p_0}\right]\int\frac{dp}{p},$$

$$\frac{f_v^2 M^2}{8\pi^2\hbar P_0}\left[3-\frac{6\mu^2}{M^2}+3\mu^2\left(1+\frac{p_0^2}{M^2}\right)\frac{1}{p_0 P_0}\lg\frac{P_0+p_0}{P_0-p_0}\right]\int\frac{dp}{p},$$

$$P_0 = \sqrt{p_0^2+M^2}$$

which are incompatible with formula (1*a*) on the one, and the corresponding results of table 3 on the other hand. The latter are found from the above expressions by putting $p_0 = 0$. Putting $\mu = 0$, we get for the divergent parts of the "photon" field self-energies

$$-\frac{3 f_s^2 M^2}{16\pi^2\hbar P_0}\int\frac{dp}{p} \quad , \quad \frac{3 f_v^2 M^2}{8\pi^2\hbar P_0}\int\frac{dp}{p},$$

respectively, which do agree with (1). However, further calculations show that the finite parts still would not be compatible with (1). Thus for a moving particle a more complicated prescription should be followed to obtain $W^{(1)}(\vec{p}_0)$ by direct calculation. As has been remarked, the knowledge of the prescription is not necessary, as, once $W^{(1)}$ has been defined, $W^{(1)}(\vec{p}_0)$ is fixed by (1*a*).

Note 2. *Self-energy due to charged fields.*

It was mentioned in § 1 that the self-energy of nucleons due to a given

type of interaction is, in first approximation, the same for neutral and charged fields. We will here illustrate the general argument, by computing $W^{(1)}$ for the charged f_s-field case. The field variables, now being complex, satisfy the equations

$$\vec{\Gamma} = \text{grad}\, \Omega,\ \Lambda = -\dot{\Omega},\ \text{div}\, \vec{\Gamma} + \dot{\Lambda} = \varkappa^2\, \Omega - R,\ R = f_s\, \psi^\dagger\, Q_{NP}\, \varrho_3\, \psi.$$

Q_{NP} is an operator transforming a neutron into a proton; we have $R^\dagger = f_s\, \psi^\dagger\, Q_{PN}\, \varrho_3 \psi$, where $Q_{PN} = Q_{NP}^\dagger$ transforms a proton into a neutron. The Hamiltonian is

$$H_{\text{total}} = H_{\text{mat}} + \tfrac{1}{2}\int (\vec{\Gamma}\vec{\Gamma}^\dagger + \Lambda\, \Lambda^\dagger + \varkappa^2 \Omega\, \Omega^\dagger) + \text{conj.} - \int (R\, \Omega^\dagger + R^\dagger \Omega),$$

the part of H depending only on the field variables having been chosen such that no infinite "zero point" meson charge occurs [11]). The commutation relations are

$$[\Omega(\vec{x}, t), \Lambda^\dagger(\vec{x}', t)]_- = \frac{\hbar}{i}\, \delta(\vec{x} - \vec{x}'),\ [\Omega^\dagger(\vec{x}, t), \Lambda(\vec{x}' t)]_- = \frac{\hbar}{i}\, \delta(\vec{x} - \vec{x}').$$

The first order self-energy is defined as one half of the expectancy value $\sim f_s^2$ of

$$H = -\int (R\, \Omega^\dagger + R^\dagger\, \Omega)\, d\nu.$$

We first compute $W^{(1)}_{\text{inv}}$. Ω satisfies

$$\Box\, \Omega - \varkappa^2\, \Omega = -R,$$

$\Omega^\dagger$ the conjugated equation. $\Omega^{(1)}$ and $\Omega^{(1)\dagger}$ can be found similar to (32) and we get

$$m_{\text{inv}} = -f_s^2\, [Q_{NP}\, Q_{PN} + Q_{PN}\, Q_{NP}]\, \frac{\{\varrho_3, \varrho_3\}}{\text{inv}},$$

the first (second) term between square brackets being equal to 1 (0) if the nucleon state considered is a neutron state and 0 (1) if it is a proton state. Thus for both neutron and proton, $W^{(1)}_{\text{inv}}$ is given by the first of equations (55). To obtain $W^{(1)}_{\text{fluct}}$, we develop $\Omega^{(0)}$ and $\Omega^{(0)\dagger}$ in plane waves:

$$\Omega^{(0)} = \Sigma[\Omega_+(\vec{k})\, e^{i(\vec{k}\vec{x} - \nu t)} + \Omega_-^\dagger(\vec{k})\, e^{-i(\vec{k}\vec{x} - \nu t)}],\ \text{etc.}$$

The only non-vanishing matrix elements of the Fourier amplitudes are

$$[\Omega_\pm(\vec{k})]_{n^\pm \to n^\pm + 1} = [\Omega_\pm^\dagger(\vec{k})]_{n^\pm + 1 \to n^\pm} = \sqrt{\frac{\hbar\,(n^\pm + 1)}{2\nu}},$$

$n^{\pm} \to m^{\pm}$ denoting a transition in which the number of mesons with charge $\pm e$ and wave vector $\vec{k}$ changes from $n^{\pm}$ to $m^{\pm}$. With the help of (51), where H'_j should be replaced by H, one easily finds

$$m_{\text{fluct}} = f_s^2 \left[Q_{NP} Q_{PN} + Q_{PN} Q_{NP}\right] \frac{\{\varrho_3, \varrho_3\}}{\text{inv}} \cdot \frac{E - E_0}{\hbar \nu}.$$

Consequently the fluctuation energy of proton and neutron is indeed given by the second of equations (55).

For the higher order $W^{(n)}$ there is no such simple one-to-one correspondence of intermediate states for neutral and charged interactions. Consider e.g. $W^{(2)}$; in the neutral theory we have for both proton (P) and neutron (N) the following chains of intermediate states, (Y_1 and Y_2 denote the two neutral mesons that come into play),

$$\begin{aligned}
&P \to P + Y_1 \to P + Y_1 + Y_2 \to P + Y_1 \to P,\\
&P \to P + Y_1 \to P + Y_1 + Y_2 \to P + Y_2 \to P,\\
&P \to P + Y_1 \to P \to P + Y_2 \to P;
\end{aligned}$$

the same for the neutron. In a charged theory we have, (Y^+ is a positive, Y^- a negative meson),

$$\begin{aligned}
&P \to N + Y^+ \to P + Y^+ + Y^- \to N + Y^+ \to P,\\
&P \to N + Y_1^+ \to P \to N + Y_2^+ \to P,
\end{aligned}$$

and

$$\begin{aligned}
&N \to P + Y^- \to N + Y^- + Y^+ \to P + Y^- \to N,\\
&N \to P + Y_1^- \to N \to P + Y_2^- \to N.
\end{aligned}$$

This difference between charged and neutral theory is due to the restrictions which, in the former, are imposed by charge conservation on the order in which positive and negative mesons are emitted and absorbed.

Note 3. *On the electromagnetic self-energy* $W^{(1)}(e)$.

In order to compute this without the use of a gauge transformation which separates the Coulomb from the dynamic interaction, we have to consider the expectancy value of

$$\tfrac{1}{2} \int (\varrho B - \vec{s}\vec{A})\, d\nu, \quad \varrho = e\psi^\dagger \psi, \quad \vec{s} = e\psi^\dagger \vec{\alpha} \psi,$$

B being the electromagnetic scalar potential. It has been remarked in § 5 that it is not legitimate to consider $W^{(1)}(e)$, computed in this way, as the limiting case of the Proca-field self-energy for $\varkappa = 0$, though it would give the right result. $W^{(1)}(e)$ has therefore been calculated by quantizing the electromagnetic field by means of a method due to FERMI [8]), according to which the electromagnetic field Lagrangian is chosen in such a way

that the canonically conjugated of B is not zero. For particulars we refer to a paper by ROSENFELD [12], formula (n) of which is in the following indicated by (Rn). The necessary equations will be given without proof. Apart from a difference in notation and representation, they are identical with the corresponding ones of the last cited paper:

The Lorentz condition

$$\operatorname{div} \vec{A} + \dot{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (R\,12)$$

and its time derivative, $(R\,13)$, should be considered as accessory conditions in the sense explained in § 5. By developing A similar to (15) and likewise putting

$$B = \Sigma [B^+(k)\, e^{i(\vec{k}\vec{x} - \nu t)} + B^-(k)\, e^{-i(\vec{k}\vec{x} - \nu t)}], \quad . \quad . \quad . \quad (R\,19)$$

we get the operator conditions (not identities)

$$B^{\pm}(k) = A^{\pm}(0, k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (R\,23)$$

Furthermore

$$A^+(j, k)_{n \to n+1} = A^-(j, k)_{n+1 \to n} = \sqrt{\frac{\hbar (n+1)}{2\nu}}, \quad . \quad . \quad . \quad (R\,25)$$

$$B^+(k)_{n+1 \to n} \quad = B^-(k)_{n \to n+1} \quad = \sqrt{\frac{\hbar (n+1)}{2\nu}} \quad . \quad . \quad . \quad (R\,26)$$

By means of (49) and (51) we then get in the usual way

$$m_{\text{inv}} = e^2 \, \frac{\{1, 1\} - \{\vec{a}, \vec{a}\}}{|\vec{p} - \vec{p}_0|^2 - (E - E_0)^2}$$

$$m_{\text{fluct}} = e^2 \sum_{j=0,1,2} \frac{\{\vec{a}\,\vec{\varepsilon}_j, \vec{a}\,\vec{\varepsilon}_j\} - \{1, 1\}}{|\vec{p} - \vec{p}_0|^2 - (E - E_0)^2} \cdot \frac{E - E_0}{\hbar \nu} .$$

The former equation which can also be obtained with MØLLER's method [7] is identical with (40a) with $\mu = 0$. The latter is *not* identical with (42) with $\mu = 0$, but it is easily seen that it again yields (44a), cf. the discussion after eq. (51).

Note 4. *Canonical transformation in the scalar theory.*

The field equations

$$\vec{\Gamma} = \operatorname{grad} \Omega, \; \Lambda = -\dot{\Omega}, \; \operatorname{div} \vec{\Gamma} + \dot{\Lambda} = \varkappa^2 \Omega - R, \; R = f_s \, \psi^\dagger \varrho_3 \psi,$$

follow from the Hamiltonian

$$H = H_{\text{mat}} - \int R \, \Omega \, dv + \tfrac{1}{2} \int (\vec{\Gamma}^2 + \Lambda^2 + \varkappa^2 \Omega^2) \, dv$$

The static field equations are

$$\vec{\Gamma}^0 = \text{grad}\, \Omega^0,\ \Lambda^0 = 0,\ \text{div}\, \vec{\Gamma}_0 = \varkappa^2\, \Omega_0 - R^0,\ R^0 = f_s\, \psi^\dagger\, \psi.$$

To separate the static interaction, we employ a canonical transformation (24) where now

$$K = -\int \Omega^0\, \Lambda\, dv.$$

The transformation yields

$$\widetilde{H} = H_{\text{mat}} + \tfrac{1}{2} \int (\vec{\Gamma}^2 + \Lambda^2 + \varkappa^2\, \Omega^2)\, dv + V + W,$$

where the right member is expressed in the new variables, and

$$V = -\tfrac{1}{2} \int\int R^0(\vec{x})\, R^0(x')\, \chi(|\vec{x} - \vec{x}'|\, dv\, dv'$$

$$W = \int r\, \Lambda\, dv,\ r(\vec{x}) = -i f_s \int dv'\, \chi(|\vec{x} - \vec{x}'|)\, \{\psi^\dagger \varrho_2 (\vec{\sigma}\vec{\nabla})\psi - (\vec{\nabla}\psi^\dagger)\varrho_2 \vec{\sigma}\psi\}_{\vec{x}'}.$$

The new field variables satisfy

$$\vec{\Gamma} = \text{grad}\, \Omega,\ \Lambda = -\dot{\Omega} + r,\ \text{div}\, \vec{\Gamma} + \dot{\Lambda} = \varkappa^2\, \Omega.$$

By means of these equations, one finds after some calculation, using identities like (36)

$$m_{\text{inv}} = -f_s^2\, \frac{\{\varrho_3, \varrho_3\}}{\text{inv}},\quad m_{\text{fluct}} = f_s^2\, \frac{\{\varrho_3, \varrho_3\}}{\text{inv}} \cdot \left(\frac{E - E_0}{\hbar \nu}\right)^3.$$

the latter being not identical with the second of equations (55). However, both give the same result for the f_s-fluctuation energy for a particle at rest.

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CHAPTER II.

On the theory of the electron.

Summary.

§ 1. Introductory remarks on classical theory. — § 2. The situation in quantum theory; statement of the problem. — § 3. The quantum electromagnetic self-energy. — § 4. Self-energy due to vector f-field. — § 5. Case of e- and vector f-field. — § 6. Subtractive vector f-field. — § 7. Criticism of the introduction of subtractive fields. — § 8. f-fields of other type. — § 9. The universal length. — Appendix: Force on surface element of finite electron.

§ 1. *Introductory remarks on classical theory.*

As is well known, the MAXWELL–LORENTZ theory, which provides us with a successful description of electromagnetic phenomena in the classical domain as long as distances of the order of the "classical electron radius" do not come into play, exhibits grave difficulties in the small distance region. This essentially finds its origin in that the theory does not involve a consistent model of the electron, which in particular is demonstrated by the fact that the electromagnetic self-energy of a point electron is infinite. In an attempt to remove this difficulty within the framework of classical theory, it has recently been suggested by STÜCKELBERG [1]) to consider the electron to be the point source not only of the electromagnetic field, but furthermore of a second field of the scalar type, the variables of which satisfy an equation of the form

$$(\Box - \varkappa^2)\,\psi = \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus, although the electron has zero radius, the theory involves a fundamental length $\varkappa^{-1}$, the range of the field, which should not exceed an order of magnitude of 10^{-13} cm, so as not to disturb the perfect accordance which exists between electromagnetic theory and experiment for larger distances.

The static field energy of a system of particles interacting through the intermediary of such a scalar field is given by

$$-\frac{f^2}{8\pi}\sum_{i,k}\frac{e^{-\varkappa r_{ik}}}{r_{ik}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

r_{ik} being the distance between i-*th* and k-*th* particle. The constant f determines the strength of the coupling between the particles and the field, to which we from now on will refer as f-field. f has the dimensions of a

charge, and is expressed in Heaviside units. The static energy of the electromagnetic field (e-field) is

$$\frac{e^2}{8\pi}\sum_{i,k}\frac{1}{r_{ik}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining the self-energy terms of (2) and (3), viz. those terms for which $i = k$, it follows that the self-energy due to both e- and f-field has a finite value W provided that

$$e^2 = f^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value then is

$$W = \frac{e^2\varkappa}{8\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Putting W equal to the rest energy of the electron:

$$W = mc^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

one gets

$$\frac{1}{\varkappa} = \frac{e^2}{8\pi mc^2},$$

i.e. one half of the electron radius; according to this relation, $\varkappa^{-1}$ is of the acceptable order of magnitude of 10^{-13} cm (see above). Thus there are two fundamental relations between the constants of this theory: one between the charges only, arising from the condition of finite self-energy, and one between $\varkappa$, m, and e, following from the assumption that the total field self-energy for an electron at rest equals c^2 times its rest mass.

