Analytical treatment of the polytopes regularly derived from the regular polytopes.

(Sections II, III, IV).

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Section II: Polytopes and nets derived from the measure polytope.

A. *The symbol of coordinates.*

46. The distance r between two points P, P', the ordinary rectangular coordinates of which are $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu'_1, \mu'_2 \ldots, \mu'_n$ is represented by the formula

$$r^2 = \sum_{i=1}^{n} (\mu_i - \mu'_i)^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad 2).$$

Now we repeat here the question of art. 1:

"Under what circumstances will the series of points obtained by giving to the set of coordinates $\mu_1, \mu_2, \ldots, \mu_n$ a determinate set of values taken in all possible permutations form the vertices of a polytope all the edges of which have the same length, say unity?"

The answer is nearly the same as that given in art. 1:

"If the n values $a_1, a_2, \ldots, a_n$ are arranged in decreasing order, so that we have

$$a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq a_{k+1} \geqq \ldots \geqq a_n,$$

the difference $a_k - a_{k+1}$ of any two adjacent values must be either $\frac{1}{2}\sqrt{2}$ or zero."

The proof runs on the same lines as that given in art. 1. The geometrical result can be stated in the following general form:

"Under the conditions stated, the polytope the vertices of which are represented by the symbol

$$(a_1, a_2, a_3, \ldots, a_n)$$

is the same as that obtained in the first section for $n - 1$ and $a_k - a_{k+1}$ either one or zero. It is a derivative of the regular simplex the vertices of which determine on the n axes OX_i of coordinates positive segments OA_i, $(i = 1, 2, \ldots, n)$, of the same length $b = \sum_{1}^{n} a_i$".

This simple result, in close connection with the new deduction of formula 1), shows us that we shall have to enlarge the scope of our symbol of coordinates in order to find something new.

47. We remember that the symbols $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}\sqrt{2}, 0, 0]$ represent the coordinates of the vertices of cube and octahedron with edge unity, if the *square* brackets indicate that all the permutations of the values they include must be taken, each value being affected successively either by the *positive* or by the *negative* sign. Moreovor $\frac{1}{2}[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $-\frac{1}{2}[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ can represent in the same way the two tetrahedra, the vertices of which form together the vertices of the cube $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$, if by the coefficient $\frac{1}{2}$ we indicate the vertices with an even, by the coefficient $-\frac{1}{2}$ the vertices with an odd number of negative coordinates.

In connection with this we amplify the question of art. 1 as follows: "Under what circumstances will the symbols

$$[a_1, a_2, \ldots, a_n], \quad \pm\frac{1}{2}[a_1, a_2, \ldots, a_n]$$

represent the vertices of polytopes in S_n, all the edges of which have the same length, say unity?"

The answer to this question runs as follows:

THEOREM XXVIII. "If the values $a_1, a_2, \ldots, a_n$ are arranged in decreasing order, a_p being the smallest non vanishing one, and if a_k, a_{k+1} represent any couple of adjacent unequal ones, we must have in the case of the first symbol $[a_1, a_2, \ldots, a_n]$

$$\left.\begin{array}{lll} \textit{either}\ p = n, & a_n = \frac{1}{2}, & a_k - a_{k+1} = \frac{1}{2}\sqrt{2} \\ \textit{or}\quad p < n, & a_p = \frac{1}{2}\sqrt{2}, & a_k - a_{k+1} = \frac{1}{2}\sqrt{2} \end{array}\right\},$$

in the case of the second symbol $\pm\frac{1}{2}[a_1, a_2, \ldots, a_n]$

$$p = n, a_{n-1} = a_n = \tfrac{1}{4}\sqrt{2}, a_k - a_{k+1} = \tfrac{1}{2}\sqrt{2}."$$

Proof. The part of the proof concerned with the common value $\frac{1}{2}\sqrt{2}$ of the difference $a_k - a_{k+1}$ of two unequal adjacent digits is the same as that given in art. 1. So we have to add only a few words about the values of a_n in the case of the first and of a_{n-1} and a_n in the case of the second symbol.

Symbol $[a_1, a_2, \ldots, a_n]$. In the supposition $a_k - a_{k+1} = \frac{1}{2}\sqrt{2}$ the length of the edge of the polytope is unity. Therefore the distance $2a_n$ between the points

$$\begin{array}{l} P \ldots x_1 = a_n, x_2 = a_1, x_3 = a_2, \ldots \\ Q \ldots x_1 = -a_n, x_2 = a_1, x_3 = a_2, \ldots \end{array}$$

which are transformed into each other by inverting the sign of

a_n, must be unity, which gives $a_n = \frac{1}{2}$, unless P and Q coincide which happens for $a_n = 0$. So in the case $p = n$ we have $a_n = \frac{1}{2}$.

In the case $p < n$ we consider the points

$$P \ldots x_1 = a_p, x_2 = 0\,, x_3 = a_1, x_4 = a_2, \ldots$$
$$Q \ldots x_1 = 0\,, x_2 = a_p, x_3 = a_1, x_4 = a_2, \ldots$$

passing into each other by interchanging x_1 and x_2. The distance $a_p \sqrt{2}$ between these points is unity for $a_p = \frac{1}{2}\sqrt{2}$.

Symbol $\pm \frac{1}{2}[a_1, a_2, \ldots, a_n]$. Here a_n differs from zero; for the supposition $a_n = 0$ is incompatible with the division of the vertices represented by the symbol $[a_1, a_2, \ldots, a_n]$ into the two groups $\pm \frac{1}{2}[a_1, a_2, \ldots a_n]$, the inversion of the sign of zero having no effect whatever.

Here the point

$$P \ldots x_1 = a_n\,, x_2 = a_{n-1}, x_3 = a_{n-2}, \ldots$$

must be considered in combination with the points

$$Q \ldots x_1 = a_{n-1}, x_2 = a_n\,, x_3 = a_{n-2}, \ldots$$
$$R \ldots x_1 = -a_n\,, x_2 = -a_{n-1}, x_3 = a_{n-2}, \ldots$$

corresponding with it as to the coordinates $x_3, x_4, \ldots x_{n+1}$, as these points Q, and R are the nearest ones to P obtainable either by interchanging two digits or by inverting the signs of two digits. Now we have under these circumstances

$$PQ^2 = 2\,(a_n - a_{n-1})^2, \quad PR^2 = 4\,(a^2_n + a^2_{n-1}),$$

from which ensues $PQ < PR$. So we must have $PQ = 0, PR = 1$, giving $a_n = a_{n-1} = \frac{1}{4}\sqrt{2}$.

48. In the case of the first symbol $[a_1, a_2, \ldots, a_n]$ we are confronted with two possibilities, as we have to choose between $a_n = \frac{1}{2}$ and $a_n = 0$, i. e. between a group containing the measure polytope $[\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2}]$ and an other group containing the cross polytope $[\frac{1}{2}\sqrt{2}, 0, \ldots 0]$. Do the two regions lying on different sides of the limiting demarcation line cover the same ground as the group of the measure polytope on one side and the group of the cross polytope on the other? The answer to this question depends on the manner of deduction of these two groups. If we follow closely the geometrical manner of deduction developed by M[rs]. STOTT the contraction forms derived from the measure polytope *do* possess coordinate symbols winding up in zero, whilst on the other hand the form derived from the cross polytope by means of a set of expansions under which e_{n-1} occurs are represented by coordinate symbols containing no zero. These two exceptional facts which

prove the close relationship between the progeniture of the two patriarchs, cube and octahedron, can be extended so as to make the two families quite *identical* with each other; to that end we have only to derive from each of the two, cube and octahedron, *all* the expansion and contraction forms, the number of which amounts in S_n to $2^n - 1$. This important fact, which will be proved later on, enables us to treat in the second and third sections the forms with the symbols $[a_1, a_2, \ldots, \frac{1}{2}]$ and $[a_1, a_2, \ldots, 0]$ successively, without being obliged to postpone the study of the corresponding nets built up by forms of both groups.

In order to avoid fractions we will multiply the digits by two in this section and the next one; under this circumstance the last digit is unity or zero, the difference $a_k - a_{k+1}$ of two unequal adjacent digits is $\sqrt{2}$ and the symbol represents a polytope with edge 2. Moreover in order to simplify the symbols we will write p' for $1 + p\sqrt{2}$ and put if possible $\sqrt{2}$ outside the brackets, substituting e. g. $[11100]\sqrt{2}$ for $[\sqrt{2}, \sqrt{2}, \sqrt{2}, 0, 0]$.

49. For $n = 2, 3, 4, 5$ we have successively in the symbols explained in the memoir of Mrs. Stott: [1])

	$n = 2.$		
$[1\ 1] = p_4$	$[1'1] = e_1 p_4 = p_8$		$[10]\sqrt{2} = ce_1 p_4 = \overline{p}_4$
	$n = 3.$		
$[1\ 1\ 1] = C$	$[1'1\ 1] = e_2 C = RCO$		$[100]\sqrt{2} = ce_2 C = O$
$[1'1'1] = e_1 C = tC$	$[2'1'1] = e_1 e_2 C = tCO$	$[110]\sqrt{2} = ce_1 C = CO$	$[210]\sqrt{2} = ce_1 e_2 C = tO$
	$n = 4.$		
$[1\ 1\ 1\ 1] = C_8$	$[2'2'1'1] = e_1 e_2 C_8$		$[2110]\sqrt{2} = ce_1 e_2 C_8$
$[1'1'1'1] = e_1 C_8$	$[2'1'1'1] = e_1 e_3 C_8$	$[1110]\sqrt{2} = ce_1 C_8$	$[2110]\sqrt{2} = ce_1 e_3 C_8$
$[1'1'1\ 1] = e_2 C_8$	$[2'1'1\ 1] = e_3 e_3 C_8$	$[1100]\sqrt{2} = ce_2 C_8$	$[2100]\sqrt{2} = ce_2 e_3 C_8$
$[1'1\ 1\ 1] = e_3 C_8$	$[3'2'1'1] = e_1 e_2 e_3 C_8$	$[1000]\sqrt{2} = ce_3 C_8$	$[3210]\sqrt{2} = ce_1 e_2 e_3 C_8$
	$n = 5.$		
$[1\ 1\ 1\ 1\ 1] = C_{10}$	$[2'2'1'1\ 1] = e_2 e_3 C_{10}$		$[22100]\sqrt{2} = ce_2 e_3 C_{10}$
$[1'1'1'1'1] = e_1 C_{10}$	$[2'1'1'1\ 1] = e_2 e_4 C_{10}$	$[11110]\sqrt{2} = ce_1 C_{10}$	$[21100]\sqrt{2} = ce_2 e_4 C_{10}$
$[1'1'1'1\ 1] = e_2 C_{10}$	$[2'1'1\ 1\ 1] = e_3 e_4 C_{10}$	$[11100]\sqrt{2} = ce_2 C_{10}$	$[21000]\sqrt{2} = ce_3 e_4 C_{10}$
$[1'1'1\ 1\ 1] = e_3 C_{10}$	$[3'3'2'1'1] = e_1 e_2 e_3 C_{10}$	$[11000]\sqrt{2} = ce_3 C_{10}$	$[33210]\sqrt{2} = ce_1 e_2 e_3 C_{10}$
$[1'1\ 1\ 1\ 1] = e_4 C_{10}$	$[3'2'2'1'1] = e_1 e_2 e_4 C_{10}$	$[10000]\sqrt{2} = ce_4 C_{10}$	$[32210]\sqrt{2} = ce_1 e_2 e_4 C_{10}$
$[2'2'2'1'1] = e_1 e_2 C_{10}$	$[3'2'1'1'1] = e_1 e_3 e_4 C_{10}$	$[22210]\sqrt{2} = ce_1 e_2 C_{10}$	$[32110]\sqrt{2} = ce_1 e_3 e_4 C_{10}$
$[2'2'1'1'1] = e_1 e_3 C_{10}$	$[3'2'1'1\ 1] = e_2 e_3 e_4 C_{10}$	$[22110]\sqrt{2} = ce_1 e_3 C_{10}$	$[32100]\sqrt{2} = ce_2 e_3 e_4 C_{10}$
$[2'1'1'1'1] = e_2 e_3 C_{10}$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4 C_{10}$	$[21110]\sqrt{2} = ce_2 e_3 C_{10}$	$[43210]\sqrt{2} = ce_1 e_2 e_3 e_4 C_{10}$

[1]) For the deduction of the e and c symbols from the symbol of coordinates compare the part D of this section; here $\overline{p}_4$ means: p_4 turned 45° about the centre.

In Table IV added at the end of this memoir are put on record for $n = 3, 4, 5$, the different polyhedra and polytopes deduced from the measure polytope. Of this table the first column contains the symbols of deduction of the polytope from measure polytope and cross polytope — with the first of which we are concerned in this section only — and the third the symbol of coordinates. The second and the following columns will be explained farther on.

Here we have $[1100]\sqrt{2} = C^{(2)}_{24}$, $[1000]\sqrt{2} = C^{(2)}_{16}$, $[10000]\sqrt{2} = C^{(2)}_{32}$.

Remark. If we invert the sign of all the coordinates of a vertex V of the polytope we get the coordinates of an other vertex V' of that polytope for which the centre of the segment PP' is the origin of coordinates O. So, all the forms derived analytically from the measure polytope admit central symmetry, as the geometrical deduction by means of the operations e and c requires it.

B. *The characteristic numbers.*

50. In the case of the simplex the direct method for the determination of the characteristic numbers proceeding regularly from vertices to edges, from edges to faces, etc. was preceded by an easier method fulfilling the exigencies of the cases $n = 4$ and $n = 5$, working from both sides, the vertex side and the side of the limiting element of the highest number of dimensions; in this case of the measure polytope we will do likewise. [1])

Here also the number of vertices is easily found. If all the n digits of the symbol of coordinates are different it is $2^n . n!$; of the two factors 2^n and $n!$ of this product the first is due to the power of choosing arbitrarily the signs of the n digits, whilst the second corresponds to the power of permutating them. This product must be divided by $2!$ for any two, by $3!$ for any three digits being equal, etc.

In order to be able to find the number of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$) we have to prove here the

THEOREM XXIX. "The non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the polytope deduced from the measure polytope of S_n must all of them have the same *absolute* value."

The difference between this theorem and the corresponding one for the simplex (theorem II of art. 6) lies in the addition of the word "absolute", therefore printed in italics. This amplification is necessary here, in connection with the power of assigning to each of the n digits of the coordinate symbol either the positive or the negative sign. But the proof runs quite in the same lines. If in the case of the polytope $[1 + 2\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2}, 1]$ we start from the equation $2x_1 - x_2 = p$ and try to determine the

[1]) The treatment of the offspring of the measure polytope with which we are concerned now — and of that of the cross polytope which comes next — will be copied as much as possible from Section I.

vertices of the polytope for which the expression $2x_1 - x_2$ becomes either a maximum or a minimum we find the maximum $3+5\sqrt{2}$ for $x_1 = 1+2\sqrt{2}$, $x_2 = -(1+\sqrt{2})$ and the minimum $-(3+5\sqrt{2})$ of the same absolute value for $x_1 = -(1+2\sqrt{2})$, $x_2 = 1+\sqrt{2}$. So, for values of p between $3+5\sqrt{2}$ and $-(3+5\sqrt{2})$ the space $2x_1 - x_2 = p$ intersects the polytope, whilst it cannot contain a limiting body but at most a limiting face only for the extreme values $\pm(3+5\sqrt{2})$ of p, as each of the two couples of equations $x_1 = 1+2\sqrt{2}$, $x_2 = -(1+\sqrt{2})$ and $x_1 = -(1+2\sqrt{2})$, $x_2 = 1+\sqrt{2}$ determines a plane. Here too, as far as the vertices of the polytope are concerned, any linear equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents k different equations if the non vanishing coefficients c_i admit k different *absolute* values. Here too the theorem is not reversible. As to the theory of the determination of the number of faces ($n = 4$) and the number of limiting bodies ($n = 5$) compare the end of art. 6.

Remark. In accordance with the central symmetry of the polytope $[a_1, a_2, \ldots, a_n]$ any two parallel spaces S_{n-1}, represented by the equations $x_i + x_k + x_l + \ldots = \pm p$ and lying therefore on different sides at the same distance from the origin, bear either both or none of them a limit $(l)_{n-1}$ of the polytope. So, in the determination of the limits $(l)_{n-1}$ we can restrict ourselves here to the equations $x_i + x_k + x_l + \ldots =$ maximum.

51. We now treat at full length two examples, one in S_4 and one in S_5.

Example $[1+2\sqrt{2},\ 1+\sqrt{2},\ 1+\sqrt{2},\ 1]$ [1]).

The number of vertices is $2^4 . 4!$ divided by $2!$, i. e. $16 . 24 : 2 = 192$.

The number of the edges passing through each vertex is five. For the pattern vertex

$$1+2\sqrt{2}\ ,\ 1+\sqrt{2}\ ,\ 1+\sqrt{2}\ ,\ 1$$

is adjacent to the five vertices

$$\left.\begin{array}{cccc} 1+\sqrt{2}\ , & 1+2\sqrt{2}\ , & 1+\sqrt{2}\ , & 1 \\ 1+\sqrt{2}\ , & 1+\sqrt{2}\ , & 1+2\sqrt{2}\ , & 1 \\ 1+2\sqrt{2}\ , & 1\ , & 1+\sqrt{2}\ , & 1+\sqrt{2} \\ 1+2\sqrt{2}\ , & 1+\sqrt{2}\ , & 1\ , & 1+\sqrt{2} \\ 1+2\sqrt{2}\ , & 1+\sqrt{2}\ , & 1+\sqrt{2}\ , & -1 \end{array}\right\},$$

[1]) In vol. XI of the „Wiskundige Opgaven" we have recently treated the polytope $[1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1]$ and its projections on its four kinds of axes (problem 78) and deduced the symbol of characteristic numbers of the polytope $[1+(n-1)\sqrt{2},\ 1+(n-2)\sqrt{2}, \ldots,\ 1+\sqrt{2}, 1]$ of S_n (problem 80). For the latter point compare also my paper „On the characteristic numbers of the polytopes $e_1 e_2 \ldots e_{n-2} e_{n-1} S(n+1)$ and $e_1 e_2 \ldots e_{n-2} e_{n-1} M_n$ of space S_n" (Mathematical congress, Cambridge, August 1912).

which may be indicated by the brackets and the negative sign after 1 in the symbol

$$1 + 2\sqrt{2}\ ,\ 1 + \sqrt{2}\ ,\ 1 + \sqrt{2}\ ,\ 1(-).$$

So the number of edges is $\frac{192 \times 5}{2} = 480$.

In order to find spaces which may contain limiting bodies we have to consider the equations

$$\begin{array}{lll} a) \ldots & \pm x_1 & = 1 + 2\sqrt{2}, \\ b) \ldots & \pm x_1 \pm x_2 & = 2 + 3\sqrt{2}, \\ c) \ldots & \pm x_1 \pm x_2 \pm x_3 & = 3 + 4\sqrt{2}, \\ d) \ldots & \pm x_1 \pm x_2 \pm x_3 \pm x_4 & = 4(1 + \sqrt{2}). \end{array}$$

a). The equation $x_1 = 1 + 2\sqrt{2}$ gives us for the other coordinates the system represented by $x_2, x_3, x_4 = [1 + \sqrt{2}, 1 + \sqrt{2}, 1]$, i.e. an $e_1\, C$. This $t\, C$ presents itself 2. 4 times, as in the equation $\pm x_i = 1 + 2\sqrt{2}$ the sign may be either positive or negative (factor 2), while the index i may be any of the four indices 1, 2, 3, 4 (factor 4).

b). The condition $x_1 + x_2 = 2 + 3\sqrt{2}$ gives $x_1, x_2 = (1 + 2\sqrt{2}, 1 + \sqrt{2})$ and $x_3, x_4 = [1 + \sqrt{2}, 1]$, i.e. we have for the coordinates in their natural order of succession

$$x_1, x_2, x_3, x_4 = (1 + 2\sqrt{2}, 1 + \sqrt{2})[1 + \sqrt{2}, 1]$$

representing an octagonal prism P_8 with end planes parallel to $O(X_3\, X_4)$ and edges normal to these planes parallel to the lines $x_1 + x_2 =$ constant in $O(X_1\, X_2)$; this P_8 occurs 2^2. 6 times, as we dispose in $\pm x_i \pm x_j = 2 + 3\sqrt{2}$ over two couples of signs (factor 2^2) and the pair of indices i, j stands for any of the combinations of the four indices by two (factor 6).

c) In the supposition $x_1 + x_2 + x_3 = 3 + 4\sqrt{2}$ we find in the same way

$$x_1, x_2, x_3, x_4 = (1 + 2\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2})\, [1],$$

i.e. a triangular prism P_3 occurring 2^3. 4 times.

d) Finally for $\Sigma x = 4(1 + \sqrt{2})$ we get

$$x_1, x_2, x_3, x_4 = (1 + 2\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2}, 1),$$

which — compare the last result of art. 46 — is a CO, occurring 2^4 times.

So, all in all we have got the limiting bodies

$$8\, tC\ ,\ 24\, P_8\ ,\ 32\, P_3\ ,\ 16\ CO;$$

so their number is 80.

As the numbers of faces of tC, P_8, P_3, CO are respectively 14, 10, 5, 14, the total number of faces is

$$\tfrac{1}{2}\,(8 \times 14 + 24 \times 10 + 32 \times 5 + 16 \times 14) = 368.$$

So the final result is (192, 480, 368, 80), in accordance with the law of Euler.

Remark. In the case of the measure polytope C_8 of S_4 represented by [1, 1, 1, 1] the spaces represented by

$$\begin{array}{ll} a) \ldots\ldots & x_1 = 1 \\ b) \ldots\ldots & x_1 + x_2 = 2 \\ c) \ldots\ldots & x_1 + x_2 + x_3 = 3 \\ d) \ldots\ldots & x_1 + x_2 + x_3 + x_4 = 4 \end{array}$$

contain respectively a limiting cube, a face, an edge, a vertex of C_8. So we find here in the case of the chosen example

8 tC	of	body	import,
24 P_8	„	face	„
32 P_3	„	edge	„
16 CO	„	vertex	„

52. *Example* **[1 + 3 √2, 1 + 2 √2, 1 + 2 √2, 1 + √2, 1]**.

The number of vertices is $2^5 . 5! : 2! = 32 . 120 : 2 = 1920$.

The number of edges passing through each vertex is six, as can be derived from the symbol

$$1 + 3\sqrt{2},\ 1 + 2\sqrt{2},\ 1 + 2\sqrt{2},\ 1 + \sqrt{2},\ 1\ (-),$$

containing five brackets and the negative sign after 1. So the number of edges is $\dfrac{1920 \times 6}{2} = 5760$.

In this case the limiting polytopes can only lie in spaces S_4 with equations of the form

$$\begin{array}{lll} a) \ldots & \pm x_1 & = 1 + 3\sqrt{2}, \\ b) \ldots & \pm x_1 \pm x_2 & = 2 + 5\sqrt{2}, \\ c) \ldots & \pm x_1 \pm x_2 \pm x_3 & = 3 + 7\sqrt{2}, \\ d) \ldots & \pm x_1 \pm x_2 \pm x_3 \pm x_4 & = 4 + 8\sqrt{2}, \\ e) \ldots & \pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 & = 5 + 8\sqrt{2}, \end{array}$$

corresponding respectively to

$a)\,..2\ .\ \ 5$ polytopes $(1+3\sqrt{2})[1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1]$,
$b)\,..2^2.10$ „ $(1+3\sqrt{2},\ 1+2\sqrt{2})[1+2\sqrt{2},\ 1+\sqrt{2},\ 1]$,
$c)\,..2^3.10$ „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2})[1+\sqrt{2},\ 1]$,
$d)\,..2^4.\ 5$ „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2})[1]$,
$e)\,..2^5.$ „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1)$,

Of these groups of polytopes the first, of polytope import, can be studied by itself; it proves to be a form with the characteristic numbers (192, 384, 248, 56), an $e_1 e_2 C_8$. The second group consists of prisms on $[1+2\sqrt{2},\ 1+\sqrt{2}, 1] = tCO$ as base, the third group of prismotopes (3 ; 8), the fourth group of prisms on $(1+3\sqrt{2},\ 1+2\sqrt{2}, 1+2\sqrt{2},\ 1+\sqrt{2}) = CO$ as base. According to art. 46 the fifth group, of vertex import, contains forms $e_1 e_3 S(5)$. So we find

$$10\, e_1 e_2 C_8 + 40 P_{tCO} + 80\,(8;3) + 80\, P_{CO} + 32\, e_1 e_3 S(5) = \\ = 242 \text{ polytopes},$$

and, as $e_1 e_2 C_8$, P_{tCO}, (8; 3), P_{CO}, $e_1 e_3 S(5)$ admit respectively 56, 28, 11, 16, 30 limiting bodies

$$\tfrac{1}{2}(10 \times 56 + 40 \times 28 + 80 \times 11 + 80 \times 16 + 32 \times 30) = \\ = 2400 \text{ polyhedra}.$$

So, according to the law of Euler, the number of faces is 6000, and the final result a (1920, 5760, 6000, 2400, 242). [1])

53. We pass now to the more direct method going straight on from vertices to limits with the highest number of dimensions, and apply it to the second example

$$[\mathbf{1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1}]$$

of the preceding article. But in order to make the symbols less clumsy and thereby the method more manageable we represent once more $1+p\sqrt{2}$ by p'.

The number of vertices was and remains 1920.

According to the symbols the edges split up into four groups, viz. $(3'\,2')$, $(2'\,1')$, $(1'\,1)$, $[1]$. Here $(3'\,2')$ means that any determinate pair of coordinates each affected by a given sign take the interchangeable values $3'$ and $2'$, the other coordinates retaining the same values; whilst $[1]$ means that any determinate coordinate takes successively the values $+1$ and -1, the other coordinates remaining unaltered.

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[1]) **The fourth and the sixth column of Table IV contain the characteristic numbers and the limiting elements of the highest number of dimensions. The meaning of the second column, of the small subscripts in column four and of the fraction in column five, will be explained later on.**

It is easy to calculate the numbers of edges of each group. Through the pattern point with the coordinates 3′, 2′ 2′ 1′, 1 pass — on account of the two digits 2′ — *two* edges (3′ 2) and (2′ 1′), and *one* edge (1′ 1) and [1]. So there are in toto
1920 edges (3′ 2′), 1920 edges (2′ 1′), 960 edges (1′ 1), 960 edges [1], i. e. 5760 edges.

Remark. We may notice that [1] with one digit only is equivalent, as to the representation of edges, to (3′ 2′), (2′ 1′), (1′ 1) with two digits. This difference is explained by the different character of the symbols: the digits between square brackets have given *absolute* values, whilst the digits between round brackets satisfy a linear equation, the sum of the digits being constant. This difference will repeat itself throughout the whole section; so [1′ 1] is a face, an octagon, and (3′ 2′ 2′) is a face, a triangle, etc.

By applying the notions of "unextended" and "extended" symbols, of the "syllables" of these symbols, etc., given for the offspring of the simplex in art. 9, to the group of polytopes deduced from the measure polytope we easily extend this direct method to faces. According to the symbols the faces split up into eight groups, viz: the triangles (3′ 2′ 2′) and (2′ 2′ 1′), the squares (3′ 2′) (2′ 1), (3′ 2′) (1′ 1), (3′ 2′) [1], (2′ 1′) [1], the hexagon (2′ 1′ 1) and the octagon [1′ 1]. In the pattern vertex P concur one of each of the two groups of triangles, one octagon and — on account of the two digits 2′ — two of each of the four groups of squares, two hexagons. So we find

$$1920\left(\frac{2 \text{ triangles}}{3}+\frac{8 \text{ squares}}{4}+\frac{2 \text{ hexagons}}{6}+\frac{1 \text{ octagon}}{8}\right)$$
$= 1280$ triangles $+$ 3840 squares $+$ 640 hexagons $+$ 240 octagons, i. e. 6000 faces.

According to the symbols the limiting bodies split up into nine groups:

(3′ 2′ 2′ 1′), (3′ 2′ 2′)(1′ 1), (3′ 2′ 2′) [1], (3′ 2′) (2′ 1′ 1), (3′ 2′) (2′ 1′) [1], (3′ 2′) [1′ 1], (2′ 2′ 1′ 1), (2′ 2′ 1′) [1], [2′ 1′ 1],

i. e. taken in the same order of succession, of

CO, P_3, P_3, P_6, C, P_8, tT, P_3, tCO.

So we find through P

$$CO + 3\, P_3 + 2\, P_6 + 2\, C + 2\, P_8 + tT + 2\, tCO$$

and therefore in toto

$$1920\left(\frac{CO}{12}+\frac{3\,P_3}{6}+\frac{2\,P_6}{12}+\frac{2\,C}{8}+\frac{2\,P_8}{16}+\frac{tT}{12}+\frac{2\,tCO}{48}\right)$$
$$=160\,CO+960\,P_3+320\,P_6+480\,C+240\,P_8+160\,tT+80\,tCO$$

i. e. 2400 limiting polyhedra.

According to the symbols the limiting polytopes split up into five groups viz. $(3'\,2'\,2'\,1'\,1)$, $(3'\,2'\,2'\,1')[1]$, $(3'\,2'\,2')[1'\,1]$, $(3'\,2')[2'\,1'\,1]$, $[2'\,2'\,1'\,1]$, i. e., taken in the same order of succession, of $e_2\,e_3\,S(5)$, P_{CO}, $(3\,;\,8)$, P_{tCO}, $e_1\,e_2\,C_8$.

So we find through P

$$e_2\,e_3\,S(5)+P_{CO}+(3\,;8)+2\,P_{tCO}+e_1\,e_2\,C_8$$

and therefore in toto

$$1920\left(\frac{e_2\,e_3\,S(5)}{60}+\frac{P_{CO}}{24}+\frac{(3\,;8)}{24}+\frac{2\,P_{tCO}}{96}+\frac{e_1\,e_2\,C_8}{192}\right)=$$
$$=32\,e_1\,e_3\,S(5)+80\,P_{CO}+80\,(3\,;\,8)+40\,P_{tCO}+10\,e_1\,e_2\,C_8,$$

i. e. the same 242 polytopes found in the preceding article.