Another way of formulating the difficulties inherent in the classical Lorentz theory is: the energy momentum tensor T_{ik} of the total system should satisfy the relation: $\partial T^{ik}/\partial x^k = 0$. However, this condition is, on Maxwell–Lorentz theory, well known to be incompatible with the presence of charged particles in the system. This incompatibility can be overcome by adding to the electromagnetic tensor a tensor of other origin in order to let the total tensor have zero divergence; furthermore, the additional tensor should have zero $i4$-elements ($i \neq 4$) in the system in which the electron is at rest *).

An example of a tensor which satisfies the just mentioned requirements is the tensor of internal stresses of POINCARÉ, who, in fact, was the first to point out that the solution of the electron problem might be sought in this direction **).

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*) Cf. W. PAULI, *Relativitätstheorie*, § 63, Teubner 1921; R. BECKER, *Theorie der Elektrizität*, Bd. II, § 66, Teubner 1933.

**) Cf. H. POINCARÉ, *La mécanique nouvelle*, Gauthier-Villars 1924; H. A. LORENTZ, *Theory of Electrons* §§ 180—181.

Now the energy-momentum tensor of the f-field does satisfy these requirements too: its $i4$-elements, $i \neq 4$, like those of the energy momentum tensor of any field, vanish in the rest system, as they are the components of the "f-Poynting vector", while the condition of zero divergence is closely connected with the "convergence relation" (4). In order to show this, consider a spherical electron with finite radius r and with uniform superficial charge distribution such that the total charges are e and f. In order to have zero total divergence, it is sufficient to show that, in the rest system of the electron, the forces on a surface element cancel each other. In the Appendix it is shown that this is true if

$$e^2 = f^2 \cdot g(\varkappa r) \quad ; \quad g(\varkappa r) \to 1 \text{ if } r \to 0. \quad . \quad . \quad . \quad . \quad (4c)$$

Thus (4) expresses the stability condition in the limit of a point electron.

Another theory of the same kind has been proposed by BOPP [2], who introduces an f-field of the vector type with vector (no tensor) interaction. Instead of (2), we have for the static interaction in this case

$$\frac{f^2}{8\pi} \sum_{i,k} \frac{e^{-\varkappa r_{ik}}}{r_{ik}}.$$

In order to obtain a finite result in combination with (3), it is therefore necessary to take the Hamiltonian, or, generally, the energy momentum tensor of this field with the minus sign, an artifice which leads to such disagreeable consequences as a field energy which is not positive definite. This theory again gives rise to the relations (4), (4a), the assumption (4b) again is made, while it can be shown that (4c) holds too, cf. also the Appendix.

It is a characteristic feature of the LORENTZ electron theory as well as of those of STÜCKELBERG and BOPP, that the equation of motion of the electron is derived from the condition that the integral, over a narrow tube surrounding the world line of the electron, of the divergence of the energy momentum tensor density of the total system vanishes. These theories may therefore be called unitary — though not in the sense of MIE — as the equation of motion is, for given initial conditions of the source functions, essentially a consequence of the equations of the fields. Consequently, the electron mass has to be introduced into the theory as a function of the field variables, which, in their turn, now must be considered as functions of the kinematical variables of the electron which enter the theory by means of the source functions. It will be remembered from the LORENTZ theory that a convenient definition of m is provided by

$$\int \mathfrak{T}^1_{.4}(0)\, dV = mc^2,$$

where $\mathfrak{T}_{ik}(0)$ is the energy momentum tensor in the electron rest system. As the left member is equal to W, this then justifies the definition (4b).

§ 2. *The situation in quantum theory; statement of the problem.*

The present quantum field theories, based on the assumption of zero radius of the elementary particles, likewise give rise to an e-self-energy of the electron, which, like the field self-energy of any elementary particle, turns out to be infinite, but here this disturbing feature reveals itself in a far more complicated way than in classical theory. This has in particular become clear from WEISSKOPF's investigations [3, 4]: Besides the particle having a self-energy solely due to its charge, which is quite analogous to what is called self-energy in classical theory such properties which have no classical counterparts, as its spin, interference effects due to the hole theory interpretation of the vacuum, as well as typical fluctuation effects, originating from the quantization of the fields created by the particle, also are intricately involved in the determination of the total quantum field self-energy which turns out to be entirely different from its classical counterpart. In fact one may say that the field self-energy of the electron is fundamentally a quantum effect, because by introducing the quantum of action h there enters in the theory of the electron a new quantity; $\sqrt{hc}$ which has the same dimensions as the electric charge e but which is well known to be much larger than e. Thus quantum theory demands a complete revision of the whole self-energy problem.

In this chapter it will be examined, whether the assumption that the electron does, besides the electromagnetic field, also create another field of short range, will lead to a finite self-energy in a relativistic quantum theory, i.e. whether such a theory will yield convergence relations, similar to (but not necessarily identical with) (4). Furthermore, the possibility of the establishment, analogous to (4*b*), of a relation involving m, and e (all final outcomes here depending on e only through some power of the fine structure constant) will be discussed. In this connection a fundamental difference between quantum and classical theory must be observed at the outset:

Indeed, the theory which will be outlined hereafter, is, contrary to classical theory, non-unitary: while in classical theory the equation of motion of the electron represents a secondary feature, because it is in some way or other derived from the field equations, its quantum mechanical analogue, the wave equation, cannot be obtained from the equation of the fields (now also involving certain commutation relations between the field variables), as long as the electron is considered as a point, the coordinates of which are taken as quantum variables. On this assumption, the impossibility of a unitary treatment has been shown by PRYCE [5] *). Thus, the wave

*) The main point is that the commutation relations between position and momentum coordinates of the electron are incompatible with those of the field variables if one defines the electron momentum as integral over space of the field momentum density. Furthermore, the spin properties of the electron cannot, as must be required from a really unitary theory, solely be derived from the properties of the fields.

equation, which involves m, has to be introduced as independent basic postulate into the theory, so that no definition a posteriori of m in terms of field variables is possible in the same sense as in classical theory.

The non-unitarity of the theory is expressed by the occurrence in the Hamilton operator H of the total system of electron and fields of a term H_m which involves m explicitly:

$$H = H_m + H_{\text{fields}} + H_{\text{coupling}};\ H_m = \int \psi^\dagger \left(\frac{\hbar c}{i} \vec{\alpha} \vec{\nabla} + \varrho_3 mc^2 \right) \psi \, dV.$$

H_m denotes the operator of a "free" Dirac electron, H_{fields} is that part of the Hamilton operator which depends only on the field variables, while H_{coupling} involves both ψ and the field variables.

Now the total quantum mechanical self-energy W_{tot} is defined as

$$W_{\text{tot}} = \bar{H} = mc^2 + W$$

the overlining denoting the expectancy value for the state in which the electron is at rest, and it is an important consequence of the non-unitary aspect of the theory that the field self-energy W itself will now depend on m. Thus, if we assume the charge f to be eliminated by means of a convergence relation, we generally have in a two field theory:

$$W = W\left(\frac{e^2}{\hbar c}, m, \varkappa \right).$$

Still, the theory might be interpreted in such a way as to yield an electron mass which, similar to the situation in classical theory is equal to the total field self-energy, *viz.* by assuming the parameter m occurring in H_m to have a provisorily undetermined value which then is fixed by putting the final result for the self-energy of an electron at rest equal to mc^2:

$$W\left(\frac{e^2}{\hbar c}, m, \varkappa \right) = mc^2 \ldots\ldots\ldots \quad (4d)$$

Considered in this way, (4c) represents an implicit equation for m from which an expression of m in terms of $e^2/\hbar c$ and $\varkappa$ can be found *).

It has been observed in connection with the classical equation (4*b*), that a relation like (4*d*) can only be maintained, however, if it leads to a sensible value of $\varkappa$. To investigate this, we have to insert into (4*d*) the experimental value of m and then to express $\varkappa$ in terms of $e^2/\hbar c$ and m. Now it is shown in § 10 that

$$W \sim \alpha \, lg \frac{\hbar \varkappa}{mc} . mc^2 \ldots\ldots\ldots \quad (4e)$$

*) The c-number $C = mc^2$ should then be subtracted from H in order that the expectancy value of H shall correctly be equal to mc^2. *Added in proof:* A more detailed analysis of the relation of the field self-energy of an elementary particle to its total mass will be given in a forthcoming paper. The results obtained there are in agreement with the present treatment of considering W as a perturbation compared to mc^2, cf. p. 52. The transformation properties of W will also be more fully discussed.

where α is the fine structure constant. Hence (4*d*) yields

$$\frac{1}{\varkappa} \sim \frac{\hbar}{mc} \cdot e^{-\frac{1}{\alpha}} \sim 10^{-68}\ cm.$$

Thus the range of the field and, correspondingly, the mass of the field quanta, would be of a quite unfamiliar order of magnitude.

There is, however, an alternative way to obtain a theory involving one main mass contribution, viz. by considering the field self-energy to be a small perturbation compared with the "mechanical" mass m, which implies that m can practically be identified with the experimental mass. This can be achieved by putting

$$\varkappa^{-1} \sim \frac{e^2}{4\pi mc^2} = 2{,}8.10^{-13}\ cm,$$

for then W is, according to (4*d*) $\sim$ 0,01 mc^2. The further discussion is given in § 10, where it is shown that it is not trivial that W can be considered as a perturbation.

More arguments against the unitary treatment and in favour of the method, which will be followed hereafter, of treating the field self-energy as a perturbation on the a priori fixed mass m, are provided by the results obtained in a subsequent paper on the theory of nucleons. There it is found:

1. A theory of the mass difference of proton and neutron can be given by assuming the proton to interact with the f-field, too, *but only if the range of the f-field is of the order of the classical electron radius.*

2. If one tries to compensate by means of convergence relations the divergences due to the interaction of nucleons with meson fields, it turns out that the remaining finite part of the mesic self-energy is $\ll Mc^2$ ($M \approx M_{\text{neutron}} \approx M_{\text{proton}}$), whether the unitary method is followed or not, *for here we have no liberty to dispose of the magnitude of the field range, which is* $< h/Mc$. Thus the consideration of the field self-energy of the electron as a perturbation unifies the connection between field self-energy and mass for all Dirac particles.

For details we refer to Chapter III.

The unitary method will therefore be discarded and we base the further considerations on the perturbation concept, i.e. on the relation $W \ll mc^2$.

The more technical aspects of the problems on hand have been developed in the previous chapter which will be cited as *I*. For reasons to be explained in § 8 we shall, in this chapter, in particular be dealing with the results obtained in *I* on the f-interactions; as to the notations used for the various types of fields and interactions, the reader is begged to refer to the table in *I*, § 1.

After a short survey of the properties of the e-self-energy in § 3, the consequences of the introduction, similar to BOPP's classical theory, of a "subtractive" f-field of the vector type will be discussed according to the scheme just outlined (§ 6); to this purpose we will however, first treat the "additive" vector field (§§ 4—5). It appears that the subtractive vector theory can be developed more completely than theories involving f-fields of other type, but, as previously mentioned, it has the dis-

advantage of a non-definite field energy, the consequences of which are discussed in some detail (§ 7). Since these are, in quantum theory, even more serious than in classical theory, the conception of subtractive field has to be discarded, and we turn to the investigation of other than vector f-fields (§ 8), which leads to the result that an f-field of the scalar type yields, at any rate in first approximation, to finite results; there turns out to be no correspondence between this theory and STÜCKELBERGS classical formalism outlined in § 1. Finally, § 9 is devoted to a further discussion of the place of the universal length $\varkappa^{-1}$ within the present scheme.

§ 3. *The quantum electromagnetic self-energy.*

An investigation of its properties has been made by WEISSKOPF [3, 4]) who arrived at the following main results:

a) Hole theory *). The self-energy of any order diverges at most logarithmically.

b) One-electron theory. Here no such general statements can be made. The first order contribution appears to involve a quadratic and a linear divergence.

Furthermore, it was shown in I that

$$W^{(n)}(\vec{p}) = W^{(n)}\sqrt{1-\beta^2}, \quad \ldots\ldots\ldots \quad (5)$$

(cf. I eq. (1a)), where from now on we indicate the value of the n-*th* order self-energy for a particle with momentum $\vec{p}$ by $W^{(n)}(\vec{p})$ and the corresponding quantity for $p = 0$ by $W^{(n)}$. In particular (5) also applies to the e-self-energy $W_e^{(n)}$.

One may ask why the factor $(1 + \beta^2/3)$, familiar from classical theory does not occur in the transformation formula for W_e: this is just because the classical theory, contrary to quantum theory, is unitary, i.e. the energy momentum tensor density governing the behaviour of the total system of the electron and its field is given by

$$\mathfrak{T}_{ik} = -F_{il}F_k^{\,l} + \tfrac{1}{4}\,g_{ik}\,F_{pq}F^{pq},$$

(F_{ik} is the e-field), which does not involve the dynamical electron variables explicitly, and which, of course, does not contain the (electromagnetic) mass of the electron at all. This quantity, it will be remembered, is first introduced into the theory by the definition

$$T_{.4}^{4}(0) = \int \mathfrak{T}_{.4}^{4}(0)\,dV = mc^2.$$

*) As well known, the concepts of hole theory lead to a non-linear electrodynamics of the vacuum. This is expressed by the occurrence of terms in the Hamiltonian being non-linear in the potentials and field strengths, (the terms H_2, H_3 and H_4 of HEISENBERG [6])). These terms do not give rise to electron self-energy contributions, because their expectancy value for the states "vac ± 1" is the same as that for "vac", as they do not depend on the occupation numbers of the electron states.

Supposing the electron to move in the 1-direction, the factor $(1 + \beta^2/3)$ is then obtained from the well known formula

$$T^4_{.4} = \frac{T^4_{.4}(0) - \beta^2\, T^1_{.1}(0)}{\sqrt{1-\beta^2}},$$

and from symmetry considerations, noting that the trace of $\mathfrak{T}_{ik}$ vanishes.

§ 4. *Self-energy due to vector f-field.*

Consider now a particle with spin ½ which we assume to be the (point) source of a neutral short range field of the vector type, the variables of which satisfy a wave equation of the form (1), ("neutral vector meson field"). We take the four-vector source function to be proportional to a constant f which represents the corresponding "charge" of the particle. If the universal length, thus introduced, tends to infinity, we obtain the e-case if $f = e$.

Now it was shown in I, § 7c *) that the f-self-energy diverges at most logarithmically to any order of approximation and that the divergent term is independent of $\varkappa$. This means that the divergence occurring in the *n–th* order contribution due to the f-field is identical with that of the e-field provided that $e^{2n} = f^{2n}$. Consequently, a sufficient condition for the f-divergences of all orders to be identical with the corresponding e-divergences is

$$e^2 = f^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As (5) also holds for the f-field, it follows that the condition (6), as expressing the identity of e- and f-divergences, is relativistically invariant.