54. If we exclude once more the "petrified" syllables (11), (111), etc. introduced in art. 9 we can state the:

THEOREM XXX. "We obtain the extended symbols of all the groups of d-dimensional limits $(P)_d$ with different symbol of any given n-dimensional polytope $(P)_n$ derived from the measure polytope M_n of space S_n, if we split up the n digits of the pattern vertex in all possible ways, *either* into $n-d$ *or* into $n-d+1$ groups of adjacent digits, place all these groups with exception of the last one of the second case between round and this last one between square brackets, and consider these bracketed groups as the syllables of the extended symbol."

Proof. As in art. 10 we represent the $n-d$ different syllables in round brackets by $(\,.\,.\,)^{k_1}, (\,.\,.\,)^{k_2}, \ldots, (\,.\,.\,)^{k_{n-d}}$. So, in the first case we have the relation $k_1+k_2\ldots+k_{n-d}=n$, whilst addition of the syllable $[\,.\,.\,]^{k'}$ with k' digits leads in the second case to the condition $k_1+k_2+\ldots+k_{n-d}+k'=n$. In both cases we suppose in order to fix the ideas that to $(\,.\,.\,)^{k_1}$ correspond the coordinates $x_1, x_2, \ldots, x_{k_1}$, to $(\,.\,.\,)^{k_2}$ the coordinates $x_{k_1+1}, x_{k_1+2}, \ldots, x_{k_1+k_2}$, etc. and in the second case to $[\,.\,.\,]^{k'}$ the coordinates $x_{n-k'+1}$, $x_{n-k'+2}, \ldots, x_n$.

Here too the proof splits up into three parts. As the first case can be deduced from the second by supposing $k'=0$, we indicate the alterations which the three parts of the proof of art. 10 have to undergo for the second case only.

a). The polytope obtained is a $(P)_d$.

By the exclusion of petrified syllables we are sure here too that any syllable $(\ldots)^k$ with k digits allows the vertex, the coordinates of which are the n digits of the symbol of $(P)_n$, to coincide successively with all the vertices of a definite $k - 1$-dimensional polytope $(P)_{k-1}$ situated in a space S_{k-1} determining equal segments on k of the n axes OX_i. Moreover the unique syllable $[\ldots]^{k'}$ with k' digits allows that vertex to coincide successively with all the vertices of a definite k'-dimensional polytope $(P)_{k'}$ situated in a space $S_{k'}$ parallel to the space of coordinates $S'_{k'}$ containing the k' axes OX_i, where i is successively $n - k' + 1$, $n - k' + 2$, .. , n. The spaces bearing these $n - d + 1$ polytopes $(P)_k$, $(k = k_1, k_2, \ldots k_{n-d})$, and $(P)_{k'}$ are by two normal to each other. For $(P)_{k_1}$ lies in the space $S_{k_1} = O(X_1 X_2 \ldots X_{k_1})$, $(P)_{k_2}$ lies in the space $S_{k_2} = O(X_{k_1+1} X_{k_1+2} \ldots X_{k_1+k_2})$, etc. and now the spaces $S_{k_1}, S_{k_2}, \ldots, S_{k_{n-d}}, S_{k'}$ form a set of coordinate spaces containing together all the axes OX_i once, i. e. they are by two perfectly normal to each other. So, as two spaces lying in spaces perfectly normal to each other are themselves perfectly normal to each other, the spaces bearing the $n - d + 1$ polytopes found above partake by two of that property. So the polytope under consideration is a prismotope with $n - d + 1$ constituents and this prismotope is a $(P)_d$; for its number of dimensions is the sum of the numbers $k_1 - 1$, $k_2 - 1, \ldots, k_{n-d} - 1$, k' of the dimensions of the constituents, i. e. the sum of the numbers $k_1, k_2, \ldots k_{n-d}$ diminished by $n - d$, i. e. n diminished by $n - d$, i. e. d.

b). The $(P)_d$ *obtained is a limit of* $(P)_n$.

According to the manner in which $(P)_d$ is obtained the coordinates of its vertices satisfy the $n-d$ mutually independent equations

$$x_1 + x_2 + \ldots + x_{k_1} = p_1, x_{k_1+1} + x_{k_1+2} + \ldots + x_{k_1+k_2} = p_2, \text{ etc.},$$

if p_1 is the sum of the first k_1 digits of the pattern vertex, p_2 the sum of the next k_2 digits, etc. As in art. 10 these equations can be written in the form

$$\sum_{i=1}^{k_1} x_i = p_1, \sum_{i=1}^{k_1+k_2} x_i = p_1 + p_2, \ldots, \sum_{i=1}^{k_1+k_2+\ldots+k_{n-d}} x_i = p_1 + p_2 + \ldots + p_{n-d},$$

representing $n-d$ limiting spaces S_{n-1} of $(P)_n$, as each of the right hand members is a maximum. For the rest of this part we refer to art. 10.

c) By means of the theorem we obtain all the limits $(P)_d$ of $(P)_n$.

For this part compare also art. 10.

55. We apply the notion of end digits and middle digits of the syllables, introduced in art. 12, to the syllables in round brackets occurring in the symbols of the polytopes deduced from the measure polytope, in order to be able to repeat theorem XXX, in a version connected with the more practical unextended symbols, in the following form:

THEOREM XXX′. "We obtain the unextended symbol of a polytope $(P)_d$ the vertices of which are vertices of the given $(P)_n$, if we put the lowest k digits of the pattern vertex between square brackets, where k takes successively one of the values $0, 1, 2, \ldots, d$, and place before it, of the $n - k$ remaining digits, between round brackets either one group of $d - k + 1$ interchangeable digits, or two groups containing together $d - k + 2$ interchangeable digits, or three groups containing together $d - k + 3$ interchangeable digits, etc., this process winding up where the total number of groups is $n - d + k$ for $n < 2d - k + 1$ and d for $n > 2d - k - 1$".

"This $(P)_d$ will be a limiting polytope of $(P)_n$, if the syllables between round brackets satisfy the two following conditions:

1°. each syllable with middle digits exhausts these digits of the symbol of $(P)_n$,

2°. no two syllables without middle digits have the same end digits".

The proof of this new version can be deduced fron the articles 10, 12 and 54.

By means of theorem XXX′ we deduce the limits $(P)_6$ of the polytope $(P)_{10}$ represented by the symbol [**5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1**], of which — as is easily shown[1]) — the $(P)_9$ of art. 12 represented by (5443322210) is the limit g_0 of vertex import. If we put together the different $(P)_6$ for which the k has the same value we find for $k = 0$ the 58 polytopes given in art. 12 and for $k = 1, 2, \ldots, 6$ successively groups of 33, 11, 9, 6, 2, 1, i. e. in toto 120 polytopes. If for brevity the last syllable — between square brackets — is put at the head of each group, these are

[1]) In rectangular coordinates the polytope g_0 is (5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1) which may be simplified by passing to parallel axes with the point 1, 1, ..., 1 as origin, i. e. by subtracting a unit from all the coordinates. If we then bear in mind that according to art. 1 we have to divide the coordinate values by $\sqrt{2}$ if we pass to barycentric coordinates on account of the new unit of length, we find (5443322210).

From this relation between a polytope deduced from the measure polytope and its polytope of vertex import can be deduced generally that the number of these polytopes in S_n, the measure polytope itself included, is $C + 2N + 1$, where C and N represent the numbers of central symmetric and of non central symmetric polytopes in S_{n-1} of simplex extraction, the simplex itself included.

$k = 1$, last syllable [0]

(544332), — (54433) (21) — (5443) (322), (5443) (32) (21), (5443) (221), — (544) (3322), (544) (332) (21), (544) (3222), (544) (322) (21), (544) (32) (221), (544) (2221), — (54) (43322), (54) (4332) (21), (54) (433) (221), (54) (43) (3222), (54) (43) (332) (21), (54) (43) (32) (221), (54) (33222), (54) (3322) (21), (54) (332) (221), (54) (32221), — (443322), — (44332) (21), — (4433) (221), — (443) (3222), (443) (322) (21), (443) (32) (221), — (433222), — (43322) (21), — (4332) (221), — (433) (2221), — (43) (32221), — (332221)

$k = 2$, last syllable [10]

(54433), — (5443) (32), — (544) (322), (544) (322), — (54)(4332), (54) (43) (322), — (44332), — (443) (322), — (43322), — (43) (3222), — (33222)

$k = 3$, last syllable [210]

(5443), — (544) (32), — (54) (433), (54) (43) (32), — (4433), — (443) (32), — (4332), — (43) (322), — (3322)

$k = 4$, last syllable [2210]

(544), — (54) (43), — (443), — (433), — (43) (32), — (332)

$k = 5$, last syllable [22210]

(54), — (43)

$k = 6$, *only* syllable [322210].

We remark, that in general the k of the theorem indicates how many of the axes of the rectangular system of coordinates are parallel to the space S_6 bearing the $(P)_6$. For $d = n - 1$, i. e. if we determine the limits of the highest number of dimensions, the k is at the same time the index of the symbol g_k indicating the import. For comparison we put side by side in the next table the different g_k of the polytope $(P)_{10}$ just treated and those of its polytope of vertex import

(5443322210) . . . . . . g_0	
(544332221)[0]. . . . . g_1	(544332221) . . . . g_8
(54433222)[10]. . . . . g_2	(54433222)(10). . . . . g_7
(5443322)[210]. . . . . g_3	(5443322)(210). . . . . g_6
(544332)[2210]. . . . . g_4	(544332)(2210). . . . . g_5
(54433)[22210]. . . . . g_5	(54433)(22210). . . . . g_4
(5443)[322210]. . . . . g_6	(5443)(322210). . . . . g_3
(544)[3322210]. . . . . g_7	(544)(3322210). . . . . g_2
(54)[43322210]. . . . . g_8	(54)(43322210). . . . . g_1
[443322210]. . . . . g_9	(443322210). . . . g_0

From the examples given in the art. 51 and 52 it is clear that in the enumeration of the limits of the highest number of dimensions we proceed from $k = n - 1$ to $k = 0$; this principle has been followed too in column five of Table IV.

C. *Extension number and truncation integers and fractions.*

56. THEOREM XXXI. "The new polytopes, all with half edges of length unity, can be found by means of a regular extension of the measure polytope followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

This theorem is an immediate consequence of that given in art. 50 (theorem XXIX) about the equality of the absolute value of the non vanishing coefficients c_i of the coordinates x_i in the equation $\pm c_1 x_1 \pm c_2 x_2 \pm \ldots = p$ of a limiting space S_{n-1} of the polytope. As to the proof we can refer to art. 15.

The extension number is always equal to the largest digit of the symbol of coordinates. So, if in the case $[2' 1' 1]$ of tCO of three-dimensional space the cube $[1\,1\,1]$ with edge 2 is extended to the cube $[2' 2' 2']$ with edge $2(1 + 2\sqrt{2})$ it is precisely large enough to enable us to deduce $[2' 1' 1]$ from it by truncation; for the limit of face import lies in the space $\pm x_i = 2'$. Likewise in the case $[\sqrt{2}, \sqrt{2}, 0, 0]$ of C_{24} in S_4, which symbol winds up in zero, we have to extend the eightcell $[1\,1\,1\,1]$ to $[\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}]$ by multiplying its linear dimensions by $\sqrt{2}$, etc,

The manner in which the amount of truncation is measured most easily can be explained as follows. If the measure polytope $M_n^{(2)} = [\overbrace{11\ldots1}^{n}]$ of S_n with centre O is extended to $M_n^{(2\varepsilon)} = [\overbrace{\varepsilon\,\varepsilon\ldots\varepsilon}^{n}]$, ε being the extension number, and this extended $M_n^{(2\varepsilon)}$ is truncated at a k-dimensional limit $M_k^{(2\varepsilon)}$ with centre M by a space S_{n-1} normal to OM cutting in R any edge PQ of $M_n^{(2\varepsilon)}$ one end point P of which belongs to $M_k^{(2\varepsilon)}$, then $\frac{PR}{PQ}$ is considered as the "truncation fraction". Now, as we will prove immediately, PR is always a multiple of $\sqrt{2}$ with half the edge of $M_n^{(2)}$ as unit, whether the symbol of coordinates of the polytope deduced from $M_n^{(2\varepsilon)}$ by truncation terminates in unity or in zero; so, in the relation $PR = q\sqrt{2}$ the multiplicator q which is integer may be called the "truncation integer". So the truncation fraction $\frac{q\sqrt{2}}{2\varepsilon}$ is irrational if the symbol of coordinates of the polytope winds up in 1 and rational if the last digit of that symbol is zero.

57. If we indicate the truncation numbers corresponding successivily to a truncation at a vertex, an edge, a face, ... by $\tau_0, \tau_1 \tau_2, \ldots$ and p' stands once more for $1 + p\sqrt{2}$ we have:

THEOREM XXXII. "If $[m'_0, m'_1, m'_2, \ldots, m'_{n-1}]$ is the symbol of coordinates of a polytope deduced from the measure polytope $M_n^{(2)}$ of S_n — where m'_{n-1} stands for either 1 or 0 — the truncation numbers $\tau_0, \tau_1, \tau_2, \ldots$ are

$$\tau_0 = n\, m_0 - \sum_{i=0}^{n-1} m_i,\quad \tau_1 = (n-1)\, m_0 - \sum_{i=0}^{n-2} m_i,\quad \tau_2 = (n-2)\, m_0 - \sum_{i=0}^{n-3} m_i, \ldots \text{"}$$

Proof. Here m'_0 is the extension number. Now, if we wish to calculate τ_k and we take for the vertices P and Q of the extented measure polytope $[m'_0, m'_0, \ldots, m'_0]$ the points $m'_0, m'_0, \ldots, m'_0$ and $- m'_0, m'_0, \ldots, m'_0$ differing in the sign of x_1 only, we have to apply the theorem of page 27 (art. 17) with respect to the equation $x_1 + x_2 + \ldots + x_{n-k} = c_t$, $(t = 1, 2, 3)$, where c_t is determined by the condition that this space is to contain successively the points P, Q and the pattern vertex $m'_0, m'_1, m'_2, \ldots, m'_{n-1}$ of the polytope under consideration. So we find

$$c_1 = (n-k)\, m'_0,\quad c_2 = (n-k-2)\, m'_0,\quad c_3 = \sum_{i=0}^{n-k-1} m'_i$$

and therefore

$$\frac{PR}{PQ} = \frac{(n-k)\, m'_0 - \sum_{i=0}^{n-k-1} m'_i}{2\, m'_0}$$

But, as $2m'_0$ is PQ, the numerator is PR. As the rational part of $(n-k)\, m'_0$ is equal to that of $\sum_{i=0}^{n-k-1} m'_i$, viz. $n-k$ for $m'_{n-1} = 1$ and zero for $m'_{n-1} = 0$, this numerator is a multiple of $\sqrt{2}$, as we have stated at the end of the preceding article. So we find $\tau_k = (n-k)\, m_0 - \sum_{i=0}^{n-k-1} m_i$, as the theorem has it.

In the case of the polytope P_{10} represented by $[5'4'4'3'3'2'2'2'1'1]$ and in the case of $[5443322210]$ we get

$$\tau_0 = 24,\ \tau_1 = 19,\ \tau_2 = 15,\ \tau_3 = 12,\ \tau_4 = 9,\ \tau_5 = 6,\ \tau_6 = 4,$$
$$\tau_7 = 2,\ \tau_8 = 1.$$

But the extension number of the first polytope is $1 + 5\sqrt{2}$, that of the second is 5.