§ 5. *Case of e- and vector f-field.*

From these considerations we infer that, if an electron would be the source of the e- *as well as* of the vector f-field, its self-energy of *n–th* order would, always in the positon theory, contain a logarithmically divergent (multiple) integral which stands multiplied with $a_n(e^{2n} + f^{2n})$, the factor a_n not depending on $\varkappa$ nor on e and f. This integral, however, forms only a part of the total *n–th* order divergence which now occurs. In fact, as the *n–th* order contribution is now generally brought about by means of virtual processes in which p photons and q f-quanta come into play, $p + q = n$, $p = 0, 1, \ldots, n$, there must, more generally, occur divergent integrals $\sim a_{nm} e^{2(n-m)} f^{2m}$, $m = 0, \ldots, n$. From the reasoning of I, § 7, it will be immediately clear that these integrals will at most diverge logarithmically, while all a_{nm} will be independent of $\varkappa$. Furthermore, the relative magnitude of the a_{nm} for varying m but constant n depends only on the relative magnitude of the domains in momentum space involved.

*) In loc. cit. f_v corresponds with the f of this section.

Taking all this into account, it is easily seen *) that the total logarithmic divergence of n–th order stands multiplied with $a_n(e^2 + f^2)^n$.

§ 6. *Subtractive vector f-field.*

We now have to go one step further to attain a theory of the Dirac electron which is in some respects the quantum mechanical analogue of the classical linear theory of the electron proposed by BOPP [2]: let us consider the electron, besides of the e-field, to be the source of a "subtractive" f-field, that is to say, we take that part of the energy momentum tensor (operator) of the total system which refers to the f-field, and to the interaction of the f-field with the electron field, with the minus sign. Obviously this will alter nothing of what has been said about the *type* of the divergences, this change makes itself felt only with respect to the coefficient with which the integral stands multiplied. We show that this coefficient becomes

$$a_n (e^2 - f^2)^n.$$

The fundamental difference with the case of the "additive" f-field consists in the different domains of the values which the energies of the f-quanta may take in the intermediate states. Indeed these values range from $\hbar c\varkappa$ to $+\infty$ for the case of the additive field, but from $-\hbar c\varkappa$ to $-\infty$ for the subtractive field. Now, the logarithmic divergent integral which stands proportional with $e^{2(n-m)}f^{2m}$ is, in the additive case,

$$\sim \int^{\infty} \frac{dp_1}{p_1} \int^{\infty} \frac{dp_2}{p_2} \cdots \int^{\infty} \frac{dp_{n-m}}{p_{n-m}} \int^{\infty} \frac{dq_1}{q_1} \cdots \int^{\infty} \frac{dq_m}{q_m}, \quad \ldots \quad (7)$$

the q_i denoting absolute values of the momenta $\vec{q}_i$ of the f-quanta. As $q_i^2 = \varepsilon_i^2/c^2 - \hbar^2\varkappa^2$, where ε_i is the f-quantum energy:

$$\frac{dq_i}{q_i} = \frac{d\varepsilon_i}{\varepsilon_i}\left(1 + \frac{\hbar^2 c^2 \varkappa^2}{\varepsilon_i^2} + \ldots\right) \equiv \varphi(\varepsilon_i)\, d\varepsilon_i.$$

Thus the integral $\int^{\infty} q_i^{-1}\, dq_i$ may be replaced by $\int^{\infty} \varphi(\varepsilon_i) d\varepsilon_i$, where $\varphi(\varepsilon_i)$

*) The argument runs as follows: Consider the contribution to the divergent term, due to e-interaction only, of the part of momentum space which lies between p_1 and $p_1 + dp_1$, ..., $p_n + dp_n$. This can be written as

$$\text{const. } e^{2n} \cdot \frac{dp_1}{p_1} \cdots \frac{dp_i}{p_i} \cdots \frac{dp_n}{p_n}.$$

Replace the e-contribution to the i–th shell by the corresponding f-contribution. The contribution to the divergent term is now $\sim e^{2n-2}f^2$, but is, for the rest, identical with the former one. The replacement can be performed in $\binom{n}{1}$ ways, the relative magnitude of the total contribution $\sim e^{2n-2}f^2$ as compared to that $\sim e^{2n}$ is thus $\binom{n}{1}$. Now replace the contribution to two shells by f-contributions. the factor of relative magnitude now is $\binom{n}{2}$; etc.

is an odd function. The corresponding integral in the subtractive case will then be $\int_{-\infty}\varphi(\varepsilon_l)d\varepsilon_l = -\int^{\infty}\varphi(\varepsilon_l)d\varepsilon_l$. The integral (7) therefore $= (-1)^m$ times the integral for the subtractive case which proves our assumption.

Consequently the condition (6) is sufficient to obtain finite results in every approximation *). From what has been said before on the invariance of the identity between e- and f-divergences, it follows that the "convergence condition" (6) is relativistically invariant.

§ 7. *Criticism of the introduction of subtractive fields.*

A definitely unsatisfactory feature of the above theory lies in the rather artificial way in which the minus sign has to be introduced with regard to the f-field energy. The free f-field has consequently to be considered as a superposition of plane waves with which correspond quanta of negative energy. An electron therefore has a chance of emitting a negative f-quantum in going through a static field ("f-bremsstrahlung"), thereby passing itself to a state of higher energy. Though the total f-radiation probability can be estimated to be small, compared to that for e-radiation, if $\varkappa$ is large, the relative number of processes in which an electron emits an f-quantum thus being very much less than for ordinary bremsstrahlung, it is hard to conceive how such f-effects should be possible at all. It is true that, as shown by BOPP [2]), they do not occur in the case of a classical electron describing a closed orbit: in such a case, the work done in overcoming e- and f-radiation damping is in fact always positive. But this result, however interesting, is by no means sufficient to remove the objection.

Pair formation and annihilation involving f-quanta instead of photons are impossible because these processes require a positive amount of energy to be absorbed or emitted. Still, if one assumes the negative energy levels to be (nearly all) occupied, as in the hole theory, there is a possibility for the creation of pairs which has no analogue in the e-case and which presents new difficulties: if a static field is present, an electron of negative energy may be scattered by it in emitting a negative f-quantum, the energy of which may be such that the electron has positive energy in its final state. This process thus results in the formation of a pair as well as of an f-quantum (its inverse can therefore be ignored, being a more body problem). Consequently, the infinite distribution of negative energy electrons would become unstable in the presence of static fields, however slowly varying. The theory sketched in §§ 6—7 must therefore be considered as to be inconsistent with the very existence of an infinite electron distribution in vacuum; we thus discard the idea of substractive fields.

§ 8. *f-fields of other type.*

Another way of solving the self-energy difficulties in position theory which might lead to finite results without the device of a subtractive field,

*) This also holds if the multiple divergent integral (7) would be less than n-fold.

suggests itself by considering the introduction of an f-field of other type. In fact, it was shown in I, § 7*d* that all f-self-energies diverge at most logarithmically to any order of approximation and that the divergent term is independent of $\varkappa$. Consequently, other f-fields might compensate the divergences due to the e-field and that in a relativistically invariant way on account of (5). It cannot be ascertained without explicit calculation, though, whether the corresponding "convergence relations" have any physical meaning. Indeed the condition for convergence of the *n–th* approximation will have the form

$$e^2 = a_n f^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In order to be consistent, a theory of this kind imposes two conditions on the a_n:

1. They should be independent of n.
2. a_n should be positive.

A general criterion for these conditions to be satisfied would require the knowledge of the connections between the e-field and the f-fields of various types, with regard to the coefficients of their respective logarithmic divergences of any order; on account of their $\varkappa$-independence, it would be sufficient to establish such connections between the various "photon" fields. I have not succeeded in arriving at general results.

On the other hand, one can, as an orientation, perform explicit calculations to a certain approximation. This might, in any case, be instructive with regard to f-fields which have to be excluded. Such computations have been performed in I for the first order approximation of the self-energy. Let us first consider the f_s-case. From I, table 3, it is seen that the log f_s-divergence compensates the log e-divergence if

$$e^2 = \tfrac{1}{2} f_s^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The factor $\frac{1}{2}$ provides a striking example for the fact that this theory stands in no correspondence with STÜCKELBERG's classical theory: In the latter, the self-energy due to e- and scalar f-field for a point electron at rest is finite, as we have seen in § 1, eq. (4), if $e^2 = f^2$.

It would be of great interest to know whether this factor $\frac{1}{2}$ also occurs in the higher order approximations, i.e. whether the log divergence in *n–th* approximation stands multiplied with $(e^2 - f^2/2)^n$.

The other f-fields, envisaged here, yield, according to I, table 3:

f_{pv}-case, (pseudovector theory with pseudovector interaction): $a_1 = 5/3$,
f_{ps}-case, (pseudoscalar „ „ pseudoscalar „): $a_1 = -1/6$.

Thus, while the second one has to be discarded, the first one also may be taken, at all events in first approximation, to obtain convergence.

It has been shown in I, that the self-energy due to g-interactions involves, already in first approximation, a qu and a log divergence (on hole theory). As the e-self-energy has a log divergence only, this means that, in order to get a finite self-energy in first approximation, it would be necessary to introduce at least two g-fields; on account of its greater intricateness we therefore discard the possibility of introducing simultaneously various g-fields to ensure convergence.

Finally, one may inquire into the possibility of establishing convergence relations on one-electron theory: though we have no knowledge of the divergence properties of the higher order approximations in this case, it might anyhow be interesting to find out whether convergence in first approximation is possible. That this is not the case can be found from a closer examination of I, tables 1 and 2, however, from which it is seen, on one-electron theory: first, that $W^{(1)}_{\text{inv}}$ and $W^{(1)}_{\text{fluct}}$ (the sum of which is equal to $W^{(1)}$) have, for given type of field and interaction, divergences of different kinds which means that we have to look for combinations leading to compensations of the divergences of $W^{(1)'}_{\text{inv}}$ and of $W^{(1)}_{\text{fluct}}$ separately; secondly that all quadratic divergences occurring in the various $W^{(1)}_{\text{fluct}}$ have positive sign. As such a divergence especially occurs in the e-ase, it is impossible to establish, on one particle theory, convergence relations for the electron, whatever combination of fields is chosen.

Thus we arrive at the conclusion that only with a scalar f-field does one, at any rate in first approximation, obtain a finite self-energy on hole theory, but not on one-electron theory.

§ 9. *The universal length.*

The further physical implications of all "two field theories" essentially depend on the magnitude of $\varkappa$. Apart from the condition $\varkappa^{-1} \lesssim 10^{-13}$ cm, which has to be required because the pure e-theory at all events is known to hold well if only distances large compared with the "classical electron radius" are involved, the theory leaves the value of $\varkappa$ undetermined: the requirement of finite self-energy solely leads to a relation between the charges. In order to verify whether, as already asserted in § 2, W can indeed be considered as a perturbation if $\varkappa \sim 10^{13}$ cm^{-1}, we have to compute W in terms of the fine structure constant, $\varkappa$ and m.

But here we are faced with a difficulty: W is represented by an infinite series and we do not dispose of a method for generally determining its n-*th* term, so that, even apart from the still open question, whether (9) serves to cancel the higher order divergences, we cannot ascertain the convergence of the series as a whole. However, we will for the moment make the perhaps too optimistic assumption that the first order contribution roughly determines the order of magnitude of the total self-energy:

$$W \sim W^{(1)} = (W^{(1)}_e + W^{(1)}_{f_s}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From the expressions given in I, table 1, one obtains after some simple transformations (the charges have been expressed in Heaviside units),

$$\left.\begin{aligned} W_e^{(1)} &= D-\frac{e^2}{16\pi^2\hbar c}\cdot mc^2;\ D=\frac{3e^2}{8\pi^2\hbar c}\cdot\lim_{P\to\infty} lg\,\frac{P+\sqrt{P^2+m^2c^2}}{mc}\cdot mc^2,\\ W_{f_s}^{(1)} &= -D-\frac{f_s^2}{32\pi^2\hbar c}\cdot mc^2\left[1-\xi^2-(6\xi^2-\xi^4)\,lg\,\xi-\right.\\ &\qquad\left.-4y^4\left\{C(y^2,\xi)-\left(1-\frac{2}{\xi^2}\right)C(y^2,1)\right\}\right], \end{aligned}\right\}\quad(11)$$

with

$$\xi=\frac{\hbar\varkappa}{mc}\quad,\quad y^2=\frac{\xi^4}{4}-\xi^2,$$

$$C(\lambda^2,\mu)=\text{Principal value of}\int_0^\infty\frac{dx}{(x^2-\lambda^2)\sqrt{x^2+\mu^2}}$$

$$=-\frac{1}{2\lambda\sqrt{\lambda^2+\mu^2}}\,lg\,\frac{\sqrt{\lambda^2+\mu^2}+\lambda}{\sqrt{\lambda^2+\mu^2}-\lambda}.$$

Here y is understood to be ≥ 0, i.e. $\xi\geq 2$; the principal value integrals occur because, if $\xi\geq 2$, there is a particular intermediate state with negative energy, for which the energy denominator in the well known formula for the second order energy perturbation becomes zero; the momentum of the electron in this state is just $y.mc$. The condition $\xi\geq 2$ means that the rest-energy of the f-quantum should at least be 2 mc^2, the breadth of the energy zone, which separates positive and negative electron levels; cf. I, § 4*b*. After some transformations, the finite part of $W_{f_s}^{(1)}$ can be written as

$$-\frac{f_s^2}{32\pi^2\hbar c}\cdot mc^2\left[1-\xi^2+(\xi^4-6\xi^2)\,lg\,\xi+y\,(\xi^2-4)\,lg\,\frac{\xi^2-2y}{\xi^2+2y}\right].\quad(12)$$

On developing the logarithm one then finds

$$W^{(1)}\approx\frac{3f_s^2}{16\pi^2\hbar c}\cdot mc^2\left[lg\,\xi-\tfrac{1}{2}+O\left(\frac{lg\,\xi}{\xi^2}\right)\right]-\frac{e^2}{16\pi^2\hbar c}\cdot mc^2.\quad(13)$$

which, according to (9) , is $\approx 0{,}01_5\ mc^2$ if $\varkappa\sim 10^{13}\ \text{cm}^{-1}$. This is the result quoted and partly discussed in § 2. In order to examine whether the occurrence of the lg ξ-term in (13) is specific for the *scalar* field, we have computed the finite parts of the self-energies due to the other f-interactions. The results are:

$$\text{vector}\quad-\frac{f_v^2}{16\pi^2\hbar c}\cdot mc^2\left[1-\xi^2+\xi^4\,lg\,\xi+y\,(\xi^2+2)\,lg\,X\right]$$

$$\text{ps. vector} - \frac{f_{pv}^2}{16\pi^2\hbar c} \cdot mc^2 \left[1 - \xi^2 + (\xi^4 - 8\xi^2)\, lg\, \xi + y\,(\xi^2 - 6)\, lg\, X \right]$$

$$\text{ps.scalar} - \frac{f_{ps}^2}{32\pi^2\hbar c} \cdot mc^2 \left[1 - \xi^2 + (\xi^4 - 2\xi^2)\, lg\, \xi + y\xi^2\, lg\, X \right]$$

where the argument X of the logarithm is the same as that of the last term of (12). Developing gives

$$\left.\begin{matrix} \text{scalar} \\ \text{vector} \\ \text{ps. vector} \\ \text{ps. scalar} \end{matrix}\right\} W^{(1)} \sim \frac{3f^2}{8\pi^2\hbar c} \cdot mc^2 \cdot lg\, \xi \cdot \left\{\begin{matrix} \frac{1}{2} \\ -1 \\ \frac{6}{5} \\ -\frac{1}{6} \end{matrix}\right\}. \quad . \quad . \quad (14)$$

Thus in all cases the lg ξ-term is the principal term; the same has been verified to be the case for g-interactions. Another feature of the regularities of the self-energies on hole theory is demonstrated by comparing the coefficients behind the brace with the constants a_1 of § 9: the ratio of the finite parts of the self-energy is the same as that of the corresponding f^2 to e^2 in the convergence relations.

We should like to point out that it is by no means trivial that the self-energy can be considered as a small perturbation for the conveniently chosen order of magnitude of $\varkappa$. For if we would have taken $\varkappa$ to be, say, $\sim 10^{20}$ cm^{-1}, W would have been $\sim 0{,}1$ mc^2, which is already a very inconvenient value, as it would necessitate us to distinguish quantitatively between experimental mass and m. In fact, only the two extremely opposite points of view: either all mass is due to field self-energy, or the latter is small, lead to a theory involving only one mass characteristic for the electron itself, as has already been pointed out in § 2. It seems therefore gratifying that the "perturbation method" is characterized by $e^2/4\pi$ mc^2 being roughly the upper limit for $\varkappa^{-1}$, while it is at the same time its lower limit on the general physical grounds discussed in § 1, so that we may state: a consistent theory in which the self-energy can be considered as a small perturbation compared with mc^2 is possible if, and only if, $\varkappa^{-1}$ has roughly the same order of magnitude as the classical electron radius.

The results obtained can therefore be summarized as follows: The electron has, at all events in first approximation, a finite self-energy if we assume it to be the source of a scalar field of short range, besides the electromagnetic field; the higher order divergences of the f-field are, like those due to the e-field, all of logarithmic type, but we have not been able to prove that these divergences generally compensate each other. The theory is non-unitary in the sense expounded in § 2, the field self-energy is considered as a small perturbation, which is possible if the range of the field, which is the new fundamental constant of the theory, is chosen to be roughly of the same order as the classical electron radius. Further, this

theory does not exhibit any correspondence with a theory in which the self-energy singularities are eliminated by means of classical considerations and where the scheme thus obtained is subsequently subjected to quantization, because the two fundamental problems to which the present theory gives rise: the determination of a "convergence condition" and of a relation between m, $\varkappa$ and the fine structure constant, require for their solution that one from the outset shall take into account the interaction of the electron with the infinite vacuum distribution of electrons and with the e- and f-"zero fields".

Appendix.

Force on surface element of finite electron in a classical picture.

We take

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = -ct;\ \partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

The field equations of the scalar theory are

$$\Gamma_\mu = -\partial_\mu \Omega,\ \partial_\mu \Gamma^\mu = \varkappa^2 \Omega - R.$$

The first set defines the f-field vector as four-gradient of the scalar potential Ω. R is the "source". The energy momentum tensor is

$$\mathfrak{T}_{\mu\nu} = -\Gamma_\mu \Gamma_\nu + \tfrac{1}{2} g_{\mu\nu} (\Gamma_\varrho \Gamma^\varrho + \varkappa^2 \Omega^2).$$

Its divergence, the space components of which represent the f-force density, is

$$\partial_\nu \mathfrak{T}^{\mu\nu} = -\Gamma^\mu R.$$

The field equations of the vector field of short range are

$$F_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu,\ \partial_\mu F^{\mu\nu} = \varkappa^2 \varphi^\nu - s^\nu$$

φ_μ is the four-vector potential, s_μ is the "charge-current density". The energy momentum tensor is

$$\mathfrak{T}_{\mu\nu} = -F_{\mu\varrho} F_\nu^{\ \varrho} - \varkappa^2 \varphi_\mu \varphi_\nu + \tfrac{1}{4} g_{\mu\nu} [F_{\varrho\sigma} F^{\varrho\sigma} + 2\varkappa^2 \varphi_\varrho \varphi^\varrho],$$

hence

$$\partial_\nu \mathfrak{T}^{\mu\nu} = F^{\mu\nu} s_\nu.$$

The second member of this equation does not explicitly depend on $\varkappa$ and is therefore also obtained on e-theory. The components of the total force density therefore are

$$\left.\begin{array}{ll} \text{on STÜCKELBERG's theory:} & F^{\mu\nu}_{(0)} s^{(0)}_\nu - \Gamma^\mu R \\ \text{on BOPP's theory:} & F^{\mu\nu}_{(0)} s^{(0)}_\nu - F^{\mu\nu}_{(\varkappa)} s^{(\varkappa)}_\nu \end{array}\right\} \mu = 1, 2, 3.$$

The quantities marked with (0) refer to the e-field ($\varkappa = 0$), those with ($\varkappa$) to the subtractive vector f-field. The difference in relative sign of e- and f-force density in BOPP's theory occurs because there the f-energy momentum tensor is to be subtracted from the e-tensor.

In the case of a charge distribution which is at rest, these expressions become, in obvious vector notation

$$\varrho_{(0)}\vec{E}_{(0)} - \vec{\Gamma}R \quad , \quad \vec{\Gamma} = -\operatorname{grad}\Omega,$$

$$\varrho_{(0)}\vec{E}_{(0)} - \varrho_{(\varkappa)}\vec{E}_{(\varkappa)} \quad , \quad \vec{E} = -\operatorname{grad} V,$$

with

$$\triangle\,\Omega - \varkappa^2\Omega = -R',$$

$$\triangle\,V_{(\varkappa)} - \varkappa^2\,V_{(\varkappa)} = -\varrho_{(\varkappa)},$$

$$\triangle\,V_{(0)} = -\varrho_{(0)}.$$

R' is the function in which R goes over for zero velocity. Taking a spherical electron with finite radius r and with uniform superficial charge density $\varrho_{(0)} = e/4\pi r^2$, the e-force on a surface element $d0$ is, taking for the field strength at the surface the mean of its outer and inner value:

$$\frac{1}{2}\cdot\frac{e\vec{r}}{4\pi r^3}\cdot\frac{e}{4\pi r^3}\cdot dO.$$

Supposing further the electron to have a uniform superficial f-charge density $f/4\pi r^2$, due to which it is the source of either scalar or subtractive vector f-field, we have for Ω (scalar case) as well as for $V_{(\varkappa)}$ (vector case) in a point at a distance ξ from the electron centre

$$\frac{f}{4\pi}\cdot\frac{e^{-\varkappa r}}{r}\cdot\frac{\sinh\varkappa\xi}{\varkappa\xi},\ \xi\leqslant r\ ;\ \frac{f}{4\pi}\cdot\frac{e^{-\varkappa\xi}}{\xi}\cdot\frac{\sinh\varkappa r}{\varkappa r},\ \xi\geqslant r.$$

So the corresponding field strengths are

$$\frac{f\vec{\xi}}{4\pi\xi^2}\left[\frac{\sinh\varkappa\xi}{\varkappa\xi} - \cosh\varkappa\xi\right]\frac{e^{-\varkappa r}}{r},\ \xi < r$$

$$\frac{f\vec{\xi}}{4\pi\xi^3}\,[1+\varkappa\xi]\,e^{-\varkappa\xi}\cdot\frac{\sinh\varkappa r}{\varkappa r},\ \xi > r.$$

Therefore the total force on a surface element of the electron becomes, in STÜCKELBERG's as well as in BOPP's theory

$$\frac{1}{2}\cdot\frac{\vec{r}}{4\pi r^3}\cdot\frac{dO}{4\pi r^2}\,\{e^2 - f^2\,g\,(\varkappa r)\}$$

with

$$g(\varkappa r) \equiv \frac{1-(1+\varkappa r)\, e^{-2\varkappa r}}{\varkappa r} \to 1 \text{ if } r \to 0.$$

This is the result quoted in § 1.

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CHAPTER III.

On the self-energy of nucleons and the theory of nuclear forces.

Summary.

§ 1. Introduction. — § 2. Statement of the problem. — § 3. The proton-neutron convergence relations; their connection with the electron convergence relation. — § 4. The mass difference of proton and neutron. — § 5. The nucleon convergence relations. — § 6. Introduction of the F-field. — § 7. The magnitude of the mesic self-energy. — § 8. Concluding remarks.

§ 1. *Introduction.*

In Chapter I (referred to as I in the following), the self-energy properties have been examined of particles with spin $\frac{1}{2}$ which are the source of one or more neutral or charged fields of either scalar (s), vector (v), pseudovector (pv) or pseudoscalar (ps) type. For the present purposes, some of the main results obtained in I will here be summarized.

a) The self-energy W due to an arbitrary field and to either f- or g-interaction (for this notation see I, § 1), can be developed in an infinite series with respect to a dimensionless constant "(charge)$^2/\hbar c$". Calling the *n–th* term in the development for a particle with momentum $\vec{p}$ $W^{(n)}(\vec{p})$, we have (cf. I, § 1)

$$W(\vec{p}) = \sum_{n=1}^{\infty} W^{(n)}(\vec{p}); \quad W^{(n)}(\vec{p}) = W^{(n)} \sqrt{1-\beta^2}, \quad . \quad . \quad . \quad (1)$$

$W^{(n)}$ referring to zero momentum. (1) is true in the "one particle theory" as well as in the "hole theory" interpretation of the negative energy levels of the spin $\frac{1}{2}$ particle concerned. According to (1), the divergence properties of the self-energy can uniquely be found from the case that the particle is at rest; in what will follow we can therefore, without loss of generality, confine ourselves to the problem for $\vec{p} = 0$.

Whereas (1) is independent of the alternative points of view which may be held with regard to the occupation of the vacuum levels, the further general results, to the survey of which we now turn, only apply to hole theory. As this case will mainly be dealt with in the present chapter, we will, if not explicitly stated otherwise, understand by $W^{(n)}$ the *n–th* order self-energy under the assumption that in vacuum all negative energy levels are completely occupied.

b) It has been found in I, § 7, that, whether the interaction concerned be of the neutral or the charged type, or a mixture of both, all f-self-

energies diverge at most logarithmically to any order of approximation, while the situation with respect ot $W_g^{(n)}$ is fundamentally more complicated. This finds its origin in that the "g-sources" are proportional to a coupling constant which, unlike in the f-case where this constant has the dimensions of a charge (called f), here has the dimensions of charge (g) times length. In particular it was found that, generally, $W_g^{(1)}$ exhibits a quadratic and a logarithmic divergence. Furthermore, if both f- and g-interaction of the same field type are present, there occur, in the v- and ps-case, self-energy terms proportional to fg, which also diverge quadratically and logarithmically. Finally, it has been found in I that terms of direct g-interaction lead to self-energy contributions which, in first approximation, again yield divergences of the two above mentioned types. For this notion of "direct self-energy" cf. esp. I, § 6, 4°.

c) It has been explained in Chapter II, (further on referred to as II) how the property of $W_f^{(n)}$ to diverge at most logarithmically, irrespective of the field type, may lead to a finite self-energy of the electron on the assumption that this particle creates, besides the electromagnetic (e-) field, another, or in principle more than one other, f-field of short range, provided that a certain relativistically invariant relation between e and the charges f, which we call convergence relation, can be satisfied. As this relation generally is of the form

$$F^{(n)}(e^2, f_s^2, \ldots, f_{ps}^2) = 0, \quad \ldots\ldots\ldots \quad (2)$$

(n) denoting the order of $W^{(n)}$, we may state the conditions for (2) to be satisfied as follows: first, the coefficients of the various powers of $e, \ldots f_{ps}$ should be such as to allow real values for $e, \ldots f_{ps}$ *); secondly, the relations (2) for $n = 1, 2, \ldots$ should be mutually compatible. If, as has been done in II, only one f-field is introduced besides the e-field, $F^{(1)}$ becomes (cf. II § 8)

$$e^2 + a_i f_i^2 = 0, \quad i = s, v, pv \text{ or } ps \ldots\ldots\ldots \quad (3)$$

and the requirements, just mentioned, now imply: first, $a_i < 0$; secondly, (3) substituted in $F^{(n)}$, $n > 1$, should yield an identity. Whereas a_i can be determined by calculating explicitly the first order self-energy concerned, which led to the result that $a_i < 0$ if $i = s, pv$ and $a_i > 0$ if $i = v, ps$, the investigation of the second condition is hampered by the difficulty that we do not dispose of a method enabling us, without the impracticable computation of $W^{(n)}$ for any n, by means of perturbation theory, to find an explicit expression for the coefficient of its divergent part. Thus we had to content ourselves with the establishment of (3), and to leave open the question, whether (3) with $a_i < 0$ suffices to eliminate divergence of higher order.

*) $F^{(n)}$ is a homogeneous polynomial in the charges of degree $2n$.

d) It is an essential feature of the present quantum field theories (cf. II, § 2), that they are necessarily non-unitary, i.e. that the wave equation of the particles cannot be obtained from the equations governing the behaviour of the fields they create. It has been shown in II how, consequently, W depends, besides on $e^2/\hbar c$ and $f^2/\hbar c$ (the latter being eliminable by (3) in the case of one f-field) and on the f-field range, also on the mass parameter M occurring in the Dirac equation. It was then remarked that, in principle, it is possible to give a unitary treatment of the electron mass m, *viz.* by considering it to be a priori undetermined and then to be fixed by the implicit equation $W = mc^2$. However, arguments were given in II, § 2 for the inapplicability of this method which was therefore discarded in favour of the alternative way of treating the field self-energy, *viz.* as a small perturbation compared to mc^2, m being the mass of the electron as occurring in the Dirac equation. This appeared possible only if the range of the f-field is roughly of the same order as the classical electron radius, cf. II, § 9.

§ 2. *Statement of the problem.*

It is the aim of the present chapter to discuss the self-energy problem for the "heavy Dirac particles": proton and neutron on similar lines as has been done for the electron *).

It is well known how the idea, originally put forward by HEISENBERG [1]) that protons and neutrons are the sole constituents of the atomic nucleus, has led to a rapid advance of our theoretical knowledge concerning nuclear physics. In order to understand such fundamental features as the β-transitions and the exchange properties of the neutron proton interaction, it is, on the proton-neutron picture of the nucleus, convenient to consider proton and neutron as different states, with respect to an e-charge coordinate, of one and the same particle, now generally called nucleon. To this concept, which at present underlies all considerations within the nuclear domain, we will in this paper refer as "nucleon concept".

The theory of nuclear forces, after passing through a half-phenomenological stage, was founded on the concept of interaction through the intermediary of fields by the remark of YUKAWA [2]) that a complex field (now called meson field) satisfying the relativistic Schrödinger–Gordon equation leads to a short range exchange force between nucleons. Since 1938 the various a priori possible [3]) meson field theories have been the subject of many detailed investigations. Though the question which meson field or combination of fields is the most suitable to account, if possible,

*) As the wave function of the proton satisfies the Dirac equation, this implies, in the hole theory interpretation, the existence of its anti-particle. The anti-proton has never been observed experimentally, but this in itself cannot be considered as a serious argument against the assumption that the proton is a spin ½ particle, as, on account of its large mass, the cross-section for proton-anti-proton pair formation by a photon (of energy $\gtrsim 1{,}8 \cdot 10^9$ eV) passing through matter is very small compared to the probability for negaton-positon formation by a photon of the same energy.

for all phenomena of short range nuclear interaction, as well as to establish the generally assumed connection with the properties of the penetrating component of cosmic radiation, is far from settled [4]), the basic idea of employing meson fields for this purpose anyhow seems fertile. It is, however, beyond the present purposes to enter into a detailed discussion of the merits and defects of the various current meson theories of nuclear forces.