Remark. In the application of the method of measuring the amount of truncation introduced for the simplex to the measure polytope we experience that the truncation fraction may become an improper fraction. This means that the point of intersection R of the truncating space S_{n-1} with the edge PQ lies on PQ produced at the side of Q.

If we wish to avoid this inconvenience we can determine the amount of truncation in the following new way. If O is once more the centre of the polytope and M the centre of the limit $M_k^{(2s)}$ of the extended measure polytope $M_n^{(2s)}$ at which the truncation is to take place, whilst the truncating space S_{n-1} normal at OM cuts OM in P, we may consider $\frac{PM}{OM}$ as measure for the amount of truncation. Then we find

$$\frac{PM}{OM} = \frac{(n-k)\,m'_0 - \sum_{i=0}^{n-k-1} m'_i}{(n-k)\,m'_0},$$

from which it ensues that the new truncation fraction is deduced from the old one by multiplication by $\frac{2}{n-k}$.

But instead of altering our method of measuring the amount of truncation we prefer to put up with the inconvenience indicated. So in Table IV the truncation numbers are indicated, after the extension number where $q' = 1 + q\sqrt{2}$ and $q'' = q\sqrt{2}$, according to the original system in column seven.

D. *Expansion and contraction symbols.*

58. We now prove the theorem:

THEOREM XXXII. "The expansion e_k, $(k = 1, 2, 3, \ldots, n-1)$, applied to the measure polytope $M_n^{(2)}$ of S_n changes the symbol of coordinates $[\overbrace{1, 1, \ldots, 1}^{n}]$ of that polytope into an other symbol which can be obtained by adding $\sqrt{2}$ to the first $n-k$ digits.

Proof. The operation of expansion e_k is performed by imparting to all the limits $M_k^{(2)}$ of $M_n^{(2)}$ a translational motion, to equal distances away from the centre O of $M_n^{(2)}$, each $M_k^{(2)}$ moving in the direction of the line OM joining O to its centre M, these M_k^2 remaining equipollent to their original position, the motion being extended over such a distance that the two new positions of any vertex which was common to two adjacent $M_k^{(2)}$ shall be separated by the length 2 of an edge.

Now if we move the limit $M_k^{(2)}$ for which we have

$$x_1 = x_2 = .. = x_{n-k} = 1, \quad x_{n-k+1}, x_{n-k+2}, .., x_n = [\overline{1, 1, \ldots, 1}^{k}]$$

in the manner described in the direction of the line joining O to its centre M, for which

$$x_1 = x_2 = .. = x_{n-k} = 1, \quad x_{n-k+1} = x_{n-k+2} = .. = x_n = 0,$$

to a λ times larger distance from O we get a new position of this $M_k^{(2)}$ characterized by

$$x_1 = x_2 = .. = x_{n-k} = \lambda, \quad x_{n-k+1}, x_{n-k+2}, .., x_n = [\overline{11 \ldots 1}^{k}],$$

in which it is a limit $M_k^{(2)}$ of the new polytope $[\overline{\lambda\lambda \ldots \lambda}^{n-k}\, \overline{11 \ldots 1}^{k}]$ and according to the last ten lines of art. 48 this polytope belongs to the progeniture of $M_n^{(2)}$ if we have $\lambda = 1 + \sqrt{2}$. So the result is $[\overline{1'1' \ldots 1'}^{n-k}\, \overline{11 \ldots 1}^{k}]$, which proves the theorem, and we find by the way:

THEOREM XXXIII. "In the expansion e_k the limits $M_k^{(2)}$ of $M_n^{(2)}$ are moved away from the centre to a distance always equal to $1 + \sqrt{2}$ times the original distance."

This comes true, for $1 + \sqrt{2}$ is the first digit of the symbol of coordinates of the new polytope and, as we found in art. 56, this first digit represents the extension number.

As the distance OM was $\sqrt{(n-k)}$ it becomes $(1 + \sqrt{2})\sqrt{(n-k)}$.

Remark. We may express the influence of the operation e_k on the symbol $[\overline{11 \ldots 1}^{k}]$ without interval between the digits by saying that it creates an interval $\sqrt{2}$ between the $n + k^{th}$ and the $n + k + 1^{st}$ digit.

59. THEOREM XXXIV. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $M_n^{(2)}$ on its symbol $[\overline{11 \ldots 1}^{n}]$ is found by adding together the influences of each of the expansions taken separately."

Proof. We begin by combining two expansions only.

In the succession of two expansions the subject of the second is to be what its original subject has become under the influence of the first. So in the case $e_2 e_1 C$ of the cube C (fig. 13[a]) the original subject of e_2 (the square) is transformed by e_1 into an octagon (fig. 13[b]) and now the octagon is moved out, in the case $e_1 e_2 C$ the linear subject of e_1 (the edge) is transformed by e_2 into a square (fig. 13[c]) and now this square is moved out; in both cases the result (fig. 13[d]) is the same, a tCO. In general, for $k > l$, in the case $e_k e_l M_n^{(2)}$ the subject $M_k^{(2)}$ of e_k is transformed by e_l into an $M_k^{(2')}$, while in the case $e_l e_k M_n^{(2)}$ the subject $M_l^{(2)}$ of e_l is transformed by e_k into an $n-1$-dimensional limit g_l of the import l. Here also the geometrical condition "that the two new positions of any vertex shall be separated by the length of an edge" makes the distance over which the second motion of any of these pairs has to take place equal to the distance described in the first motion of the other pair; i. e. if $M_l^{(2)}$ is a limit of the limit $M_k^{(2)}$ of $M_n^{(2)}$ and A is a vertex of that $M_l^{(2)}$, the segments described by A in transforming $M_n^{(2)}$ into the two polytopes $e_k e_l M_n^{(2)}$ and $e_l e_k M_n^{(2)}$ are the two pairs of sides, with the length $\sqrt{2(n-k)}$ and $\sqrt{2(n-l)}$, of a rectangle leading from A to the opposite vertex A'. So we find the coordinates of A' by adding to the coordinates of A the variations corresponding to the motions due to each of the operations e_k, e_l taken separately. So, in the case of three or more expansions we will have to use the extension of this rule to parallelopipeda and parallelotopes; to this geometrical composition of motions always corresponds the arithmetical addition of influences. So the general rule is proved.

By the way we still find the theorem:

THEOREM XXXV. "The operation e_k can still be applied to any expansion form deduced from $M_n^{(2)}$ in the symbol of coordinates of which the $n-k^{th}$ and the $n-k+1^{st}$ digit, i. e. the k^{th} and the $k+1^{st}$ digit counted from the end, are equal"

This theorem enables us to find immediately the expansion symbols of an expansion form deduced from $M_n^{(2)}$ with given coordinate symbol. We show this by the example **[5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1]** of art. 55.

In [5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1] five intervals occur, viz, if we represent the p^{th} digit from the end by d_p between (d_1, d_2), (d_2, d_3), (d_5, d_6), (d_7, d_8), (d_9, d_{10}). So we find $e_1 e_2 e_5 e_7 e_9 M_{10}$.

60. By means of the operations e_k we can deduce from $M_n^{(2)}$ all the possible polytopes the square bracketed symbol of coordinates of which winds up in a unit. If we wish to deduce from $M_n^{(2)}$ also all the forms with a square bracketed symbol ending in zero — which is a desideratum as to the treatment of the nets — we have to introduce the operation c of *contraction*. The subject of this contraction

is the group of limits $(l)_{n-1}$ of vertex import, sometimes denoted by g_0, the vertices of which form exactly all the vertices of the expansion form, each vertex taken once, and now the operation c consists in this: all these limits undergo a translational motion, of the same amount, towards the centre O of the expansion form, by which any of these limits gets a vertex or some vertices in common with some of the others. By this contraction the edges of the expansion form parallel to the axes of coordinates are annihilated.

We have now the general theorem:

THEOREM XXXVI. "By applying the contraction c to any expansion form all the digits of the symbol of coordinates of this form are diminished by a unit".

This theorem, which shows that the preceding one still holds for contraction forms deduced from $M_n^{(2)}$, is almost self evident. So, as the motion of the limit g_0 lying in that part of S_n where all the coordinates are positive takes place in the direction of the line making in that part of S_n equal angles whith the n axes, all the coordinates of the pattern vertex diminish by the same amount, and this process has to go on untill the smallest of the digits disappears. For then we once more obtain a polytope the symbol of coordinates of which satisfies the laws of the first part of theorem XXVIII (art 47).

Remark. By combining the theorems XXXV and XXXVI we can find the symbol in the operators c and e_k of *any* form deduced from $M_n^{(2)}$. But this process can be simplified by introducing the operation e_0 which transforms the centre O of $M_n^{(2)}$ considered as an infinitesimal measure polytope $M_n^{(0)}$ into $M_n^{(2)}$. Then the contraction symbol c can be shunted out by substituting $e_k\, e_l \ldots e_m\, M_n^{(0)}$ for $c\, e_k\, e_l \ldots e_m\, M_n^{(2)}$, but this implies that we replace $e_k\, e_l \ldots e_m\, M_n^{(2)}$ by $e_0\, e_k\, e_l \ldots e_m\, M_n^{(0)}$. This remark will be useful in part F of the next section (compare theorem LIII).

E. *Nets of polytopes.*

61. The theory of the nets derived from $M_n^{(2)}$ is based entirely on the consideration of the most simple of these nets, the net $N(M_n^{(2)})$ of the measure polytope itself. So we begin by the analytical representation of that net $N(M_n^{(2)})$.

By means of the symbol $[2a_1+1, 2a_2+1, \ldots, 2a_n+1]$ the net of $M_n^{(2)}$ is decomposed into its measure polytopes, if $a_1, a_2, \ldots, a_n$ are arbitrary integers and the heavy square brackets mean that in order to obtain a definite $M_n^{(2)}$ of the net we have to permutate and to take with either of the two signs the units printed in heavy type only. Of the M_n^2 brought to the fore by this symbol itself the centre is the point $2a_1, 2a_2, \ldots, 2a_n$. So $[2a_1, 2a_2, \ldots, 2a_n]$ may be called the "frame" of the net, and this symbol may be written quite as well with round or even without brackets, as the faculty of taking for the a_i all possible integer values includes permutation and changing of signs.

62. If we consider the net $N(M_n^{(2)})$ as a polytope [1]) of S_{n+1} with an infinite number of limits $(l)_n$ which instead of bending round in S_{n+1} fills S_n, we can apply to this polytope the expansions $e_1, e_2, \ldots, e_n$ and the contraction c, either separately or in possible combination; in this simple way the measure polytope nets $e_1 N(M_n)$, $e_2 N(M_n)$, etc. have been determined by M[rs]. STOTT. We introduce the corresponding analytical considerations by the following:

THEOREM XXXVII. "Let any expansion or expansion and contraction form $(P)_n$ of $M_n^{(2)}$ be represented by the symbol of coordinates $[a_1, a_2, \ldots, a_{n-1}, a_n]$. Let $M_n^{(2a)}$ be the measure polytope with edge $2a$ concentric and coaxial to this $(P)_n$ and $N(M_n^{(2a)})$ the net of measure polytopes to which the $M_n^{(2a)}$ belongs. Let us suppose in each of the ∞^n measure polytopes of this net a concentric polytope equipollent to $(P)_n$. Then the vertices of all the ∞^n polytopes obtained in this manner cannot form together the vertices of a net, if a differs from a_1 and from $a_1 + 1$."

This theorem of a negative tendency can be proved thus. If we call two $(P)_n$ "adjacent" if the measure polytopes $M_n^{(2a)}$ concentric to them have this position with respect to each other, i. e. if these $M_n^{(2a)}$ are in $M_{n-1}^{(2a)}$ contact, and we consider the limits $(l)_{n-1}$ of the highest import of any two adjacent $(P)_n$ deduced from the common $M_{n-1}^{(2a)}$ of the two $M_n^{(2a)}$ concentric with these $(P)_n$, we see at once that these limits g_{n-1} coincide for $a = a_1$, whilst they are at edge distance from each other and form therefore the end polytopes of a prism for $a = a_1 + 1$. In all other cases two adjacent $(P)_n$ are either too near to each other or too far apart.

What we shall have to show farther is this that the vertices of the ∞^n polytopes $(P)_n$ do form together the vertices of a net in each of the cases $a = a_1$ and $a = a_1 + 1$. We prepare the general proof of this assertion by indicating by the special case of the threedimensional net of truncated cubes $[1 + \sqrt{2}, 1 + \sqrt{2}, 1]$ included in larger cubes $M_3^{(2a)}$, where $a = 2 + \sqrt{2}$, how the other constituents are to be found. This will give us occasion to introduce

[1]) Compare art. 39.

some new geometrical terms by the use of which the expression of general laws will be simplified.

In fig. 14 is represented in heavy lines one of the tC with centre O and an eighth part of the $M_3^{(2a)}$ surrounding it, viz. that part lying in the octant of the positive coordinates taken in the directions OV'_1, OV'_2, OV'_3. Now we make to correspond to the different limiting elements of the surrounding cube the limiting elements of the tC into which the first are transformed if the tC is deduced from the surrounding cube by truncation at vertices, edges and faces. So the triangle ABC of vertex import corresponds to the vertex V, the edge AA' (or the face of edge import which replaces it in an other case) corresponds to the edge VW_1, the octagonal face $B'BCC' \ldots$ corresponds to the face W_2VW_3. Then by reflecting the triangle ABC into the three faces of $M_3^{(2a)}$ through the corresponding vertex V as mirrors and by dealing in the same way with the edge AA' with respect to the two faces through the corresponding edge VW_1 and with the face $B'BCC' \ldots$ with respect to the corresponding face W_2VW_3 we get successively the eight triangular faces of an RCO with V, the four upright edges of a P_4 with V_1, the two end planes of a P_8 with V'_1 as centre. We simplify these expressions by saying that "multiplication" of the triangle ABC round V, of the edge AA' round VW_1, of the face $B'BCC' \ldots$ round W_2VW_3 generates the indicated polyhedra RCO, $P_4 = C$, P_8.

In fig. 14 have been represented in ordinary lines the RCO generated by the triangle ABC, the three cubes generated by the edges AA', BB', CC' and the three P_8 generated by the faces $B'BCC'..$, $C'CAA'..$, $A'ABB'..$ From this diagram it is clear that the indicated RCO, C, P_8 fill up the interstitial space between the tC, i. e. that the net bearing in ANDREINI's memoir the number 22 exists; we facilitate the inspection of this diagram by adding a stereoscopic representation of it. [1])

The deduction of the coordinate symbols of the new constituents RCO, C, P_8 from those of the tC and its surrounding cube shows us, what we have to do in general in order to obtain the coordinate symbols of the new constituents.

We begin with RCO obtained by multiplying the triangle ABC round V. In order to get the representation of the triangle ABC with respect to the original axes we have to replace the square brackets of the symbol $[1+\sqrt{2}, 1+\sqrt{2}, 1]$ of tC by round ones. In order to represent that triangle with respect to new axes

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[1]) The effect is enhanced if we place it so, as to have the small arrow at the left.

VV_1, VV_2, VV_3 we have to replace the digits of $(1+\sqrt{2}, 1+\sqrt{2}, 1)$ by their complements to $a = 2+\sqrt{2}$, giving $(1, 1, 1+\sqrt{2})$, i. e. $(1+\sqrt{2}, 1, 1)$. In order to multiply the last triangle round the new origin V we have to return to square brackets. So $[1+\sqrt{2}, 1, 1]$ is the symbol of the new constituent RCO. We repeat that the digits of this new symbol are the complements to $a = 2+\sqrt{2}$ of the digits of the "groundform" tC taken in inversed order.