We can thus, proceeding on similar lines as in II, confront these theories with the self-energy problem of the nucleons: Indeed, we have found in the case of the electron that, with certain combinations of fields of the types $s, \ldots, ps$ correspond, at all events in first approximation, finite self-energies of the particle by which they are created. The assumption of a short range field being produced by the electron constitutes an essential new feature within the framework of the present picture of elementary particles, but in the nucleon case the situation is different as there the existence of one or more short range fields has already been postulated in order to account for the nuclear forces. Thus, the idea of attaining finite self-energies by means of suitable combinations of fields naturally leads one to inquire whether any of the (mixtures of) meson fields, may, besides describing more or less satisfactorily the nuclear interaction phenomena, moreover serve to obtain finite nucleon self-energies. From this point of view we will discuss the current meson theories of nuclear forces in section 5; however one remark of general order may already here be made on this subject:

Whereas we have already seen in connection with (2) that the convergence relations of various order, for a given type of divergence, should be mutually compatible, we meet in the case of g-self-energies with a new feature: according to I, § 7*e*, every new approximation will generally involve convergence relations for divergences of new order, or, in other words, the number of convergence relations will increase the further the approximation is pushed. Therefore, the compatibility must now be required of the convergence relations for the divergences of various types. On account of the difficulties referred to in § 1, it does, with the present methods, not seem feasible to attain to decisive conclusions on this question. Anyhow, we shall confine ourselves in this chapter to the consideration of the first order convergence relations, thereby making use of the explicit expressions for the first order self-energies obtained in I *).

*) Due to their interaction with the electron-neutrino field, the nucleons also have an "electron-neutrino self-energy". Whether this interaction is "direct" or brought about through the intermediary of mesons, the lowest self-energy contribution is of the second order, viz. proportional to the square of the "electron-neutrino charge" times the square of the mesic charge. Thus the first order convergence relations do not involve the electron-neutrino charges of the nucleons. As these quantities are very small compared to the mesic charges, it is, for the rest, improbable that this self-energy will, in any theory, play an essential rôle, as regards convergence as well as their contribution to a finite nucleon self-energy.

Before we are able to turn to the problem of the connection between theories of nuclear forces and the concept of convergence relations, another problem necessarily has to be tackled first, however: in fact, the nucleon concept, in its form enunciated above is incompatible with the very notion of convergence relations. This can be seen as follows:

As we know (see I) that the e-self-energy leads to a log divergence in first approximation, there is certainly a relation

$$e^2 + X = 0$$

which has to be fulfilled to ensure convergence of the first order proton self-energy *); X denotes the totality of terms arising from the log divergences due to the proton creating a certain set of meson fields. Thus X depends on the particular set of fields chosen; but whatever this choice may be, there will be, as a consequence of the previously given form of the nucleon concept, a corresponding relation

$$X = 0,$$

necessary *) to let the first order neutron self-energy be convergent. And so neutron and proton relation are clearly incompatible. Thus, if we assume that the finiteness of the proton and neutron self-energy is brought about by convergence relations, the nucleon concept must be restated in the following way:

> Proton and neutron are, with respect to the electric charge and at least one other parameter, different states of one and the same particle.

Therefore, the present problem can be stated more precisely as follows:

α) The compatibility has to be required of the convergence relations for proton and neutron. In § 3 it is shown that this can be fulfilled by assuming the proton to be the source of the same scalar f-field as supposed in II to be created by the electron.

β) Is it possible, once the compatibility mentioned in α has been established, to choose a mixture of meson fields responsible for nuclear interaction in such a way that the mesic nucleon charges satisfy certain convergence relations, while, moreover, the fields considered allow of the interpretation of — to put it mildly — a reasonable number of nuclear and cosmic ray phenomena? It may be noted that, if such a set of fields can be found, the existence of convergence relations leads to a reduction of the number of constants of the theory **).

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*) This relation is necessary but not sufficient, as we shall see in § 3.

**) It should be remarked at the outset that there is at least one discrepancy between current theory and experiment on which the present considerations can throw no new light: as has been pointed out by YUKAWA [5], it is inconceivable that the experimentally found order of magnitude of the cross-section for the scattering of fast mesons by nucleons can be accounted for by the assumption that it is some mixture of meson fields and not a single one which is responsible for nuclear interaction.

The total Hamiltonian governing the behaviour of the nucleons and their surrounding fields can be written as

$$H = H_{\text{mat}} + H_{\text{fields}} + H_{\text{coupling}}$$

H_{fields} being that part of H which only involves variables of the fields created by the nucleons, while H_{coupling} depends both on the nucleon and on the field variables. For H_{mat} we take

$$H_{\text{mat}} = \int \psi^{\dagger} \left(\frac{\hbar c}{i} \vec{\alpha} \vec{\nabla} + \beta Mc^2 \right) \psi \, dv.$$

Thus we let only *one* "mechanical" mass M enter explicitly in H, i.e. we do not introduce different mechanical masses for neutron and proton. This departure from the customary treatment finds its justification in the following fact: the *effective* mass M_P for proton (M_N for neutron) is now defined as the expectancy value of H, for the proton (neutron) state of the particle present, in terms of the mechanical mass:

$$M_P c^2 = (Mc^2 + W_P)$$
$$M_N c^2 = (Mc^2 + W_N), \text{ but unequal to } M_P c^2,$$

due to the fact that W_P and W_N, the expectancy values of H_{coupling} for the proton and neutron state respectively, are not identical. Now it will be shown in § 4 that $\triangle$, the empirical mass difference of proton and neutron can, to sign as well as to order of magnitude, just be accounted for by the difference of W_P and W_N (and that in first approximation independently of the particular choice of meson fields responsible for nuclear interaction). It will be clear beforehand that the scalar f-field created by proton as well as by electron (see above) plays into this result; indeed it appears from the considerations of § 4 that $\triangle$ is equal to about minus two electron masses if the range of the f-field has, not roughly as was already implied by the results of II. § 9, but rather precisely the order of magnitude of the classical electron radius. These results therefore corroborate the self-energy treatment of the electron given in II, the whole scheme shows a satisfactory consistency.

Furthermore, it will appear in § 7 that the self-energy due to the meson fields cannot be adapted to be made either large or small — as could a priori be done for the f-field self-energy of the electron — by suitably fixing the ranges $1/\varkappa_l$ of the fields. After having eliminated the divergent parts by means of the convergence relations, we have, in fact, to insert into the remaining finite part the experimentally known order of magnitude of the $\varkappa_l$ and to see what comes out. Now it appears that in all cases, for f_l- as well as for g_l-interactions, the result is that the contribution is $\ll Mc^2$.

Thus, while with regard to the magnitude of the self-energy one could,

in the electron case, consider a priori two possibilities: either the electron mass is wholly due to the self-energy, or the latter is a small perturbation compared to the former, in the nucleon case the situation is different: here the self-energy simply turns out to be a perturbation. This means that the mechanical mass is practically equal to the experimental proton (and neutron) mass.

As regards nuclear interaction problems, the occurrence in H_{mat} of M instead of both M_P and M_N does not lead to any material modification. For it means that the most convenient system in e.g. the deuteron problem is that in which the centre of gravity of the mechanical masses is at rest. But this system is, for all means and purposes, identical with the actual centre of gravity system; in fact, calculations are usually performed in the first mentioned system.

We now turn to a general discussion of the compatibility of proton and neutron convergence relations (problem α), after which in § 4 an expression for the mass difference of proton and neutron is derived. We then are able to examine the nucleon convergence relations as a whole (problem β): § 5. The current theories are found not to satisfy the convergence requirements. It is then proposed (§ 6) to introduce, somewhat similarly to the solution of the difficulties for the electron, advocated in II, a neutral scalar field to ensure convergence without influencing the main aspects of the interaction problem. It appears from the convergence relations that this field is more strongly coupled with the nucleons than the other meson fields, from which it is inferred that its range is shorter than those of the other fields. Possibilities of experimental tests of this assumption are considered. § 7 consists of the discussion of the magnitude of the mesic self-energy of the nucleons, while § 8 reviews the situation.

§ 3. *The proton-neutron convergence relations; their connection with the electron convergence relation.*

In first approximation, an arbitrary mixture of f- and g-interactions gives rise to a quadratic and a logarithmic divergence. Consequently, there are two convergence relations for the proton as well as for the neutron which generally can be written as

$$\left.\begin{aligned} e^2 + L^\nu(f,g,\varkappa,n) + L^\gamma(f,g,\varkappa,n) &= 0 \\ Q^\nu(g,\varkappa,n) \quad + Q^\gamma(g,\varkappa,n) \quad &= 0 \end{aligned}\right\}\ \text{proton} \quad \left\{\begin{matrix}(4\,P)\\(5\,P)\end{matrix}\right.$$

$$\left.\begin{aligned} L^\nu(f,g,\varkappa,n) + L^\gamma(f,g,\varkappa,n) &= 0 \\ Q^\nu(g,\varkappa,n) \quad + Q^\gamma(g,\varkappa,n) \quad &= 0 \end{aligned}\right\}\ \text{neutron} \quad \left\{\begin{matrix}(4\,N)\\(5\,N)\end{matrix}\right.$$

In $(4P)$, which serves to eliminate the logarithmic divergence in the proton self-energy, L^ν is a function of the constants $f, \ldots, n$ referring to the neutral meson fields, while L^γ is the same function of the constants of the charged fields, as follows from the connection between $W^{(1)\nu}$ and $W^{(1)\gamma}$ found in I, § 1. The fields and direct interactions of which $f, \ldots n$ are representative

are those responsible for nuclear interaction (for the precise definition of the constants n see I, eq. (58), (58a), (58b)). Besides these, the proton also creates an e-field, due to which a term e^2 will occur in the convergence relation for the logarithmic, but not for the quadratic divergence (5P), as we know that the e-interaction, belonging to the f-group, leads to a logarithmic divergence only. In (5P) Q^ν and Q^γ are defined similarly to L^ν and L^γ. Furthermore, the nucleon concept implies, that, apart from the e-properties of the proton, neutron and proton are identical. Thus apart from a term e^2 missing in (4N), the convergence relations (4N), (5N) will be identical with (4P), (5P).

The incompatibility of (4P) and (4N) has already been alluded to in § 2, where it was stated that, besides the electric charge there must be another parameter which takes different values for the neutron and proton state of the nucleon.

Now there is a condition, not yet taken into account, which has to be imposed on the nucleon convergence relations: this will lead to the introduction of a parameter of the desired properties and, it may be noted, make the compatibility of the first order convergence relations independent of the particular choice of meson fields describing the nuclear forces. This condition is obtained by considering the mechanism by which, within the framework of the present theories, the processes of β-radioactivity can be described. These are twofold: first, it may be assumed that there exists a direct coupling between nucleon- and electron-neutrino field, giving rise to

$$P \rightarrow N + e^+ + n, \qquad N \rightarrow P + e^- + n' \quad . \quad . \quad . \quad . \quad (6)$$

where e^+ (e^-) is a positon (negaton) and n (n') an (anti)neutrino. Secondly, β-decay may be brought about by the disintegration of a positive or negative meson, emitted in an intermediate state by proton and neutron:

$$P \rightarrow N + Y^+, \quad Y^+ \rightarrow e^+ + n; \quad N \rightarrow P + Y^-, \quad Y^- \rightarrow e^- + n'. \quad (7)$$

Now it has been shown in II how the assumption that the electron has an f-coupling of the scalar type leads, at all events in first approximation, to a finite negaton and positon self-energy. Denoting the corresponding coupling constant by f (instead of f_s), the convergence relation for the electron is (cf. II, eq. (9))

$$e^2 - \tfrac{1}{2} f^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

obviously, the theory is independent of the relative sign of e and f. We choose e.g. the negaton to have charges $-e$, $-f$; consequently those of the positon are e, f. We now shall require: first, that for (6) and (7) not only the (already satisfied) conservation of e-charge, but also that of f-charge be true; secondly, the compatibility of proton, neutron *and* electron convergence relation. To examine this, we put the f-charge of neutrino, neutron and proton equal to $\lambda_n f$, $\lambda_N f$ and $\lambda_P f$ respectively, where

λ_n, λ_N and λ_P may take the values ± 1 or zero. The first mentioned condition yields

$$\lambda_P = \lambda_N + \lambda_n + 1,$$

the second

$$1 - \lambda_P^2 + \lambda_N^2 = 0.$$

Elimination of λ_P gives

$$\lambda_N = \tfrac{1}{2}\left\{\frac{1}{\lambda_n + 1} - (\lambda_n + 1)\right\} = 0, \pm 1, \ldots$$

with the unique solution $\lambda_n = \lambda_N = 0$, so $\lambda_P = 1$. Thus our two conditions suffice to determine uniquely the f-charges of all particles concerned, also that of the meson: $Y^{\pm}$ has the charge $\pm (1 + \lambda_n) f = \pm f$.

Summarizing the results we have found: for all electrically charged elementary particles, an f-charge is inseparably connected with the e-charge. The compatibility of the convergence relations of proton and neutron is, independent of specific assumptions on nuclear interaction, expressed by (8), which is at the same time the convergence relation of the electron. The nucleons have, at all events in first approximation, a finite self-energy provided that

$$L^{\nu}(f, g, \varkappa, n) + L^{\nu}(f, g, \varkappa, n) = 0, \quad \ldots\ldots \quad (9)$$

$$Q^{\nu}(g, \varkappa, n) + Q^{\nu}(g, \varkappa, n) = 0, \quad \ldots\ldots \quad (10)$$

which denote two relations between the constants of the fields responsible for the nuclear interaction. Henceforth (9) and (10) will be referred to as "the" nucleon convergence relations.

In II it was found that the range $\varkappa^{-1}$ of the f-field is, in order that the self-energy of the electron shall be small compared to its rest energy, roughly $\sim 10^{-13}$ cm. This range is of course identically the same whether the f-field is created by electron, meson or proton. This shows a way to decide experimentally whether the f-field indeed exists, *viz.* by considering the cross-section for proton-proton scattering as compared with the proton-neutron scattering.

We next proceed to show that the nucleon concept, which now has to be stated as follows:

> Proton and neutron are, with respect to their e- and f-charge, different states of one and the same particle,

leads to a mass difference of proton and neutron which has the right sign and order of magnitude.