In the case of the edge AA' and the cube derived from it we have to assume V_1, the centre of the cube, as new origin, and V_1V, $V_1V'_2$, $V_1V'_3$ as new axes. Thereby $x_1 = [1]$, $x_2 = 1+\sqrt{2}$, $x_3 = 1+\sqrt{2}$ is transformed into $x_1 = [1]$, $x'_2 = 1$, $x'_3 = 1$; so by multiplication we get $x_1 = [1]$, $x_2, x_3 = [1, 1]$ or shorter $[1]$, $[1, 1]$, which in this special case may be combined to $x_1, x'_2, x'_3 = [1, 1, 1]$ or shorter $[1, 1, 1]$, the cube.

Finally the face $A'ABB'$... represented by $x_1, x_2 = [1+\sqrt{2}, 1]$, $x_3 = 1+\sqrt{2}$ passes by multiplication into $x_1, x_2 = [1+\sqrt{2}, 1]$, $x_3 = [1]$ or shorter $[1+\sqrt{2}, 1][1]$.

So if we arrange the constituents in the order g_3, g_2, g_1, g_0 of decreasing import we get

$$\left.\begin{array}{l} g_3 = [1+\sqrt{2}, 1+\sqrt{2}, 1] \\ g_2 = [1+\sqrt{2}, \quad 1]\ [1] \\ g_1 = [\quad 1 \quad]\quad [1\ , 1] \\ g_0 = [1+\sqrt{2}, \quad 1\ , 1] \end{array}\right\},$$

the first and the last being semiregular polyhedra deduced from the cube, whilst the intermediate ones appear as prisms. We remark that the pairs of syllables of the symbols of g_2 and g_1 can be derived from the symbols of g_3 and g_0 by taking for g_2 the last two digits of g_3 and the last digit of g_0, for g_1 the last digit of g_3 and the last two digits of g_0.

Now it is obvious that in the general case of the polytope $(P)_n$ of S_n represented by $[a_1, a_2, \ldots, a_{n-1}, a_n]$ the introduced multiplication of the limits of different import, which multiplication can be performed for any value of the constant a, leads in general to $n+1$ constituents $g_n, g_{n-1}, \ldots, g_1, g_0$, represented by

$$\left.\begin{array}{lll} g_0 & = [a_1, a_2, a_3, \ldots\ldots, a_{n-2}, a_{n-1}, a_n] & \\ g_{n-1} & = [a_2, a_3, a_4, \ldots\ldots, a_{n-1}, a_n]\ [a-a_1] & \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ g_{n-k} & = [a_{k+1}, a_{k+2}, \ldots\ldots, a_n]\ [a-a_k, a-a_{k-1}, \ldots, a-a_1] & \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ g_1 & = [a_n]\ [a-a_{n-1}, a-a_{n-2}, \ldots\ldots, a-a_2, a-a_1] & \\ g_0 & = [a-a_n, a-a_{n-1}, a-a_{n-2}, \ldots\ldots, a-a_2, a-a_1] & \end{array}\right\},$$

where g_n is given, g_0 is obtained by subtracting the digits of g_n from a and taking the differences in inverted order, while the two syllables of g_{n-k} are got by taking the last $n-k$ digits of g_n and the last k digits of g_0 for $k = 1, 2, \ldots, n-1$.

63. We now prove the following problem of positive tendency completing the preceding one.

THEOREM XXXVIII. "In either of the two cases $a = a_1$ and $a = a_1 + 1$ the vertices of the ∞^n polytopes $(P)_n$ of the preceding theorem do form together the vertices of a net. The constituents of this net are obtained by means of the algorithm developed at the end of the preceding article."

We march in the direction of the proof of this general theorem:

1°. by deducing from the symbol of coordinates of the given groundform $(P)_n$ the symbol representing all the repetitions of this polytope and therefore all the vertices of the system,

2°. by deriving from this new symbol the symbols of the polytopes different from the groundform the vertices of which belong to the system (which set of new constituents will prove to be equivalent to that obtained above by the geometrical multiplication introduced above),

3°. by showing that the system of polytopes obtained in this way fills space, i. e. that there is neither overlapping, nor hole.

Symbol of the total system of vertices. The symbol of a definite repetition of the groundform is

$$[2b_1 a + \alpha_1, \; 2b_2 a + \alpha_2, \ldots, \; 2b_{n-1} a + \alpha_{n-1}, \; 2b_n a + \alpha_n], \ldots T')$$

where $b_1, b_2, \ldots, b_{n-1}, b_n$ is a definite set of arbitrarily chosen integers. So this symbol represents the total system of vertices, if the b_i denote all possible sets of integers.

From the symbol T' we deduce the frame symbol

$$[2b_1 a, \; 2b_2 a, \ldots, \; 2b_{n-1} a, \; 2b_n a], \ldots\ldots\ldots\ldots F')$$

representing the system of vertices of a net of measure polytopes $M_n^{(2a)}$, one of which has the origin as vertex and the n spaces $x_i = 0, (i = 1, 2, \ldots, n)$ as limiting spaces.

Presumptive new constituents. The most general transformation by which the total system of vertices T') passes into itself consists in a transport of $p_i a$ units from the permutable to the unmovable part of x_i, the n quantities p_i being integer. But this process is limited by the restriction that in the case of a new constituent sought the permutable parts placed within the same pair of square brackets have to satisfy the conditions of theorem XXVIII, from which it ensues that the extent of the restriction depends on the number of syllables which the symbol of any constituent may contain. This number is evidently two at most. For the process can only afford besides the original minimum digit a_n one new minimum digit, viz. zero in the case $a = a_1$ and unity in the case $a = a_1 + 1$. So we have to hunt up only new constituents the symbols of which are either monosyllabic or composed of two syllables.

If we take *all* the p_i equal to one we find

$$[(2b_1+1)a+\alpha_1-\alpha, (2b_2+1)a+\alpha_2-\alpha, \ldots,$$
$$(2b_{n-1}+1)a+\alpha_{n-1}-\alpha, (2b_n+1)a+\alpha_n-\alpha],$$

or, if we replace negative permutable parts by the positive ones of the same absolute value, rearrange these positive parts according to decreasing order and substitute for brevity β' for $2b+1$,

$$[\beta_1' a+\alpha-\alpha_n, \beta_2' a+\alpha-\alpha_{n-1}, \ldots, \beta'_{n-1} a+\alpha-\alpha_2, \beta'_n a+\alpha-\alpha_1], \ldots T')$$

winding up in zero for $a = a_1$ and in unity for $a = a_1 + 1$. So we find the repetitions of the new constituent g_0 of the last list of the preceding article. This form g_0 and the given form g_n we started from are the only constituents of measure polytope descent proper.

If we transform the first k digits of T by the transport of a units from the permutable parts to the unmovable ones and put each of the two sets of digits, the set of the k transformed ones and the set of the $n-k$ untransformed ones, between square brackets, we get after rearranging, if β'_i still replaces $2b_i + 1$ and β_i is substituted for $2b_i$

$$[\beta'_1 a+\alpha-\alpha_k, \beta'_2 a+\alpha-\alpha_{k-1}, \ldots, \beta'_k a+\alpha-\alpha_1]$$
$$[\beta_{k+1} a+\alpha_{k+1}, \beta_{k+2} a+\alpha_{k+2}, \ldots, \beta_{n-1} a+\alpha_{n-1}, \beta_n a+\alpha_n], \ldots T'')$$

revealing the new constituent

$$[a-a_k, a-a_{k-1}, \ldots, a-a_1][a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n],$$

a prismotope $(P_k; P_{n-k})$ with the constituents $(P)_k$ and $(P)_{n-k}$ represented by each of the two syllables of the symbol taken separately; if the digits of the second syllable correspond to the coordinates x_1, $x_2, \ldots, x_{n-k}$ and those of the first syllable to $x_{n-k+1}, x_{n-k+2}, \ldots, x_n$, this prismotope is the constituent g_{n-k} of the last list of the preceding article. In the latter case the different positions of $(P)_{n-k}$ are parallel to $O(X_1 X_2 \ldots X_{n-k})$, those of $(P)_k$ to $O(X_{n-k+1} X_{n-k+1} \ldots X_n)$. So we find again all the new constituents obtained formerly by geometrical multiplication.

No overlapping and no hole. By a translational motion in the direction of one of the axes over a distance $2a$ the system of vertices T) is transformed in itself; so, if the central measure polytope $[\overbrace{a, a, \ldots, a}^{n}]$ is filled exactly by the set of constituents found above, these constituents form a net By a reflection in one of the n spaces $x_i = 0$, $(i = 1, 2, \ldots, n)$, the system T) also is transformed in itself; so, if the part of the central measure polytope $M_n^{(2a)}$ containing the points with positive coordinates only is filled exactly, the constituents form a net. We indicate this part of the central measure polytope by the symbol $M_n^{(+a)}$.

We now prove the following lemma:

„Let $(P)_n^a$ be a constituent lying partially within $M_n^{(+a)}$ and $(P)_{n-1}^{a,b}$ any of its limits lying partially within $M_n^{(+a)}$. Then the set of polytopes obtained above always contains one and only one polytope $(P)_n^b$ having with $(P)_n^a$ the limit $(P)_{n-1}^{a,b}$ in common; this $(P)_n^b$ lies with respect to $(P)_n^a$ on the opposite side of $(P)_{n-1}^{a,b}$."

The condition that $(P)_n^a$ lies at least partially within $M_n^{(+a)}$ is fulfilled, if we consider that repetition of the chosen constituent the coordinates of the centre of which admit the values $+a$ and zero only. We find, if all the coordinates are zero the groundform contained in T), if all the coordinates are $+a$ a polytope contained in T'), if some coordinates are $+a$ and the other ones zero a polytope contained in T''). Now the first case, of the groundform, and the second case, of all coordinates $= +a$, are included in the third case, as we get them by putting $k = 0$ and $k = n$. So we can choose for $(P)_n^a$ the polytope

$$\underset{x_1,\, x_2,\, \ldots\ldots,\, x_k}{[a + \boldsymbol{a} - \boldsymbol{a}_k, a + \boldsymbol{a} - \boldsymbol{a}_{k-1}, \ldots, a + \boldsymbol{a} - \boldsymbol{a}_1]}\underset{x_{k+1},\, x_{k+2},\, \ldots\ldots,\, x_n}{[a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n]}$$

where the x_i placed under the two syllables indicate the coordinates to which the two sets of digits refer, and occupy ourselves with the question how to get a limit $(l)_{n-1}$ of this prismotope. Now in general the limits $(l)_{n-1}$ of the prismotope $(P_k; P_{n-k})$ present themselves in two groups, viz. if $(P)_{k-1}$ is any limit $(l)_{k-1}$ of $(P)_k$ and $(P)_{n-k-1}$ any limit $(l)_{n-k-1}$ of $(P)_{n-k}$, in the two forms $(P_{k-1}; P_{n-k})$ and $(P_k; P_{n-k-1})$. So [1]), we have to consider the two different cases

[1]) For a limit $(P)_{n-1}^{a,b}$ lying at least partially within M_n^{+a} none of the coordinates may assume values equal to or surpassing $+a$ for *all* the vertices of that limit; therefore in the first case $(P_{k-1}; P_{n-k})$ we have to place between round brackets a certain number s_1 of the *largest* digits $[a + \boldsymbol{a} - \boldsymbol{a}_i]$ where $\boldsymbol{a} - \boldsymbol{a}_i$ is taken with the reversed sign, i. e. a_{k_1} $a_{k-1}, \ldots, a_{k-s_1+1}$ taken in inverted order.

$$\left.\begin{matrix}\left[a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1}, a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1-1},\ldots, a+\boldsymbol{a}-\boldsymbol{a}_1\right] & (a_{k-s_1+1}, a_{k-s_1+2},\ldots, a_k) \\ x_1, x_2, \ldots\ldots\ldots, x_{k-s_1} & x_{k-s_1+1}, x_{k-s_1+2},\ldots, x_k \\ & \left[a_{k+1}, a_{k+2},\ldots, a_{n-1}, a_n\right] \\ & x_{k+1}, x_{k+2},\ldots, x_n\end{matrix}\right\},$$

$$\left.\begin{matrix}\left[a+\boldsymbol{a}-\boldsymbol{a}_k, a+\boldsymbol{a}-\boldsymbol{a}_{k-1},\ldots, a+\boldsymbol{a}-\boldsymbol{a}_1\right] & \\ x_1, x_2, \ldots\ldots\ldots, x_k & \\ (a_{k+1}, a_{k+2},\ldots\ldots, a_{k+s_2}) & \left[a_{k+s_2+1}, a_{k+s_2+2},\ldots, a_{n-1}, a_n\right] \\ x_{k+1}, x_{k+2},\ldots\ldots, x_{k+s_2} & x_{k+s_2+1}, x_{k+s_2+2},\ldots, x_n\end{matrix}\right\}$$

which two limits $(l)_{n-1}$ admit as centres the points

$$\left.\begin{matrix}\underbrace{aa\ldots a}_{k-s_1}\ \underbrace{t_1 t_1\ldots t_1}_{s_1}\ \underbrace{00\ldots 0}_{n-k} \\ \underbrace{aa\ldots a}_{k}\ \underbrace{t_2 t_2\ldots t_2}_{s_2}\ \underbrace{00\ldots 0}_{n-k-s_2}\end{matrix}\right\},$$

t_1 and t_2 being determined by the relations

$$s_1 t_1 = \sum_{i=k-s_1+1}^{k} a_i \quad , \quad s_2 t_2 = \sum_{i=k+1}^{k+s_2} a_i \quad ,$$

showing that we have $0 < t_i < a$ for $i = 1, 2$. So the centres of these two $(l)_{n-1}$ lie on the boundary of the measure polytope $M_n^{(+a)}$ and therefore the $(l)_{n-1}$ themselves lie partially within that measure polytope.

Now for each of the two cases there is only one constituent passing through the chosen limit $(l)_{n-1}$, viz.

$$\begin{matrix}\left[a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1}, a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1-1},\ldots, a+\boldsymbol{a}-\boldsymbol{a}_1\right] & \left[a_{k-s_1+1}, a_{k-s_1+2},\ldots, a_{n-1}, a_n\right] \\ x_1, x_2,\ldots, x_{k-s_1} & x_{k-s_1+1}, x_{k-s_1+2},\ldots, x_n \\ \left[a+\boldsymbol{a}-\boldsymbol{a}_{k+s_2}, a+\boldsymbol{a}-\boldsymbol{a}_{k+s_2-1},\ldots, a+\boldsymbol{a}-\boldsymbol{a}_1\right] & \left[a_{k+s_2+1}, a_{k+s_2+2},\ldots, a_{n-1}, a_n\right] \\ x_1, x_2,\ldots, x_{k+s_2} & x_{k+s_2+1}, x_{k+s_2+2},\ldots, x_n\end{matrix}$$

So, all we have to do yet is to investigate the position of the centres. If we indicate these points by the letters G_a, G_{b_1}, G_{b_2}, G_{ab_1}, G_{ab_2} and we remark that for these five points we have

$$x_1 = x_2 = \ldots = x_{k-s_1},\ x_{k-s_1+1} = x_{k-s_1+2} = \ldots = x_k,$$
$$x_{k+1} = x_{k+2} = \ldots = x_{k+s_2},\ x_{k+s_2+1} = x_{k+s_2+2} = \ldots = x_n,$$

we find the following list of coordinates

	$x_1,\ldots$	$x_{k-s_1+1},\ldots$	$x_{k+1},\ldots$	$x_{k+s_2+1},\ldots$
G_a	a	a	0	0
G_{b_1}	a	0	0	0
G_{b_2}	a	a	a	0
G_{ab_1}	a	t_1	0	0
G_{ab_2}	a	a	t_2	0

According to this list of the two triples (G_a, G_{b_1}, G_{ab_1}), (G_a, G_{b_2}, G_{ab_2}) of collinear points G_{ab_1} lies between G_a, G_{b_1}, and G_{ab_2} between

G_a, G_{b_1}. So the proof of the lemma is given. So neither of the two systems of constituents can admit holes.