§ 4. *The mass difference of proton and neutron.*

Generally we may write for the self-energy of the

$$\text{proton} : W_P(e, f, \varkappa; \text{nucl}; M),$$
$$\text{neutron}: W_N(e, f, \varkappa; \text{nucl}; M),$$

where the assembly of parameters of the nuclear interaction fields, being the same for proton and neutron, has been written as "nucl". (It should be noted that, in a theory which is not purely neutral, W_N depends on e, f and $\varkappa$, too as a consequence of higher order transitions.) Hence the mass difference Δ of proton and neutron is given by

$$c^2 \Delta = W_P - W_N.$$

On account of the frequently mentioned inaccessibility of the higher order terms, we replace this relation by its first order approximation. We have

$$W_P^{(1)} = w^{(1)}(e, f, \varkappa, M) + W^{(1)}(\text{nucl}, M)$$
$$W_N^{(1)} = \qquad\qquad W^{(1)}(\text{nucl}, M),$$

where $w^{(1)}$ is the sum of e- and f-self-energy, while $W^{(1)}$ (nucl, M) denotes the contribution due to the nuclear fields of force and thus is the same for proton and neutron. Thus

$$c^2 \Delta \approx w^{(1)}(e, f, \varkappa, M) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

independent of the particular choice of meson fields responsible for nuclear interaction. We introduce the dimensionless quantity

$$\zeta = \frac{\hbar \varkappa}{Mc} = \frac{m}{M} \cdot \frac{\hbar \varkappa}{mc} = \frac{m}{M} \xi \approx 5.10^{-4} \xi \quad . \quad . \quad . \quad . \quad (12)$$

where m is the electron mass and ξ the same as in the formulae of II, § 9. Furthermore, we put

$$y^2 = \frac{\xi^4}{4} - \xi^2 > 0, \quad z^2 = \zeta^2 - \frac{\zeta^4}{4}.$$

y^2 is again the same as in II, § 9; we remind that $y^2 > 0$ if $\xi > 2$. Now let us suppose for a moment that also

$$\zeta > 2, \text{ i. e. } z^2 < 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

so, according to (12):

$$\xi > 4.10^3.$$

As $\zeta > 2$, we find the right member of (11) simply by replacing in II eq. (11):

m by M, ξ by ζ, y^2 by $-z^2$, $C(y^2, 1)$ by $C(-z^2, 1)$, $C(y^2, \xi)$ by $C(-z^2, \zeta)$.

Further, all results of II, § 10 can m.m. be taken over, so we get according to II (13)

$$w^{(1)} \sim \frac{3f^2}{16\pi^2\hbar c} \cdot Mc^2 \lg \zeta = \frac{3e^2}{8\pi^2\hbar c} \cdot Mc^2 \lg \zeta > 0 \,, \quad . \quad . \quad (14)$$

which would give the wrong sign of Δ. It is important to note that, moreover, the order of magnitude of Δ would be quite wrong if we would have put the self-energy of the electron equal to mc^2. For from (13), (14) and

$$\frac{3e^2}{8\pi^2\hbar c} \cdot mc^2 \lg \xi = mc^2$$

it would follow that $\Delta \sim M$!

The situation changes completely, however, if still taking $\xi > 2$, we put, instead of (13)

$$\zeta < 2, \text{ i. e. } z^2 > 0.$$

Then, according to (12)

$$\frac{1}{2} \cdot \frac{\hbar}{mc} > \frac{1}{\varkappa} > \frac{1}{2} \cdot \frac{\hbar}{Mc}, \text{ or } 10^{-11}\, cm \gtrsim \frac{1}{\varkappa} \gtrsim 10^{-14}\, cm.$$

More precisely, we can take as upper limit of $\varkappa^{-1}$ the order of 10^{-13} cm, for a larger $\varkappa^{-1}$ would be at variance with the harmony between experiment and a purely electromagnetic theory of the interaction between electrically charged particles in the corresponding domain of distances; thus we place $\varkappa^{-1}$ roughly between 10^{-13} and 10^{-14} cm.

To find $w^{(1)}$ we now have to replace in II, (11) *)

m by M, ξ by ζ, y^2 by $+z^2$, $C(y^2, \xi)$ by $C'(z^2, \zeta)$, $C(y^2, 1)$ by $C'(z^2, 1)$,

but the integrals $C'(z^2, \zeta)$ and $C'(z^2, 1)$ are now no longer, as in II, § 9, principal value integrals, for $\zeta < 2$:

$$C'(z^2, \zeta) = \int_0^\infty \frac{dx}{(x^2 + z^2)\sqrt{x^2 + \zeta^2}} \qquad C'(z^2, 1) = \int_0^\infty \frac{dx}{(x^2 + z^2)\sqrt{x^2 + 1}}$$

and Δ becomes (all charges are expressed in Heaviside units):

$$\left.\begin{aligned} c^2 \Delta = &- \frac{e^2}{16\pi^2\hbar c} \cdot Mc^2 - \frac{f^2}{32\pi^2\hbar c} \cdot Mc^2 \cdot \\ &\cdot \left[1 - \zeta^2 + (\zeta^4 - 6\zeta^2) \lg \zeta - 4z^4 \left\{ C'(z^2, \zeta) - \left(1 - \frac{2}{\zeta^2}\right) C'(z^2, 1) \right\}\right] \end{aligned}\right\} \quad (15)$$

*) This is most easily seen by noting that

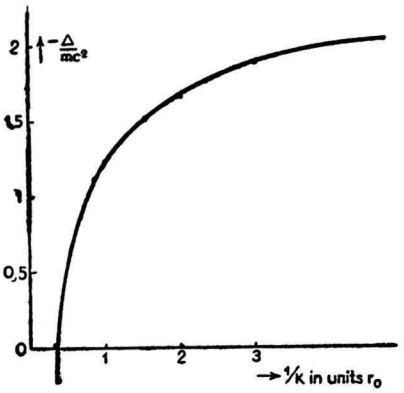
$$w^{(1)} = W_e^{(1)} + W_f^{(1)},$$

where $W_e^{(1)}$ is given by the first equation of II, (11). $W_f^{(1)}$ is given in I, table 1, *viz.* by taking together the "inv" and the "fluct" term for the f_s-case. As to the treatment of the integrals, we especially refer to I, § 4*b*.

As $z \leq 1$ and $z < \zeta$, we get, using (8):

$$\left.\begin{aligned} \Delta = -\frac{\alpha}{4\pi} \cdot M[1 + F(\zeta)],\ \alpha = \frac{1}{137}, \\ F(\zeta) = 1 - \zeta^2 + (\zeta^4 - 6\zeta^2)\, lg\, \zeta + 2z(\zeta^2 - 4)\ \text{arc tg}\ \frac{2z}{\zeta^2 - z}. \end{aligned}\right\} \quad (16)$$

In the accompanying figure, $-\Delta/mc^2$ has been plotted against the range of the f-field in units $r_0 = e^2/4\pi mc^2$. For an order of magnitude of $\varkappa^{-1}$ which is smaller than that of r_0, Δ is positive. It becomes negative for $\varkappa^{-1} = \frac{1}{3} r_0$ and has the value $-1{,}25\ mc^2$ for $\varkappa^{-1} = r_0$, $-1{,}90\ mc^2$ for



$\varkappa^{-1} = 3r_0$. For $\varkappa \to 0$ it tends to $-2{,}14\ mc^2$. Generally we may state that if $\varkappa^{-1} \approx r_0$, Δ has the right sign and order of magnitude, the experimental value being $-2{,}47\ mc^2$ *).

In II it was found that $\varkappa^{-1}$ should roughly be $\sim r_0$, to ensure that the electron self-energy shall be $\ll mc^2$, with which the present result, satisfactory in itself, is consistent.

We do not know whether the higher order contributions will be finite too, but, if so, it is reasonable to expect that they are small compared with $w^{(1)}$, as the fine structure constant $\alpha \ll 1$, so that the present result would indeed determine the order of magnitude of the whole effect to be expected theoretically. These higher order contributions then might possibly account for the fact that $|\Delta|$, according to the present first order computations, turns out to be somewhat too small.

Having thus shown that, at all events in first approximation, the possibility in principle exists of obtaining simultaneously a finite self-energy of proton and neutron, it remains to examine whether this possibility indeed can be realized, i.e., according to the program of § 2, whether the nucleon convergence relations (9) and (10) admit of solutions with which

*) This value for the mass difference is obtained from the determination of the deuteron binding energy by MYERS and VAN ATTA [4a]) combined with that of the mass difference of the hydrogen molecule and the deuterium atom by MATTAUCH [4b]). I should like to thank Dr. KUSAKA for kindly checking the calculation of Δ.

correspond mixtures of meson fields that are suitable for the description of nuclear interaction phenomena.

§ 5. *The nucleon convergence relations.*

According to (9) and (10), these are characterized by two functions L and Q of the parameters $f_i, \ldots, n_i$, from which L^ν, Q^ν and L^γ, Q^γ are found by putting $f_i = f_i^\nu, \ldots, n_i = n_i^\nu$ and $f_i = f_i^\gamma, \ldots, n_i = n_i^\gamma$ respectively. From I, table 3, we immediately obtain:

$$\left.\begin{aligned} L = &-\tfrac{1}{2} f_s^2 + f_v^2 - \tfrac{5}{3} f_{pv}^2 + \tfrac{1}{6} f_{ps}^2 + \tfrac{4}{3} g_s^2 \eta_s^2 - \tfrac{1}{3} g_v^2 \eta_v^2 (1 - \tfrac{9}{2} \eta_v^{-2}) + \\ &+ \tfrac{1}{3} g_{pv}^2 \eta_{pv}^2 (5 - \tfrac{3}{2} \eta_{pv}^{-2}) - \tfrac{2}{3} g_{ps}^2 \eta_{ps}^2 (1 - \eta_{ps}^{-2}) - 2 f_v g_v \eta_v (1 + \eta_v^{-2}) - \\ &- \tfrac{2}{3} f_{ps} g_{ps} \eta_{ps} (1 + \eta_{ps}^{-2}) + \tfrac{4}{3} n_s g_s^2 \eta_s^2 - 2 n_v g_v^2 \eta_v^2 + \cdot \\ &+ 2 n_{pv} g_{pv}^2 \eta_{pv}^2 - \tfrac{4}{3} n_{ps} g_{ps}^2 \eta_{ps}^2 \\ Q = &-g_s^2 \eta_s^2 + \tfrac{1}{4} g_v^2 \eta_v^2 - \tfrac{5}{4} g_{pv}^2 \eta_{pv}^2 + \tfrac{1}{2} g_{ps}^2 \eta_{ps}^2 + \tfrac{3}{2} f_v g_v \eta_v + \\ &+ \tfrac{1}{2} f_{ps} g_{ps} \eta_{ps} - n_s g_s^2 \eta_s^2 + \tfrac{3}{2} n_v g_v^2 \eta_v^2 - \tfrac{3}{2} n_{pv} g_{pv}^2 \eta_{pv}^2 + n g_{ps}^2 \eta_{ps}^2, \end{aligned}\right\} \quad (17)$$

in which

$$\eta_i = \frac{M}{\mu_i} \quad \ldots\ldots\ldots\ldots \quad (17a)$$

We may replace (9) by $(L_\nu + \frac{4}{3} Q_\nu) + (L_\gamma + \frac{4}{3} Q_\gamma) = 0$. For this purpose a function $L' = L + \frac{4}{3} Q$ must be introduced. As L will henceforth no more occur, we write L instead of L' and have

$$\left.\begin{aligned} L = &-\tfrac{1}{2} f_s^2 + f_v^2 - \tfrac{5}{3} f_{pv}^2 + \tfrac{1}{6} f_{ps}^2 + \tfrac{3}{2} g_v^2 - \tfrac{1}{2} g_{pv}^2 + \\ &+ \tfrac{2}{3} g_{ps}^2 - 2 f_v g_v \eta_v^{-1} - \tfrac{2}{3} f_{ps} g_{ps} \eta_{ps}^{-1}, \end{aligned}\right\} \quad (18)$$

independent of g_s and the n's. By (17) and (18) the most general form of the convergence relations is determined.

Now it must be remarked that the total self-energy is, on account of the possibility of introducing terms of direct g-interaction for any of the types s, v, pv, ps with corresponding coefficients $n_s, \ldots, n_{ps}$, not unambiguously defined. It is therefore necessary to dispose of a physical argument to determine the value of these constants $n_s, \ldots, n_{ps}$. Now it has been pointed out by MØLLER and ROSENFELD [5]) that the constants n are fixed by the requirement, which will here be used too, that the operator of *static* interaction between nucleons shall, in order to ensure the (static) binding energy of the deuteron to be finite [6]), not contain δ-interactions. This condition is the first of a minimum number which will be laid down in the present paper, in order to arrive at a convenient limitation of the choice of meson field mixtures that are a priori admitted by the most general convergence relations, *viz.* those involving the coupling constants corresponding with all possible neutral and charged interactions and the constants n, without any inter-relations between them. Thus we state

A. The static nuclear interaction shall not contain terms of the δ-type. From this it is not difficult to prove that *)

$$n_s = n_{pv} = n_{ps} = -1, \; n_v = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (19)$$

As a first simplification, we now impose the condition A on (17), i.e. we use (19). Writing Q instead of $4Q$, this gives

$$Q = g_v^2 \eta_v^2 + g_{pv}^2 \eta_{pv}^2 - 2 g_{ps}^2 \eta_{ps}^2 + 6 f_v g_v \eta_v + 2 f_{ps} g_{ps} \eta_{ps} . \quad . \quad (20)$$

In this way, both L and Q have become independent of g_s. This implies that, as regards convergence, the inclusion of a g_s-interaction in the nuclear interaction is immaterial. We will now discuss various assumptions on nuclear interaction from the point of view of the convergence relations, given by (9), (10), (18) and (20).

(α). The simplest a priori possibility is to describe nuclear forces by either one f- or one g-interaction; we will directly assume that an arbitrary mixture of the corresponding charged and neutral fields is employed.

However, it is evident from (18) and (20) that such an assumption cannot give rise to a finite nucleon self-energy. An exception should, in principle, be made for a g_s-theory, but this does not give rise to any static interaction between nucleons at all, and is therefore of no interest. In particular it follows from this result that BETHE's "single force hypothesis" [7], or, in the present terminology, a g_v-theory, is incompatible with

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*) These values of n are obtained if the total static interaction between two particles 1 and 2 has, apart from dependence on isotopic variables, the form

$$[\alpha + \beta \, \sigma^{(1)} \sigma^{(2)} + \gamma \, (\sigma^{(1)} \operatorname{grad}^{(1)}) (\sigma^{(2)} \operatorname{grad}^{(2)})] \frac{e^{-\varkappa r_{12}}}{r_{12}} + \delta\text{-terms},$$

the latter vanishing on account of (19). However, the term proportional to γ yields, on performing the spatial differentiations, also terms of the δ-type: the longitudinal δ-interaction [15]). Thus if $\gamma \neq 0$ there are still δ-terms hidden in the dipole-dipole interaction. To eliminate these from the static interaction, too, other values, n'_i, should be taken which are discussed in a paper by F. J. BELINFANTE [16]). However, it can be seen that those theories which are examined on their convergence hereafter for which $\gamma \neq 0$ do not yield convergence whether we use n_i or n'_i (in particular $n'_v = -\frac{1}{3}$, $n'_{ps} = -\frac{2}{3}$); furthermore we are ultimately interested in theories for which $\gamma = 0$ (see condition D further on), and for these our values n_i apply. To simplify the trend of the discussion we have therefore confined ourselves to the n_i given by (19) and have used these values also in discussing the theories with $\gamma \neq 0$. It will be noted that, as the n'_i differ in numerical value only slightly from the n_i, the qualitative discussions of the vector and the pseudoscalar theory in § 6 are practically unaffected by using the n'_i instead of the n_i.

One may also introduce terms of direct f-interaction into the Hamiltonian which generally are of the form $n'' l^2 \int S \ldots (f) \, S^{\cdots} \, (f) dv$, where n'' again is a numerical coefficient and l a length. As, except for the f_{pv}-case, the constants n'' should be zero according to condition A, we have thought it reasonable not to introduce direct f-interactions at all.

the idea of obtaining finite self-energies by means of convergence relations.

(β). Next we try to satisfy (18) and (20) by taking a mixture of either charged or neutral f- and g-interactions of the same type.

This yields, omitting the superscript ν or γ,

Type s: The only solution is $f_s = 0$, g_s arbitrary. This case has just been discussed.

Type v: One finds no non-trivial solutions. This especially applies to BETHE's f_v, g_v-theory *).

Types pv, ps: no non-zero solution.

Thus all purely neutral or purely charged one-field theories of nuclear forces are incompatible with the convergence relations, combined with condition A.

(γ). The following step will be to consider one-field theories involving a mixture of neutral and charged contributions. For the cases s and pv we get the same result as under (β); the same holds for v and ps if

$$\varkappa_v^\nu = \varkappa_v^\gamma, \quad \varkappa_{ps}^\nu = \varkappa_{ps}^\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In order to see whether (21) is essential with regard to the argument, we consider the case that (21) is not satisfied, but at the same time, in order not to have to deal with unnecessarily complicated formulae, impose, besides A, a new condition:

B. The theory shall be charge-independent, i.e. the forces between nucleons are independent of their state of e-charge.

Within the framework of a one-field theory, the two possibilities to which B leads are: the neutral theory, which has already been discussed from the present point of view, and the symmetrical theory, characterized by the following relations between the constants

$$|f^\nu| = |f^\gamma|, \quad |g^\nu| = |g^\gamma| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\varkappa^\nu = \varkappa^\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

for any type s—ps. Thus (21) expresses that an exactly symmetrical vector or pseudoscalar theory does not yield finite self-energies. This result is readily seen not to be affected by a slight deviation which may exist from charge-independence, which can e.g. be obtained by putting instead of (23): $\varkappa_v^\nu \sim \varkappa_v^\gamma$, $\varkappa_{ps}^\nu \sim \varkappa_{ps}^\gamma$, or by replacing (22) by $|f^\nu| = |f^\gamma|(1+\varepsilon)$, $|g^\nu| = |g^\gamma|(1+\varepsilon')$, where ε and ε' are small. As small deviations from B appear to be insignificant, we will in the following for simplicity assume B to be exactly valid.

Thus there is no meson theory involving sources of one type s—ps only, which satisfies the requirement of convergence and it is therefore necessary to examine more complicated interactions. Of course, there is a

*) The constants f_v, g_v correspond with g, f of loc. cit.

large choice of combinations which are in accordance with the convergence relations as well as with the conditions A and B. However, it is hardly of physical interest to consider all these cases in detail. We will therefore now introduce two new conditions which lead to a convenient further limitation of the number of theories which are in agreement with (18), (20) and the conditions A and B only. As such we take:

C. The theory shall give the right relative position of the ^3S- and ^1S-level of the deuteron.

D. The static interaction potential of the proton-neutron system shall not contain an $1/r^3$-singularity.

The latter condition is of basic importance in any theory which attempts to eliminate the self-energy divergences. Indeed, the requirement that a theory which is free of these divergences shall not contain infinities of other origin is but a matter of course. Now it is well known, that $1/r^3$-terms in the deuteron potential entail an infinite binding energy; thus, as the present endeavour to obtain finite self-energies doet not implicitly involve a prescription by which the $1/r^3$-terms are discarded, it is quite natural to impose condition D as an accessory requirement on the theory *).

On the next simple assumption of only two fields of different types being involved, it has been shown by MØLLER and ROSENFELD [5]) that there are only two possibilities to satisfy the conditions A—D *viz.:* a symmetrical vector-pseudoscalar theory in which **)

$$g_v^2 \eta_v^2 = g_{ps}^2 \eta_{ps}^2, \text{ (for both } \nu \text{ and } \gamma), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or, if $\varkappa_v = \varkappa_{ps}$, as in loc. cit.:

$$g_v^2 = g_{ps}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

and a neutral scalar-pseudovector theory with

$$f_{pv}^2 = g_{pv}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The latter theory is easily seen, irrespective of (25), to admit no solutions of the convergence relations, while using (24*a*), the symmetrical theory appears to yield only the trivial zero solution.

Thus the symmetrical mixed theory which on the whole gives a satisfactory account of nuclear as well as cosmic ray phenomena is, notwithstanding the fact that it does not involve $1/r^3$-singularities, still beset with the difficulty of diverging self-energies. The situation is somewhat analogous to that of the electron theory, where the perfect accordance of a

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*) If C and D would have been introduced at the outset, all one-field theories would, irrespective of the convergence relations, have to be discarded. However, in view of the interest which sometimes is attached to vector and pseudoscalar theories involving cut-off prescriptions to eliminate the r^{-3}-singularity, it seemed justified to show that, even apart from D, such theories do not satisfy the convergence relations.

**) Cf. the table in I, § 1.

purely electromagnetic theory with experiment which always is found if only distances $\lesssim 10^{-13}$ cm do not come into play, is in disharmony with the consequences of the theory with regard to the structure of the electron, as is most clearly expressed by its leading to an infinite self-energy for a point electron. Thus the possibility, discussed in II, of eliminating the divergence in the electron case by assuming the electron to be the source of a short range scalar field besides the e-field, suggests to proceed likewise in the nucleon case, i.e. to introduce a scalar short range field besides the field already attributed to the nucleons in order to describe nuclear interactions *).

However, there is one difference as regards electrons and nucleons: whereas physical phenomena connected with electrons can be distinguished in "long range" and "short range" phenomena, the former being describable by the purely electromagnetic interactions, while the latter exhibit new features on the f-field assumption, all typical nucleon effects have exclusively reference to the short range region and so the question presents itself, whether the introduction of an additional scalar field in the nucleon case will modify those effects which are, with more or less success, described by the meson fields hitherto employed. This problem is discussed in the next section.

§ 6. *Introduction of the F-field.*

A mixture of charged and neutral scalar f_s-fields leads to a term, (cf. (9) and (18))

$$-\tfrac{1}{2} f_s^{\gamma 2} - \tfrac{1}{2} f_s^{\nu 2}$$

in the convergence relation which serves to eliminate the logarithmic divergences, but leaves the other convergence relation unchanged. Although from the point of view of convergence it is immaterial whether the scalar field γ which we are now introducing, and to which we shall refer as F-field, is charged or neutral or a mixture of both, it can be seen from an investigation of the coupling of this field with the electron neutrino field — a question to which we hope to return elsewhere — that it should be neutral. This is the case we therefore consider in the following and we denote the corresponding coupling constant by F:

$$f_s^{\nu} = F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

We might, proceeding on the lines of the previous section, immediately turn to the consideration of the simplest cases which are in accordance with the condition A—D, *viz.* the symmetrical and the neutral mixed theory, either of them combined with the F-field. However, it is of some

*) This new scalar field should not be confused with that which serves to make the proton and neutron convergence relations compatible, and which, as we have seen, is in point of fact identical with the electron scalar field.

interest to discard provisorily condition D and to consider the vector and pseudoscalar theories involving a cut off r^{-3}-potential.

In the first place we remark, that the introduction of the F-field can never lead to finite self-energies within the framework of theories which for the rest are based on a "single force hypothesis", i.e. in which the nuclear forces are described by one g-interaction only. Indeed, according to (20), (10) will have the form $g^2 = 0$, irrespective of the presence of the F-field. Thus we will now consider an (F, f_v, g_v)- and a (F, f_{ps}, g_{ps})-theory. BETHE's neutral (f_v, g_v)-theory, generalized by the introduction of the F-field, yields

$$-\tfrac{1}{2}F^2 + f_v^2 + \tfrac{3}{2}g_v^2 - 2f_v g_v \eta_v^{-1} = 0$$
$$g_v^2 + 6 f_v g_v \eta_v^{-1} = 0$$

or, equivalently,

$$F^2 = 2f_v^2 + \tfrac{11}{6}g_v^2;\ |f_v| = \frac{|g_v|\,\eta_v}{6} \sim |g_v|,\ \text{if } \eta_v \sim 10.$$

Assuming a purely neutral F-field, taking its range to be the same as that of the vector field and expressing all charges in BETHE's $f \equiv g_v$, it can be seen that f, being essentially fixed by the ^1S-level, becomes $\sim \frac{1}{4}$ of its value on the single force hypothesis. This means that the cut-off should become smaller, too, in order that the non-central term in the triplet interaction shall ensure that the ^3S-level becomes lower than the ^1S-level; this is, in itself, a satisfactory feature.

A charged (f_{ps}, g_{ps})-theory, combined with the F-field gives

$$F^2 = \tfrac{1}{3}f_{ps}^2 = \tfrac{1}{3}g_{ps}^2\eta_{ps}^2 \sim 30\, g_{ps}^2,\ \text{if } \eta_{ps} \sim 10 \quad . \quad . \quad . \quad . \quad (27)$$

It is remarkable that the non-symmetrical theory under consideration, which has recently been proposed by HULTHÉN [9]), who takes a purely neutral F-field *), satisfies all requirements laid down in (27): it is approximately charge independent, because $F^2 \gg g_d^2$. Furthermore, the cut-off R_0 is, according to HULTHÉN, fixed by

$$\frac{g_{ps}^2}{\varkappa_{ps}^2 R_0^3} \sim \frac{F^2}{R_0}$$

which is necessary to obtain the correct binding energy of the deuteron. With (27) this gives

$$R_0 \sim \frac{\hbar}{Mc}$$

in accordance with HULTHÉN's result. The mass of the F-mesons is taken to be larger than that of the pseudoscalar mesons. The theory furthermore

*) The constants g, f, $\varkappa$ of loc. cit. correspond with F, g_{ps}, $\varkappa_{ps}$ respectively.

gives the right sign and order of magnitude of the deuteron quadrupole moment.

Thus, notwithstanding the objections of principle which can be raised against the device of cut-off potentials, it is noteworthy that the concept of the F-field seems anyhow qualitatively to support the basic ideas underlying theories of this kind.

We now reinclude condition D in the discussion, and thus turn to the mixed theories. Obviously, the neutral mixed theory still yields divergences, for if no convergence can be attained by a pseudovector and one scalar field, this cannot be remedied by introducing another scalar field. The symmetrical theory gives

$$F^2 = 4\,(f_v^2 + \tfrac{1}{6} f_{ps}^2 + \tfrac{11}{6} g_v^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$g_v = 2\,\eta^{-1}\,(3 f_v + \varepsilon f_{ps}), \quad \varepsilon = \pm 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

As in the previous cases, F is larger than the constants which, in the theory not involving the F-field, determine the strength of the nuclear interaction. We must therefore assume that the range of the F-field is much smaller than that of the other meson fields, for otherwise the interaction would be preponderantly of the scalar type, in contradiction with condition C. Furthermore, (29) provides a new relation between the constants of the mixed theory. We will discuss its consistency under the assumption that the F-range is such, that the F-field does not appreciably contribute to the binding energy of the deuteron. In this case, the static interaction is determined by f_v, g_v and η. Therefore (29) can be considered as defining f_{ps} in terms of these quantities. Estimations based on the position of the deuteron S-levels give *)

$$\frac{g_v^2}{4\pi\hbar c} = 0{,}56\,\eta^{-1} + 0{,}009, \frac{f_v^2}{4\pi\hbar c} = 0{,}027 \quad . \quad . \quad . \quad . \quad (30)$$

Taking the meson mass to be $= 240$ m gives, for $f_v\, g_v > 0$

$$\frac{g_v^2}{4\pi\hbar c} \approx 0{,}082, \quad |f_{ps}| \sim 2\,|g_v| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30a)$$

which seems a reasonable order of magnitude. Inserting this in (28) one gets

$$|F| \sim 3{,}5\,|g_v|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

An experimental indication, (which led HULTHÉN to consider the scalar-pseudoscalar theory mentioned above), of the presence of the F-field may perhaps be found in the results of AMALDI and collaborators [10]) who obtain an angular distribution for the scattering of high energy neutrons

*) Cf. loc. cit. [5]), eq. (106), (107) and the table of I, § 1.

by protons which is in definite disagreement with the mixed theory in its symmetrical form. It has been pointed out by HULTHÉN [11]) that this discrepancy may be overcome by the introduction of an additional neutral interaction of appreciably shorter range than that of the vector and pseudoscalar mesons of the symmetrical theory. However this may be, it is clear that experimental evidence on scattering in the high energy region may provide important clues with regard to the problem of the self-energy of nucleons and in particular with respect to its tentative solution on the lines presented here.

HULTHÉN proposes another modification which makes the theory more asymmetrical than that discussed above: he assumes a purely charged vector-pseudoscalar interaction combined with a purely neutral scalar-pseudovector interaction. It is readily seen that the corresponding convergence relations can be satisfied in a great variety of ways; more definite statements will therefore have to wait for more specified calculations.

Anyhow, the considerations of this section show, that it would seem possible to attain a theory of nuclear forces which involves, at all events in first approximation, no self-energy divergences and which is, moreover, free of r^{-3}-singularities in the deuteron potential, while we also have found that the introduction of an F-field is qualitatively reconcilable with the current theories involving cut-off. Just as for the f-field of § 3, experimental evidence on nucleon scattering in the high energy region will be able to throw more light on the validity of the present assumptions.

§ 7. *The magnitude of the mesic self-energy.*

The ratio of the meson mass and M is such that

$$\eta_l \sim 10.$$

Whether the mesons actually have a mass, say, 200 m or 300 m, is irrelevant for the general result that the self-energy is $\ll Mc^2$. One example of this we have already found in § 4, where a (neutral) scalar field was examined. Other f_l-interactions lead to quite similar results for the finite part W of the self-energy. We will here only state the results, that are readily obtained from I, table 1, (see also the footnote on p. 74)

$$W_v = -\frac{f^2}{16\pi^2\hbar c} \cdot Mc^2 \left[1+\xi^4 lg\,\xi + 4z\left(1+\frac{\xi^2}{2}\right)A\right],$$

$$W_{pv} = -\frac{f^2}{16\pi^2\hbar c} \cdot Mc^2 \left[1 + (\xi^4 - 8\,\xi^2)\,lg\,\xi + 2z\,(\xi^2-6)\,A\right],$$

$$W_{ps} = -\frac{f^2}{32\pi^2\hbar c} \cdot Mc^2 \left[1 + (\xi^4 - 2\,\xi^2)\,lg\,\xi + 2z\,\xi^2\,A\right],$$

$$\xi = \eta^{-1} \quad A = \text{arc } tg\,\frac{2z}{\xi^2} \quad z^2 = \xi^2 - \frac{\xi^4}{4} > 0.$$

For $\eta \sim 10$ all these contributions are

$$\sim \frac{1}{4\pi} \cdot \frac{f^2}{4\pi\hbar c} \cdot Mc^2 \sim \frac{f^2}{4\pi\hbar c} \cdot 0{,}1\,Mc^2.$$

In order to estimate more quantitatively the ratio of self-energy and Mc^2 the actual values of the constants f and η should be inserted. We will content ourselves with one remark of qualitative order, however, which throws new light on the convergence relations: Consider all four f-interactions to be present. The total contribution then is easily seen from (15) and the above equations to be, for all meson masses for instance equal to 274 m:

$$\sim(-0{,}33\, f_s^2 + 1{,}9\, f_v^2 - 1{,}5\, f_{pv}^2 + 0{,}5\, f_{ps}^2).$$

It is interesting to note that the alternation of signs is the same as in the convergence relation which, for this set of fields, is

$$-\tfrac{1}{2} f_s^2 + f_v^2 - \tfrac{5}{3} f_{pv}^2 + \tfrac{1}{6} f_{ps}^2 = 0.$$

This means that the existence of convergence relations tends to decrease the magnitude of the finite part of the self-energy. Thus the convergence relations favour, so to say, the treatment of the self-energy as a perturbation. This is particularly borne out by II, eq. (14) from which it is seen that, for smaller η's, the convergence relations lead to a decrease of the self-energy of at least one order of magnitude. In the case of different meson masses involved the situation is more complicated but the same general trend exists.