In order to show that no two polytopes of any of the two systems can overlap we remark that by means of the symbols T'), T''), T''') any polytope of the chosen system can be promoted to central polytope, which shows that not a single vertex can lie inside that polytope.

So we have proved completely now the theorem under consideration.

64. We now formulate the manner of deduction of all the measure polytope nets as follows:

THEOREM XXXIX. "Let $G = [a_1, a_2, \ldots, a_{n-1}, a_n]$ be the symbol of coordinates of the "groundform" of the net. Deduce from it the symbol $O = [a - a_n, a - a_{n-1}, \ldots, a - a_2, a - a_1]$ of the "opposite form", where a is either a_1 or $a_1 + 1$. Derive from these two symbols G, O the mixed symbol I_k of the "intermediate forms" represented by

$$[a_{n-k+1}, a_{n-k+2}, \ldots, a_{n-1}, a_n]$$
$$[a - a_{n-k}, a - a_{n-k-1}, \ldots, a - a_2, a - a_1],$$

of the two syllables of which the first contains the last k digits of G, the second the last $n - k$ digits of O. Then G, the forms I_k, $(k = n - 1, n - 2, \ldots, 2, 1)$, O are respectively the constituents $g_n, g_{n-1}, g_{n-2}, \ldots, g_2, g_1, g_0$ of the net."

"The frame of the constituent g_{n-k} is

$$[\beta_{n-k+1} a, \beta_{n-k+2} a, \ldots \beta_{n-1} a, \beta_n a, \beta'_{n-k} a, \beta'_{n-k-1} a, \ldots, \beta'_2 a, \beta'_1 a],$$

where we have $\beta_i = 2 b_i$ and $\beta'_i = 2 b_i + 1$, the b_i being integer and the digits of the first syllable being related to the odd, those of the second syllable being related to the even multiples of a".

"If (e, c), etc. indicates a net with an expansion groundform and a contraction opposite form, the theorem includes the four cases:

$$a_n = 1, a = a_1 \ldots\ldots\ldots\ldots\ldots (e, c),$$
$$a_n = 1, a = a_1 + 1 \ldots\ldots\ldots\ldots (e, e),$$
$$a_n = 0, a = a_1 \ldots\ldots\ldots\ldots\ldots (c, c),$$
$$a_n = 0, a = a_1 + 1 \ldots\ldots\ldots\ldots (c, e)."$$

In this theorem the deduction of the intermediate constituents differs slightly from that given in the preceding article, the two methods passing into each other by interchanging k and $n - k$, and the two syllables. In the new form the succession of the different constituents is a more regular one, as the following examples prove.

Example I. The two nets with **[5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1]** as groundform admit the constituents:

g_{10} . . [5′4′4′3′3′2′2′2′1′1]	g_{10} . . [5′4′4′3′3′2′2′2′1′1]
	g_{9} . . . [4′4′3′3′2′2′2′1′1] , [1]
g_{8} . . . [4′3′3′2′2′2′1′1] , [1 0]$\sqrt{2}$	g_{8} . . . [4′3′3′2′2′2′1′1] , [1′1]
g_{7} . . . [3′3′2′2′2′1′1] , [1 1 0]$\sqrt{2}$	g_{7} . . . [3′3′2′2′2′1′1] , [1′1′1]
g_{6} . . . [3′2′2′2′1′1] , [2 1 1 0]$\sqrt{2}$	g_{6} . . . [3′2′2′2′1′1] , [2′1′1′1]
g_{5} . . . [2′2′2′1′1] , [2 2 1 1 0]$\sqrt{2}$	g_{5} . . . [2′2′2′1′1] , [2′2′1′1′1]
g_{4} . . . [2′2′1′1] , [3 2 2 1 1 0]$\sqrt{2}$	g_{4} . . . [2′2′1′1] , [3′2′2′1′1′1]
g_{3} . . . [2′1′1] , [3 3 2 2 1 1 0]$\sqrt{2}$	g_{3} . . . [2′1′1] , [3′3′2′2′1′1′1]
g_{2} . . . [1′1] , [3 3 3 2 2 1 1 0]$\sqrt{2}$	g_{2} . . . [1′1] , [3′3′3′2′2′1′1′1]
g_{1} . . . [1] , [4 3 3 3 2 2 1 1 0]$\sqrt{2}$	g_{1} . . . [1] , [4′3′3′3′2′2′1′1′1]
g_{0} . . . . [5 4 3 3 3 2 2 1 1 0]$\sqrt{2}$	g_{0} . . . . . . [5′4′3′3′3′2′2′1′1′1]

Example II. The two nets with **[5 4 4 3 3 2 2 2 1 0]**$\sqrt{2}$ as groundform admit the constituents:

g_{10} . . [5443322210]$\sqrt{2}$	g_{10} . . . . [5 4 4 3 3 2 2 2 10]$\sqrt{2}$
	g_{9} . . . [1] , [4 4 3 3 2 2 2 10]$\sqrt{2}$
g_{8} . . . [43322210]$\sqrt{2}$, [10]$\sqrt{2}$	g_{8} . . . [1′1] , [4 3 3 2 2 2 10]$\sqrt{2}$
g_{7} . . . [3322210]$\sqrt{2}$, [110]$\sqrt{2}$	g_{7} . . . [1′1′1] , [3 3 2 2 2 10]$\sqrt{2}$
g_{6} . . . [322210]$\sqrt{2}$, [2110]$\sqrt{2}$	g_{6} . . . [2′1′1′1] , [3 2 2 2 10]$\sqrt{2}$
g_{5} . . . [22210]$\sqrt{2}$, [22110]$\sqrt{2}$	g_{5} . . . [2′2′1′1′1] , [2 2 2 10]$\sqrt{2}$
g_{4} . . . [2210]$\sqrt{2}$, [322110]$\sqrt{2}$	g_{4} . . . [3′2′2′1′1′1] , [2 2 10]$\sqrt{2}$
g_{3} . . . [210]$\sqrt{2}$, [3322110]$\sqrt{2}$	g_{3} . . . [3′3′2′2′1′1′1] , [2 10]$\sqrt{2}$
g_{2} . . . [10]$\sqrt{2}$, [33322110]$\sqrt{2}$	g_{2} . . . [3′3′3′2′2′1′1′1] , [10]$\sqrt{2}$
g_{0} . . . . [5433322110]$\sqrt{2}$	g_{0} . . . [5′4′3′3′3′2′2′1′1′1]

The nets of measure polytope extraction of the spaces S_3, S_4, S_5 are put on record in the Tables V and VI. The first column of these tables is concerned with the "name" of the net; it contains the system of operators e_k and c which are to precede the general symbot $N(M_n^2)$ in order to obtain the symbol of the net. This system of operators is in close connection with the consideration of the net of S_n as a simple polytope of S_{n+1}; for $a = a_1$ it is equal to the system of operators characterizing the groundform, for $a = a_1 + 1$ it consists of latter system completed by e_n. So of the three parts into which each of the three cases $n = 3$, $n = 4$, $n = 5$ has been subdivided, the first contains the nets (e, c), the second the nets (e, e), the third the nets (c, c). Therefore the question rises where the nets (c, e) are to be found.

The algorithm indicated in our last theorem immediately shows that by interchanging the two extreme forms with one another the intermediate constituents return in inverted order of succession. This remark suggests an answer to the question raised just now. By taking the constituents $g_n, g_{n-1}, \ldots, g_1, g_0$ contained in the second, third, ..., $n+1^{st}$, $n+2^{nd}$ column of the same horizontal line corresponding to a certain net in reversed order of succession we get the constituents $g'_n, g'_{n-1}, \ldots, g'_1, g'_0$ of a net bearing in general an other name, the operators occurring in which are inscribed in the $n+3^{rd}$ column; this net with constituents with complementary import is essentially the same as the original one. So by inverting the order of succession of the imports the three groups (e, c), (e, e), (c, c) pass into (c, e), (e, e), (c, c), in other words the first group furnishes the group (c, e), whilst each of the other groups passes into itself. We have used this fact, to which we shall have to come back in part *F* of this section, in order to simplify the Tables V and VI. So on one hand the nets (c, e) have been omitted totally, whilst on the other the number of lines of the groups (e, e) and (c, c) have been diminished by writing down the nets in a transparent systematical order and omitting at any time the net appearing already in inverted order under the preceding ones.[1])

In the column under the heading *p.* some particularities of the nets have been inscribed. By *r.* we have indicated that the net is regular, by *s. p.* that it is "semiperiodic", i. e. that the two extreme forms are the same which implies the equality of any two constituents with complementary import.

The other columns will be explained later on.

A survey of the results contained in the tables suggests the following remarks:

a). There is a great difference in character between the constituents of a simplex net proper on one hand and those of a measure polytope net. All the constituents of a simplex net proper are expansion and contraction forms of the simplex, whilst we found just now that in a measure polytope net in general only two of the constituents, the groundform and the opposite form, are expansion and extraction forms of the measure polytope.[2])

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[1]) The cases $ce_2\, N(C_8)$, $ce_3\, N(C_8)$, etc. do not figure in the first third part of Table II contained in the memoir of Mrs. Stott, as they appear already as expansion forms under either $N(C_{16})$ or $N(C_{24})$.

In order to spare room we have omitted in Table VI the column containing the name of the net taken in inversed order. For the upper and middle part it is always the symbol before M_5 under g_0 to which e_5 has been added, for the last part it is that symbol itself.

[2]) Compare for the prisms and prismotopes entering here my paper: "On the characteristic numbers of the prismotope", *Proceedings* of Amsterdam, vol. XIV, p. 424.

This difference in character implies a difference in the number of different positions a constituent of definite form may admit. In the case of a simplex net proper this number is *two* in general and only *one* if the form is central symmetric. In the case of a measure polytope net this number is *one* for the two extreme constituents, whilst the intermediate form I_k generally occurs in a number of different positions indicated by half the number of limits $M_k^{(2)}$ of $M_n^{(2)}$, i. e. in $2^{n-k-1}(n)_k$ different positions.

In the case of the simplex net we have considered as kind of constituent any polytope of the net with *equipollent* repetitions; when the partition cycle was a power cycle we have even been obliged to split up a *kind* of constituent into several *groups*, in order to keep the analytical treatment in contact with the geometrical facts. On account of the extreme transparency of the measure polytope nets we can allow ourselves to be less exacting and extend the notion of constituent here by admitting that the $2^{n-k-1}(n)_k$ different positions of the intermediate form I_k introduced above belong to the *same* constituent.

b). In order to be able to indicate the number of different constituents according to the new point of view we fall back on the different cases (e, c), (e, e), (c, c), (c, e) mentioned at the end of the last theorem. By generalizing the results of the two examples given above one finds immediately that the required number is in general $n - p + 1$, where p indicates the number of c's contained in the symbol. But this general number $n - p + 1$ is still to be considered as a maximum, i. e. under circumstances the number of constituents may become less. This decrease can be due to two different causes. If in the first place in one of the two groups (e, c), (c, c) of a net in S_n the expansion operator with the largest index is e_k, where $k < n - 1$, the constituents $g_k, g_{k+1}, \ldots, g_{n-2}$ are lacking together with g_{n-1}. If in the second place in one of the two groups (e, e), (c, c) a net is semiperiodic the equal constituents of complementary import may count for one constituent.

c). *Some* of the intermediate constituents may become measure polytopes, this being even the case with *all* the intermediate constituents of the net $e_n N(M_n)$. So by extending the notion of constituent still more the number of the different kinds of constituent is lessened in these cases, this number being unity for the net $e_n N(M_n)$.

d). By comparing the cases g_2 under $n = 4$ we remark that the prismotope $(4 ; 4)$ which is the measure polytope C_8 of S_4 is indicated by three different symbols; in the cases of the nets (e, c), of the nets (e, e), of the nets (c, c) we get successively:

$$[11] \,.\, [10]\sqrt{2}, \quad [11] \,.\, [11], \quad [10]\sqrt{2} \,.\, [10]\sqrt{2}$$

corresponding (fig. 15) to the projections

$$\left.\begin{matrix} A\,B\,C\,D \\ E\,F\,G\,H \end{matrix}\right\}, \quad \left.\begin{matrix} A\,B\,C\,D \\ A\,B\,C\,D \end{matrix}\right\}, \quad \left.\begin{matrix} E\,F\,G\,H \\ E\,F\,G\,H \end{matrix}\right\}$$

on the planes $O\,X_1\,X_2$ and $O\,X_3\,X_4$, if in these symbols the successive digits refer to x_1, x_2, x_3, x_4. Of these the second, equal to $[11\;11]$ occurs in one position only, whilst the two others admit respectively six and three positions in accordance with the splitting up of x_1, x_2, x_3, x_4 in $(x_1, x_2), (x_3\, x_4)$, in $(x_1, x_3), (x_2, x_4)$, in $(x_1, x_4), (x_2, x_3)$.

F. *Polarity*.

65. If we polarize an expansion or a contraction form derived from the measure polytope $M_n^{(2)}$ of S_n with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles [1]), to which corresponds the inverted symbol of characteristic numbers of the original polytope. Moreover, if $[a_1, a_2, \ldots, a_{n-1}, a_n]$ is the coordinate symbol of the original polytope, this symbol represents also the limiting spaces S_{n-1} of the new polytope in space coordinates.

For the manner in which the process of truncation is transformed by inversion compare page 69 of Section I.

66. We now pass to:

THEOREM XL. "Any polytope $(P)_n$ of measure polytope descent in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining

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[1]) Compare for this inversion page 68 of Section I.

By inversion of the measure polytope we find the cross polytope. Moreover we find in S_4, in the notation of the foot note of page 68, if $L\,e_1\,e_2\,e_3$ stands *now* for the "limiting bodies of the reciprocal polytope of $e_1\,e_2\,e_3\,C_8$",

$Le_1 = 64\;T(1_3, 3_{2+1})$,	$Lce_1 = 32\;P_3^2$,
$Le_2 = 96\;P^2_{2+1}$,	$Lce_2 = LC_{24} = 24\;O$,
$Le_3 = 64\;X$,	$Lce_3 = LC_{16} = 8\;C$,
$Le_1\,e_2 = 192\;T(1_{2+1}, 1_{2+1}, 2_{1+1+1})$,	$Lce_1\,e_2 = 96\;T(2_{2+1}, 2_{2+1})$,
$Le_1\,e_3 = 192$ symm. P^1_{deltoid},	$Lce_1\,e_3 = 96\;P_3^2$,
$Le_2\,e_3 = 192$ symm. P^1_{deltoid},	$Lce_2\,e_3 = 48\;P_4^1{}_{\text{(square)}}$,
$Le_1\,e_2\,e_3 = 384\;Y$,	$Lce_1\,e_2\,e_3 = 192\;T(1_3, 3_{2+1})$,

X representing a polyhedron limited by six faces, two groups of three equal deltoids connected in such a way as to give rise to an axis of period 3, and Y a tetrahedron limited by four unequal scalene triangles. For the shape of the tetrahedra Y compare problem 79 of vol. XI of the "Wiskundige Opgaven", where the projections of these tetrahedra on the four sets of axes of the polytope are given into the bargain.

this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain concentric spherical space, unless $(P)_n$ be the cross polytope $ce_{n-1}\, M_n$ in which special case all the spaces S_{n-1} pass through the centre O."