It remains to consider the g-interactions. To show the main point we take the expressions which are found in the vector theory:

1. g_v-interaction.

$$W = \frac{g_v^2}{8\pi^2\hbar c} \cdot Mc^2 \cdot \xi^{-2} \lim_{P\to\infty} \left[\frac{P^2}{M^2} + \tfrac{1}{2} - (1 - \tfrac{3}{2}\xi^2)\, lg\, P_M\right]$$
$$- \frac{g_v^2}{32\pi^2\hbar c} \cdot Mc^2 \left[1 - \xi^2 + (\xi^4 + 6\xi^2)\, lg\, \xi + 16\xi^2 z \left(1 + \frac{\xi^2}{8}\right) A\right]$$

2. Direct g_v-interaction.

$$W = \frac{3 n_v g_v^2}{4\pi^2\hbar c} \cdot Mc^2 \cdot \xi^{-2} \lim_{P\to\infty} \left[\frac{P^2}{M^2} + \tfrac{1}{2} - (1 + \xi^2)\, lg\, P_M\right].$$

3. $f_v g_v$-interaction.

$$W = \frac{3 f_v g_v}{4\pi^2\hbar c} \cdot Mc^2 \cdot \xi^{-1} \lim_{P\to\infty} \left[\frac{P^2}{M^2} + \tfrac{1}{2} - (1 + \xi^2)\, lg\, P_M\right]$$
$$+ \frac{3 f_v g_v}{4\pi^2\hbar c} \cdot Mc^2\, [\xi^3\, lg\, \xi + \xi\, z^2\, A].$$

4. f_v-interaction.

$$W = \frac{3 f_v^2}{8\pi^2\hbar c}\, Mc^2 \lim_{P\to\infty} lg\, P_M,$$
$$- \frac{f_v^2}{16\pi^2\hbar c} \cdot Mc^2\, [1 - \xi^2 + \xi^4\, lg\, \xi + z\,(\xi^2 + 2)\, A],$$

where

$$P_M = \frac{P + \sqrt{P^2 + M^2 c^2}}{Mc}.$$

If we would suppose a vector field to be present only, it is easily seen from the convergence relations that the terms in the first line (in the case of the direct self-energy: the only line) of the various W's cancel each other. Thus the terms of the second lines remain. The contribution of the fg-interaction is, for $\eta \sim 10$, readily seen to be negligible compared to that of pure f- and g-interactions which, for the same order of η, are

$$\sim \frac{1}{4\pi} \cdot \frac{f^2 \text{ or } g^2}{4\pi\hbar c} \cdot Mc^2 \sim 0{,}01\, Mc^2$$

for

$$\frac{f^2 \text{ or } g^2}{4\pi\hbar c} \sim 0{,}1$$

which is roughly the order of magnitude of the mesic coupling constants. Similar results are obtained if a mixture of fields is present. We note that in the first lines a finite term occurs, the "$\frac{1}{2}$"-term which would yield a large contribution, but that these $\frac{1}{2}$-terms just cancel out by virtue of the convergence relations.

It is particularly gratifying, with regard to the non-ambiguity of the scheme, that the direct self-energy does not contribute to the finite part at all. For, had this not been so, we could have chosen, discarding condition A, the n_l such that the self-energy would have any order of magnitude. Thus the direct self-energies are, whether the "unitary" or the "perturbation" view is held, irrelevant as concerns their mass contribution, they only play into the theory in the convergence relations.

Thus we have shown that the meson field self-energies are all $\ll Mc^2$ with which the "perturbation method" for nucleons is justified.

§ 8. *Concluding remarks.*

In briefly recapitulating the results of II and the present chapter, we divide the self-energy problems of Dirac particles into two groups:

a. The convergence problem for the electron.
b. The "compatibility problem" for the proton-neutron (cf. §§ 2 and 3).
c. The convergence problem for the nucleon (cf. §§ 2 and 5).

These are representative of that crisis of divergences which mark the relativistic quantum field theories in their present stage.

The second group comprises:

a'. The mass problem for the electron.
b'. The mass-difference problem for the proton-neutron.
c'. The mass problem for the nucleon.

By mass problem we understand the question whether the masses concerned can be derived from the "universal constants of the first kind" [12] e, $\hbar$, c and a universal length. It has often been suggested that the solution of a problem of the first group would entail that of the corresponding problem of the second group, analogous to the situation in classical electron theory, where a non zero electron radius ensures finite self-energy and at the same time allows us to consider the electron mass to be of purely electromagnetic origin. Let us call (a, a') a *simultaneous* solution of a and a'. The results of a theory based on convergence relations can then be summarized as follows:

> (a, a') is possible a priori (unitary method) but leads to contradictions with regard to (b, b'): if we would assume the mass of the electron to be entirely due to field self-energy, the proton would be twice as heavy as the neutron (cf. § 4).
>
> $(b. b')$ is possible if (a, a') is discarded: perturbation method.
>
> (c, c') is anyhow impossible: the mesic self-energy of nucleons is always $\ll Mc^2$.

Thus we see that the present theory essentially disconnects a and a', c and c'. This demonstrates a characteristic feature of the present method: with the introduction of any universal length, a charge, too, is introduced; the latter plays the decisive rôle inasmuch as convergence is concerned, while both charge and length enter the mass problem. Thus the solution of the convergence problem does not necessarily lead to that of the mass problem, contrary to classical electron theory involving an electron radius $\neq 0$, but there only a length is introduced.

On the other hand, the fact that the theory yields (b, b') might seem to support the present point of view of considering all self-energies as perturbations. For a consistent theory on these lines necessarily must account for the only *observable* perturbation *viz.* the quantity Δ. The range of the f-field of the electron, though of the same order of magnitude as the so-called classical electron radius r_0, plays a fundamentally different rôle compared to that of r_0 in classical theory: whereas the latter is determined by the condition that the (electromagnetic) self-energy shall be equal to mc^2, in accordance with the unitary aspect of classical theory, $\varkappa^{-1}$ is roughly fixed by the condition that $W \ll mc^2$, in order to attain a non-unitary treatment, cf. II, § 9.

As regards the possibilities of experimental tests of the f-field as well as the F-field hypothesis, the high energy nucleon scattering has already been alluded to. A fuller account will be given in chapter IV.

From the results up to the present it cannot be concluded whether or not all divergences proportional to higher powers of the charges remain. In this respect we recall that in one case (cf. II, §§ 5 and 6) it appeared possible to find relations between higher order divergences. In this connection, the belief which sometimes has been expressed [13], that the diffi-

culties encountered, also in interaction problems, when considering higher order effects, may at least partly be due to the inappropriateness of the perturbation theory methods, seems worth noticing. Even quite apart from the self-energy problem it would indeed seem questionable whether any physical meaning can be attached to these higher approximations in the approximation procedure.

Furthermore the theory should be generalized to the discussion of self-energy effects in the presence of external fields, amongst which the proper magnetic moments of nucleons is of foremost importance. This has been discussed by SERPE [14]), who has, in accordance with the general prescriptions of the MØLLER–ROSENFELD-theory [5]), considered the static contribution due to vector and pseudoscalar fields on a "one-nucleon theory". However, the present point of view necessitates: first, the treatment on hole theory, secondly the inclusion of non-static effects; for the distinction between static and non-static contributions cannot be maintained as far as an exact treatment of self-energy effects is considered.

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CHAPTER IV.

On some further consequences of the f-field hypothesis.

Let us first briefly recapitulate what we know already about the f-field:

1. On hole theory it ensures, at all events in first approximation, convergence of the electron self-energy provided

$$e^2 = \tfrac{1}{3} f^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

f being the charge of the electron describing the coupling with the f-field. Furthermore, this same f-field is also created by the proton; but not by the neutron.

2. The electron field self-energy (divided by c^2) should be small compared to its mechanical mass. This holds true if

$$\varkappa^{-1} \approx r_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\varkappa$ is the inverse f-field range, r_0 the classical electron radius. As a consequence of this same connection (2), the proton-neutron mass difference could be accounted for, as to sign and order of magnitude.

3. We have already pointed out in chapter III, § 3 that the production of an f-field by the proton may, in principle, be verified by comparing the proton-proton scattering with the proton-neutron scattering.

We shall now turn to the consideration of some further consequences of the f-field hypothesis.

According to (1) the static electron-proton potential becomes

$$-\frac{e^2}{r}\left[1-2\,e^{-\varkappa r}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

reaching a minimum value of $\approx 0{,}375 e^2 \varkappa$ for $r \approx 5/3\varkappa$; for smaller r the attraction goes over into a repulsion. This leads to a shift of (especially) the S-levels of hydrogen towards higher energies, which can be computed by means of a perturbation calculus [1]). The result is, ($\alpha = 1/137$, $n =$ principal quantum number)

$$\Delta\nu\,(n\,S) \approx \frac{8\alpha^6}{n^3} \cdot \frac{1}{\varkappa^2 r_0^2} \cdot \frac{mc}{h}\ \mathrm{cm}^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The following table gives the values of $\Delta\nu\ (2S)$ for various values of ξ defined by $\varkappa^{-1} = \xi\, r_0$.

ξ	$\Delta\nu(2S)$ in cm^{-1}.
1	$0{,}6.10^{-3}$
2	$2{,}6.10^{-3}$
3	$5{,}9.10^{-3}$.

R. C. WILLIAMS [2]) has reported a deviation in the fine structure of the $H\alpha$ line from theoretical expectations which has been interpreted by PASTERNACK [3]) as due to a shift in the $2S$-level of the H-atom of about 0,03 cm^{-1}. This would correspond with a range of the f-field of about 6 times the classical electron radius *). Later experiments performed by DRINKWATER, RICHARDSON and H. E. WILLIAMS [5]) do not confirm R. C. WILLIAMS' result. It is therefore not yet possible to infer anything quantitative from the experimental evidence.

It may be noted that the shift (4) is somewhat larger than that following from SOMMERFELD's perturbation potential [1]), because the latter is different from zero in a finite region, while the Yukawa potential has a small "tail". The relative influence of this tail is well known from calculations on proton-neutron scattering, in which a Yukawa potential is compared e.g. with a box potential [6]).

As far as can be judged from calculations based on BORN's approximation the potential (3) cannot remove the discrepancy between theory and experiment as regards MOTT's polarization effect in double scattering of electrons by heavy atoms [5]).

With regard to non-static effects, the first question is whether evidence can be found for the existence of free f-quanta. However, this particle is unobservable: in any theory of the electron in which the universal length is introduced as the range of a field, the accompanying field quanta are unstable as their Compton wave length necessarily is shorter than twice that of the electron, enabling them to decay into a negaton-positon pair. In the present case the life time τ_0 in the rest system is given by

$$\frac{1}{\tau_0} = \frac{f^2}{\hbar c} \cdot \varkappa c = 2\alpha c \sim 10^{21}\,\mathrm{sec}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence τ_0 is so small that the f-quanta escape detection. It is noteworthy that a theory of the type considered automatically involves a universal time as the consequence of the introduction of a universal length. According to (5) the fine structure constant thus can be expressed in terms of "universal constants of the first kind" [5]), viz. as the ratio of a universal length and a universal time (in units c).

It should furthermore be noted that the undetectability of the f-quantum is *a consequence* of the fundamental concepts of the theory, and thus should not be compared with the *ad hoc* unobservability of the neutrino.

All further modifications of current theories due to the f-field can be traced back to either one or both of the effects (3) and (5). The survey we shall now give of the modifications occurring in the domain of cosmic radiation should be considered as merely a rough outlining of the situation.

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*) In the preliminary report of this work [4]) the equation for $\triangle\nu$ is erroneously given with $\hbar$ instead of h; the statement in loc. cit. that WILLIAMS' shift corresponds with $\varkappa^{-1} = 2{,}5\ r_0$ is therefore incorrect. I am indebted to dr. KIKUCHI for pointing this out to me.

1°. Pair formation by photons in the field of a proton. The momentum absorbing field is now given by (3) which leads to a decrease of the differential cross-section by a factor

$$x = \left(1 - \frac{2}{1+\varepsilon^2}\right)^2$$

where ε is the ratio of $\hbar\varkappa$ and the momentum absorbed by the proton. Thus in the non-relativistic as well as in the extreme relativistic (ER) approximation all results remain unmodified, while a decrease occurs in the region of about 137 mc^2 (to fix thoughts we put $\varkappa^{-1} = r_0$).

On the other hand, an increase is obtained in the same energy region due to the "e—f effect": a photon of energy $\gtrsim 137\ mc^2$ may be absorbed and an f-quantum subsequently be emitted which "immediately" decays in a pair. In the ER case the cross section for the total number of produced f-quanta is

$$\sim r_0^2 \frac{mc^2}{E} lg \frac{E}{mc^2}, \quad \ldots\ldots\ldots \quad (6)$$

where E is the energy of the incoming photon. The effect is independent of screening.

2°. Bremsstrahlung of electrons. The differential cross section again decreases by the factor x. Thus on the present theory fast electrons of energy $\sim 137\ mc^2$ are more penetrating than on purely electromagnetic theory.

3°. Pair formation by electrons of energy $> 137\ mc^2$. By passing through a field (3) electrons of this energy may emit an f-quantum: "f-bremsstrahlung" which leads to the formation of a pair. By (5) this may be considered as a first order effect.

4°. Annihilation of positons. The two-photon annihilation remains unaltered. A new effect, however, is the two f-quanta annihilation for energies $> 274\ mc^2$ which effectively leads to the creation of two pairs by one pair, and the one photon- one f-quantum annihilation for energies $> 137\ mc^2$ giving a photon and a pair. In the ER case all three effects are of the order (6) where E now means the energy of the positon in the rest system of the negaton; thus they are small.

It is clear that these effects generally only can play a rôle in that cosmic ray energy region where the hard component is already of importance. In this connection BHABHA's remark may be recalled [7]) that the difference in properties of the hard component of cosmic radiation from those of the soft component may either be due to different properties of electrons at energies $\gtrsim 137\ mc^2$, or to the fact that the hard component mainly consists of other particles than electrons; or to both causes.

With regard to the f-interaction between electrons, it should be noticed that this cannot be treated by means of the customary expansion in powers

of v/c, for here the field quanta are very heavy compared to the generating particles so that the neglect of recoil as a first approximation has no sense: the effect is from the outset relativistic.

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