After the first section of this memoir had been published we perceived that the analytical proof of the corresponding theorem XXII might have been replaced by a much simpler geometrical one [1]), applicable to any polytope $(P)_n$ deduced from a regular polytope, whether simplex or not, by the operations e_k and c.

This simple geometrical proof runs as follows:

All the vertices V_i adjacent to V lie on two spherical spaces (with ∞^{n-1} points), the circumscribed one with centre O and an other with centre V and radius VV_i equal to the edge. So they lie in the spherical space (with ∞^{n-2} points) common to these two spherical spaces and therefore in the space S_{n-1} normal to VO containing this intersection. If this S_{n-1} cuts VO in P we have $2\, VP.\, VO = \overline{VV_i}^2$ from which it ensues that the distance PO is the same for all the vertices V, i. e. that the spaces S_{n-1} are the polar spaces of the vertices V with respect to a definite spherical space (with ∞^{n-1} points) round O as centre.

Moreover the special case of the cross polytope, where P coincides with O, is self evident.

67. In the section concerned with the simplex we have explained by the laws of reciprocity why it may happen that two different groups of operations of expansion applied to the simplex produce under circumstances either two polytopes equal and concentric but of opposite orientation, or the same polytope. What corresponds to this here is that any polytope derived from M_n can also be derived from the cross polytope C_{2^n} of S_n which is the reciprocal polar of M_n. As we had already occasion to remark in art. 48 we shall have to come back to this assertion in the third section.

But the state of affairs with respect to equal measure polytope nets with different expansion symbols is a quite different one. In a joint paper of Mrs. Stott and myself published two years ago [2]) it is shown geometrically that we have in general the relations:

[1]) To some of the free copies at my disposal I added a post-scriptum, containing this remark, on page 69.

[2]) Compare the second foot note of art. 38 of Section I.

$$EN = cE'e_n N', \quad Ee_n N = E'e_n N', \quad cEN = cE'N',$$

where N and N' represent polarly related regular nets of S_n, whilst the sets of operations e_k, $(k = 1, 2, \ldots, n-1)$, contained in E and E' are complementary to each other, i.e. that E' contains the operations e_{n-k} complementary to the operations e_k of E and no other one. Now, in the case of the net of measure polytopes we have $N' = N$; so we get:

THEOREM XLI. "We have the relations:

$$\begin{aligned} e_a e_b e_c \ldots e_r e_s e_t \quad NM_n^{(2)} &= ce_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} e_n NM_n^{(2)}, \\ e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)} &= e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} e_n NM_n^{(2)}, \\ ce_a e_b e_c \ldots e_r e_s e_t \quad NM_n^{(2)} &= ce_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} \quad NM_n^{(2)}, \end{aligned}$$

under the conditions

$$a + t' = b + s' = c + r' = \ldots = r + c' = s + b' = t + a' = n;$$

then the constituents $g_0, g_1, g_2, \ldots, g_{n-2}, g_{n-1}, g_n$ of the one are equal to the constituents $g'_n, g'_{n-1}, g'_{n-2}, \ldots, g'_2, g'_1, g'_0$ of the other. So the nets $e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)}$ and $ce_a e_b e_c \ldots e_r e_s e_t NM_n^{(2)}$ are semiperiodic under the conditions

$$a + t = b + s = c + r = \ldots = n.$$

In the latter cases there is an unpaired middle constituent for n even."

Proof. We prove each of the three relations by showing that the extreme constituents g_0, g_n of the net at the left of the sign of equality are equal to the constituents g'_n, g'_0 of the net at the right. But we suppose that it will do to enter into details for one of the three relations, say the second.

In the case of the net $e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)}$, where as in art. 38 we suppose the indices of the $k + 1$ factors $e_a, e_b, \ldots e_n$ to be arranged according to increasing values of the subscripts, the principal constituent g_n is, according to theorem XXXV:

$$[\overbrace{k', k', \ldots k'}^{n-t}, \overbrace{(k-1)', (k-1)', \ldots (k-1)'}^{t-s}, \overbrace{(k-2)', (k-2)', \ldots (k-2)'}^{s-r}, \ldots,$$
$$\overbrace{2', 2', \ldots 2'}^{c-b}, \overbrace{1', 1', \ldots 1'}^{b-a}, \overbrace{1, 1, \ldots 1}^{a}].$$

So we find according to theorem XXXIX for g_0 by subtraction from $k' + 1$:

$$[\overbrace{k', k', \ldots k'}^{a}, \overbrace{(k-1)', (k-1)', \ldots (k-1)'}^{b-a}, \overbrace{(k-2)', (k-2)', \ldots (k-2)'}^{c-}, \ldots,$$
$$\overbrace{2', 2', \ldots 2'}^{s-r}, \overbrace{1', 1', \ldots 1'}^{t-s}, \overbrace{1, 1, \ldots 1}^{n-t}].$$

Likewise we get for the constituents g'_n and g'_0 of the second net represented by $e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'}\, e_n N M_n^{(2)}$ the same expressions in which the $a, b, c, \ldots r, s, t$ are dashed. From this it ensues that we shall have at the same time $g'_n = g'_0$ and $g'_0 = g_n$ under the conditions

$$a = n - t',\ b - a = t' - s',\ c - b = s' - r', \ldots$$
$$s - r = c' - b',\ t - s = b' - a',\ n - t = a',$$

giving immediately

$$a + t' = b + s' = c + r' = \ldots = r + c' = s + b' = t + a' = n.$$

These conditions pass into

$$a + t = b + s = c + r = \ldots = n,$$

if the two nets coincide in a semiperiodic one [1]).

Remark. If we count as one the two nets which pass into each other by interchanging the two extreme forms (and also the two nets N and $e_n N$ of measure polytopes only) the number of measure polytope nets is $8 + 2.5 = 18$ in S_4, $16 + 2.9 = 34$ in S_5, $32 + 2.19 = 70$ in S_6, $64 + 2.35 = 134$ in S_7, $128 + 2.71 = 270$ in S_8, etc.

68. The circumstances under which polarization of a measure polytope net leads to an other measure polytope net are easily indicated. For, though in the case of a net belonging to the family (e, e) the centres of all the constituents are the groups of centres of the different limits $(l)_0, (l)_1, (l)_2, \ldots, (l)_{n-1}, (l)_n$ of the net $N(M_n^{2m})$, m being the extension number, and these points form together the vertices of a net $N(M_n^m)$, it is only $N(M_n^2)$ itself which satisfies the condition that an $M_n^{(2)}$ the vertices of which are centres of the $M_n^{(2)}$ of the net includes only one vertex of this net. So, if we discard the case $ce_2 N(M_4) = N(C_{24})$, the net $N(M_n)$ and the one deduced from it by polarization form together the only pair of *two reciprocal nets* of measure polytope descent.

In general the system of vertices of a net obtained by polarizing a measure polytope net is the combination of several groups of centres of limits $M_k^{(2m)}$ of the measure polytopes of the net $N(M_n^{2m})$, m being the extension number. So we find in S_3:

[1]) In the case of the first relation, where we do not obtain the second member by dashing the subscripts $a, b, c, \ldots, r, s, t$ of the first, the proof is a bit more complicated. Here we find for g_n the expression given above, but for g_0 — as we have to subtract from k_1 instead of $k_1 + 1$ —

$$\big|\, \overbrace{k, k, \ldots k}^{a},\ \overbrace{k-1, k-1, \ldots k-1}^{b-a},\ \overbrace{k-2, k-2, \ldots k-2}^{c-b},\ \ldots,\ \overbrace{2, 2, \ldots 2}^{s-r},\ \overbrace{1, 1, \ldots 1}^{t-s},\ \overbrace{0, 0, \ldots 0}^{n-t} \,\big|\, \sqrt{2},$$

etc.

in the case of N centres of limits M_3,
" $e_1 N, c e_1 N, c e_1 e_2 N$ " " " M_3, M_0,
" $e_2 N, e_1 e_2 N$ " " " M_3, M_1, M_0,

whilst — as we remarked above — in the cases where e_3 occurs all the groups of centres contribute to the system of vertices. In the case of the groups M_3, M_0 a space filling double pyramid on a square base may be considered as the constituent of the reciprocal net, in the case of the three groups M_3, M_1, M_0 we are obliged to consider as constituent a polyhedron (5, 9, 6) which may be got by dividing the double pyramid mentioned into four equal parts by bisecting the pairs of parallel sides of the square base [1]).

G. *Symmetry, considerations of the theory of groups, regularity.*

69. We determine the spaces of symmetry Sy_{n-1} and consider successively the case of the measure polytope M_n of S_n and that of any polytope $(P)_m$ deduced from M_n by the operations of expansion and contraction.

Case of the measure polytope. Let us suppose Sy_{n-1} is a definite space of symmetry of M_n and let A_1 be a vertex of M_n not contained in Sy_{n-1}. Then the mirror image of A_1 with respect to Sy_{n-1} is an other vertex A_2 of M_n, which implies that $A_1 A_2$ is either an edge or a central diagonal of a certain limit M_k of M_n where k may be — the case of the edge included — one of the numbers $1, 2, \ldots, n-1$. Let S_k be the space containing that M_k. Then any edge $A_1 A'$ through A_1 of M_n not belonging to M_k is normal in A_1 to S_k and therefore to $A_1 A_2$; so these $n-k$ edges $A_1 A'$ are parallel to Sy_{n-1} and M_n can be generated by prismatizing M_k in these directions, i.e. *Sy_{n-1} is a space of symmetry of M_n, if and only if its section S_{k-1} with S_k is a space of symmetry of M_k*, which condition is fulfilled in the cases $k=1$, $k=2$ only. For in all the remaining cases $k=3, 4, \ldots, n-1$ (and also for $k=n$) the two simplexes $S(k)$ the vertices of which are the groups of vertices of M_k adjacent to A_1 and to A_2 are equal but of opposite orientation, which proves that the space S_{k-1} of S_k normally bisecting $A_1 A_2$ is no space of symmetry of M_k.

For $k=1$ the line $A_1 A_2$ is an edge, for $k=2$ it is a diagonal of a face. So the two groups of spaces Sy_{n-1} are the n spaces $x_i = 0$ and the $n(n-1)$ spaces $x_i \pm x_k = 0$; so the number of spaces Sy_{n-1} is n^2.

[1]) We defer further developments about reciprocal nets to an other paper also destined to complement art. 39; compare "Nieuw Archief voor Wiskunde", vol. X, p. 273.

Case of the polytope $(P)_m$ deduced from the measure polytope. The n^2 spaces Sy_{n-1} found above are spaces of symmetry for $(P)_m$; so here again the only question is if $(P)_m$ can admit a space of symmetry Sy_{n-1} which is no Sy_{n-1} for the M_n from which $(P)_m$ has been derived. We suppose that there is such an Sy_{n-1}, represent by M'_n the mirror image with respect to that Sy_{n-1} of the M_n from which $(P)_m$ has been derived by a set of e_k and c operations, and remark now that — as Sy_{n-1} is space of symmetry for the figure consisting of $(P)_m$ and the two measure polytopes M_n, M'_n — it must be possible to derive $(P)_m$ from M'_n by the same set of operations. This particularity presents itself in the case of the octagon $e_1(p_4)$ only, as the p_4 itself may be represented either as $[1, 1]$ or as $[1, 0]\sqrt{2}$. So we find:

THEOREM XLII. "The measure polytope $[\overbrace{1\ 1 \ldots 1}^{n}]$ of S_n and the polytopes deduced from it by expansion and contraction admit n^2 spaces Sy_{n-1} of symmetry, the n spaces $x_i = 0$ and the $n(n-1)$ spaces $x_i \pm x_k = 0$. Only in the case of the plane we have to add for $e_1(p_4)$ the four new axes of symmetry passing through pairs of opposite vertices of the octagon".

70. Moreover we find: [1])

THEOREM XLIII. "The order of the group of anallagmatic displacements of the measure polytope M_n of S_n and the polytopes deduced from it by expansion and contraction is $2^{n-1}.\ n!$"

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} of symmetry included, is $2^n.\ n!$ In this extended group the first group of order $2^{n-1}.\ n!$ forms a perfect subgroup".

For $n = 2$ these general results have to be completed in the known way for the octagon."

For the simple proof we refer to the article quoted.

71. Finally we have to apply to the polytopes and nets of measure descent the scale of regularity due to Mr. ELTE. As to the theory we can only repeat here what has been remarked in the arts. 42 and 43, with omission of all that refers to the central symmetry of some of the polytopes of simplex extraction. So theorem XXV must take here the simpler form:

THEOREM XLIV. "Any two limiting elements $(l)_d$ belong to the same subgroup or to different subgroups, in the sense of the scale

1) Compare "Report of the British Association", 1894, p. 563.

of regularity, according as their symbols of coordinates are equal or different."

As the application of ELTE's scale[1]) to polytopes and nets of measure descent is rather easy it may suffice to give some examples, both of polytopes and nets.

a). *Example* **[3′ 3′ 2′ 1′ 1]**. Here we find four different groups of edges (3′, 2′), (2′, 1′), (1′, 1) (1, — 1). So the contributions to the numerator are 1 from the vertices and $\frac{1}{2}$ from the edges and the fraction is $\frac{1+\frac{1}{2}}{5} = \frac{3}{10}$, the minimum value in S_5.

b). *Example* **[3 3 2 1 0]**$\sqrt{2}$. Here three groups of edges appear, viz. $(3, 2)\sqrt{2}$, $(2, 1)\sqrt{2}$, $(1, 0)\sqrt{2}$. So we find once more $\frac{3}{10}$.

c). *Example* **[1 1 0 0 0]**$\sqrt{2}$. Only one kind of edge, viz. $(1, 0)\sqrt{2}$. So we have to examine the faces. As it is clear that we find triangles $(\sqrt{2}, \sqrt{2}, 0)\ 0\ 0$ and squares $[\sqrt{2}, \sqrt{2}]\ 0\ 0\ 0$, the degree of regularity is $\frac{4}{10} = \frac{2}{5}$.

d). *Example* $\boldsymbol{e}_4\boldsymbol{N}(\boldsymbol{M}_5^{(2)})$. The groundform [1′ 1 1 1 1] admits two kinds of edges (1′ 1) 1 1 1 and 1′ 1 1 1 [1] of a different character. So we find $\frac{1\frac{1}{2}}{6} = \frac{1}{4}$.

e). *Example* $\boldsymbol{ce}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4\boldsymbol{N}(\boldsymbol{M}_5^{(2)})$. Here we have to deal with four groups of constituents represented wich their frames in the table

$$
\begin{array}{ll}
g_5 . [43210] \ldots\ldots & (2p_1\ \ , 2p_2\ \ , 2p_3\ \ , 2p_4\ \ , 2p_5\ \ )\,4 \\
g_3 \ldots [210][10] \ldots & (2p_1\ \ , 2p_2\ \ , 2p_3\ \ , 2p_4+1, 2p_5+1)\,4 \\
g_2 \ldots [10][210] \ldots & (2p_1\ \ , 2p_2\ \ , 2p_3+1, 2p_4+1, 2p_5+1)\,4 \\
g_0 \ldots\ldots [43210] . & (2p_1+1, 2p_2+1, 2p_3+1, 2p_4+1, 2p_5+1)\,4
\end{array}
$$

So through the vertex 4, 3, 2, 1, 0 pass

$$
\left.
\begin{array}{l}
[\ \ 4,\quad 3,\quad 2,\quad 1,\quad 0] \ldots\ldots A_1 \\
[8+4,\quad 3,\quad 2,\quad 1,\quad 0] \ldots\ldots A_2 \\
[4+0, 4+1]\quad [2,\quad 1,\quad 0] \ldots\ldots B \\
[4+0, 4+1, 4+2]\quad [1,\quad 0] \ldots\ldots C \\
[4+0, 4+1, 4+2, 4+3,\quad 4+4] \ldots\ldots D_1 \\
[4+0, 4+1, 4+2, 4+3, -4+4] \ldots\ldots D_2
\end{array}
\right\},
$$

i. e. six polytopes and more in detail four cells [4 3 2 1 0] and two prismotopes [2 1 0] [1 0]. Now the edges (4 3) 2 1 0 and 4 3 2 (1 0) belong to both the prismotopes, whilst each of the edges 4 (3 2) 1 0

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[1]) We stick here to the original scale (compare *Proceedings* of Amsterdam, vol. XV, p. 200).

and 4 3 (2 1) 0 belongs to only one. So there are two different kinds of edges and we find $\frac{2}{6} = \frac{1}{3}$.

Remark. In S_n the degree of regularity is a minimum, i. e. $\frac{3}{2n}$ for a polytope and $\frac{3}{2(n+1)}$ for a net,

1°. if the symbol of the polytope or that of the groundform of the net contains no zero,

2°. if the net admits a constituent g_{n-1}.

For in both cases there are at least two kinds of edges: in the first case the edges [1], in the second case the erect edges of the prisms g_{n-1} differ in character from the remaining ones.

The results about regularity have been indicated in the Tables IV, V, VI. In Table IV the regularity fraction is contained in column 5, whilst the subscripts in column 4 give the different groups of limits $(l)_n$. In Tables V and VI in the cases $n = 4$ and $n = 5$ the last column contains the regularity fraction, the last but one [1]) the different groups of limits $(l)_k$, whilst the part $n = 3$ of Table V contains two columns more, one indicating the number of the ANDREINI diagram of the net, the other indicating the particularities of the edges passing through a vertex (see ANDREINI's list, page 30—32 of the memoir quoted in art. 22).

Section III: POLYTOPES AND NETS DERIVED FROM THE CROSS POLYTOPE.

A. *The symbol of coordinates.*

72. In this section which is so closely related to the immediately preceding one that it may be considered as a mere supplement of the latter we have to start from the cross polytope $C_{2n}^{(2)}$ of S_n represented by the symbol $[\overline{1\,00\ldots0}^{\,n-2}]\sqrt{2}$ and to remember that we are to prove by and by that there is no difference whatever between the offspring of this cross polytope and that of the measure polytope $[\overline{11\ldots1}^{\,n}]$ of S_n.

For $n = 3, 4, 5$ we have successively in the symbols of M[rs]. STOTT:[2])

[1]) The numbers of the different groups of limits $(l)_k$ for $k > 1$ have been found in the manner indicated for the simplex in Table III, but we have judged it of no importance to insert an analogous table for the measure polytope.

[2]) For the deduction of the e and c symbols from the symbols of coordinates compare part D of this section.

In Table IV second column are inscribed the e and c symbols of the polytopes deduced from the cross polytope corresponding to the symbols of coordinates of the third column.

$n = 3.$

$[100]\sqrt{2} = O$	$[1'1\,1] = e_2\, O = RCO$		$[1\,1\,1] = ce_2\, O = C$
$[210]\sqrt{2} = e_1\, O = tO$	$[2'1'1] = e_1 e_2\, O = tCO$	$[110]\sqrt{2} = ce_1\, O = CO$	$[1'1'1] = ce_1 e_2\, O = tC$

$n = 4.$

$[1\,000]\sqrt{2} = C_{16}$	$[3\,2\,1\,0]\sqrt{2} = e_1 e_2 C_{16}$		$[2\,2\,1\,0]\sqrt{2} = ce_1 e_2 C_{16}$
$[2\,100]\sqrt{2} = e_1\, C_{16}$	$[2'1'1\,1] = e_1 e_3\, C_{16}$	$[1100]\sqrt{2} = ce_1 C_{16} = C_{24}$	$[1'1'1\,1] = ce_1 e_3\, C_{16}$
$[2\,110]\sqrt{2} = e_2\, C_{16}$	$[2'1'1'1] = e_2 e_3\, C_{16}$	$[1110]\sqrt{2} = ce_2 C_{16}$	$[1'1'1'1] = ce_2 e_3 C_{16}$
$[1'111] = e_3\, C_{16}$	$[3'2'1'1] = e_1 e_2 e_3 C_{16}$	$[1111] = ce_3 C_{16} = C_8$	$[2'2'1'1] = ce_1 e_2 e_3 C_{16}$

$n = 5.$

$[1\,0\,000]\sqrt{2} = C_{32}$	$[3\,2\,2\,1\,0]\sqrt{2} = e_2 e_3 C_{32}$		$[2\,2\,2\,1\,0]\sqrt{2} = ce_2 e_3 C_{32}$
$[2\,1\,000]\sqrt{2} = e_1\, C_{32}$	$[2'1'1'1\,1] = e_2 e_4 C_{32}$	$[1\,1\,000]\sqrt{2} = ce_1\, C_{32}$	$[1'1'1'1\,1] = ce_2 e_4 C_{32}$
$[2\,1\,100]\sqrt{2} = e_2\, C_{32}$	$[2'1'1'1'1] = e_3 e_4 C_{32}$	$[1\,1\,100]\sqrt{2} = ce_2\, C_{32}$	$[1'1'1'1'1] = ce_3 e_4 C_{32}$
$[2\,1\,110]\sqrt{2} = e_3\, C_{32}$	$[4\,3\,2\,1\,0]\sqrt{2} = e_1 e_2 e_3 C_{32}$	$[1\,1\,110]\sqrt{2} = ce_3\, C_{32}$	$[3\,3\,2\,1\,0]\sqrt{2} = ce_1 e_2 e_3 C_{32}$
$[1'1\,111] = e_4\, C_{32}$	$[3'2'1'1\,1] = e_1 e_2 e_4 C_{32}$	$[1\,1\,111] = ce_4\, C_{32}$	$[2'2'1'1\,1] = ce_1 e_2 e_4 C_{32}$
$[3\,2\,100]\sqrt{2} = e_1 e_3\, C_{32}$	$[3'2'1'1'1] = e_1 e_3 e_4 C_{32}$	$[2\,2\,100]\sqrt{2} = ce_1 e_2\, C_{32}$	$[2'2'1'1'1] = ce_1 e_3 e_4 C_{32}$
$[3\,2\,110]\sqrt{2} = e_1 e_3\, C_{32}$	$[3'2'2'1'1] = e_2 e_3 e_4 C_{32}$	$[2\,2\,110]\sqrt{2} = ce_1 e_3\, C_{32}$	$[2'2'2'1'1] = ce_2 e_3 e_4 C_{33}$
$[2'1'111] = e_1 e_4\, C_{33}$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4 C_{32}$	$[1'1'111] = ce_1 e_4\, C_{32}$	$[3'3'2'1'1] = ce_1 e_2 e_3 e_4 C_{33}$

B. *The characteristic numbers.*

73. From the preceding section concerned with the measure polytope can be gathered the symbols with the characteristic numbers of the polytopes deduced from the cross polytope, the symbols of coordinates of which wind up in a unit, as these polytopes also belong to the offspring proper of the measure polytope. So we have only to add a couple of examples about polytopes, the symbols of coordinates of which end in zero.

Example **[2110]**, method working from two sides [1]).

The number of vertices is $2^3 . 4!$ divided by $2!$, i. e. $8 . 24 : 2 = 96$.

The number of the edges passing through the pattern vertex is six, for this vertex is united by edges to the vertices:

$$1210, \quad 2011, \quad 201-1,$$
$$1120, \quad 2101, \quad 210-1.$$

So the number of edges is $\dfrac{96.6}{2} = 288$.

In order to find spaces containing limiting bodies we consider successively the equations:

$$\pm x_1 = 2, \quad \pm x_1 \pm x_2 = 3, \quad \pm x_1 \pm x_2 \pm x_3 \pm x_4 = 4.$$

The equations $\pm x_i = 2$ give 8 forms $[1\,1\,0]$, i. e. 8 CO of vertex import.

---

[1]) In the two examples we omit the common factor $\sqrt{2}$.

The equations $\pm x_i \pm x_j = 3$ give 24 forms (2 1) [1 0], i. e. 24 P_4 of edge import.

The equations $\Sigma \pm x_i = 4$ give 16 forms (2 1 1 0), i. e. 16 *CO* of body import.

So, we find 24 *CO* and 24 *C*, i. e. 48 polyhedra, and therefore $\frac{1}{2}(24 \times 14 + 24 \times 6) = 240$ faces.

So the result is (96, 288, 240, 48) in accordance with the law of Euler.

Example **[3 2 1 1 0]**, direct method.

The number of vertices is $2^4 . 5!$ divided by 2!, i. e. 1920 : 2 = 960.

The edges split up into three groups (32), (21), (10). Through the pattern vertex pass: *one* edge (32), *two* edges (21) — on account of the two digits 1 — and *four* edges (10) — on account of the two digits 1 and of the faculty to make the last digit to correspond either to $+x_5$ or to $-x_5$.

So there are in toto

480 edges (32), 960 edges (21), 1920 edges (10),

i. e. 3360 edges.

The faces split up into six groups, viz. the triangles (211) and (110), the squares (32)(10), (21)(10) and [10] and the hexagon (321).

In the pattern vertex concur:

one triangle (211),
two triangles (110), on account of $\pm x_5$,
four squares (32) (10), on account of the two digits 1 and of $\pm x_5$,
„ „ (21) (10), „ „ „ „ „ „ „ „ „ „ „,
two „ [10], „ „ „ „ „ „ „,
two hexagons (321), „ „ „ „ „ „ „ [1]).

So we find:

$$960 \left(\frac{3 \text{ triangles}}{3} + \frac{10 \text{ squares}}{4} + \frac{2 \text{ hexagons}}{6} \right)$$
$$= 960 \text{ triangles} + 2400 \text{ squares} + 320 \text{ hexagons},$$

i. e. 3680 faces.

The limiting bodies split up into the seven groups:

$$(3211) = tT, (321)(10) = P_6, (32)(110) = P_3, (2110) = CO,$$
$$(32)[10] = (21)[10] = P_4, [110] = CO.$$

[1]) In the case [10] the difference between $+x_5$ and $-x_5$ has no effect, on account of the square brackets.

Of these seven polyhedra concur, on account of the reasons given above, in the pattern vertex in the indicated order:

$$1\, tT, \quad 4\, P_6, \quad 2\, P_3, \quad 2\, CO, \quad 2\, P_4, \quad 2\, P_4, \quad 1\, CO.$$

So we find:

$$960\left(\frac{tT}{12}+\frac{3\, CO}{12}+\frac{4\, P_6}{12}+\frac{4\, P_4}{8}+\frac{2\, P_3}{6}\right)$$
$$= 80\, tT + 240\, CO + 320\, P_6 + 480\, P_4 + 320\, P_3,$$

i. e. 1440 limiting polyhedra.

Finally the limiting polytopes split up into four groups:

(32110), (321)[10], (32)[110], [2110]

and so we find:

$$32\, e_1 e_3 S(5), \quad 80\,(6\,;4), \quad 40\, P_{CO}, \quad 10\, c\, e_1 e_3 C_8,$$

i. e. 162 limiting polyhedra.

So the result is (960, 3360, 3680, 1440, 162) in accordance with the law of Euler.

With respect to the import we have still to add that we pass to the complementary import, if a polytope of the measure polytope family is regarded as a polytope of cross polytope descent. So in the first of the two examples where the cross polytope import has been indicated the result is complementary to that registered in Table IV read from left to right.

C. *Extension number and truncation integers and fractions.*

74. Theorem XLV. "The new polytopes, all with edges of length unity, can be found by means of a regular extension of the cross polytope followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

For the proof we refer to the arts. 15 and 56.

Here the limit $(l)_{n-1}$ of the highest import, i. e. g_{n-1}, corresponds to the equation $x_1 + x_2 + \ldots + x_n =$ constant. So the extension number is the sum of the digits of the new polytope divided by the sum of the digits of the cross polytope, i. e. by $\sqrt{2}$. So the extension number of $[3'3'2'1'1]$ is $5 + 9\sqrt{2}$ divided by $\sqrt{2}$, i. e. $9 + \frac{5}{2}\sqrt{2}$.

We can stick here to the method of measuring the amount of the different truncations on the edges. But we must point out a difficulty underlying this method. So, in the case of truncation of an octahedron (fig. 16) at the edge BC, it makes a difference whether we choose BA or BC' as the edge on which we determine the amount of truncation. For if we move the truncating plane (through BC normal to OM, where M is the midpoint of BC) parallel to itself untill it passes through O it contains the other extremity A of the edge BA, while it bisects the edge BC'. This difficulty can be overcome by stipulating that the edge to be chosen may not contain a vertex opposite to one of the vertices of the limit at which the truncation takes place. But this implies always that we measure quite as well on the line MO joining the centre of that limit to the centre of the polytope. So if the truncating space cuts MO in P the amount of truncation is $\frac{MP}{MO}$. Now the complement $\frac{PO}{MO}$ of this quantity can be deduced immediately from the symbol of coordinates $[a_1, a_2, \ldots, a_n]$ of the cross polytope form considered. If we suppose that the truncation takes place at the limit $(l)_{k-1}$ of the corresponding extended cross polytope $[1, \overline{0, \ldots, 0}^{\,n-1}]\sum\limits_1^n a_i$ lying in the space represented by $x_1 + x_2 + \ldots + x_k =$ constant it is immediately evident that $\frac{PO}{MO}$ is equal to the quotient of the sum $\sum\limits_1^k a_i$ of the first k digits of the symbol of the truncated polytope by the corresponding sum of the extended cross polytope, i. e. by $\sum\limits_1^n a_i$. So from $\frac{PO}{MO} = \frac{\sum\limits_1^k a_i}{\sum\limits_1^n a_i}$ we deduce:

$$\text{amount of truncation} = \frac{MP}{MO} = \frac{\sum\limits_{k+1}^n a_i}{\sum\limits_1^n a_i}.$$

We illustrate this theory by the example $[3'3'2'1'1]$ for which we have determined above the extension number. Here we find moreover

$$\sum_2^5 a_i = 4 + 6\sqrt{2},\ \sum_3^5 a_i = 3 + 3\sqrt{2},\ \sum_4^5 a_i = 2 + \sqrt{2},\ a_5 = 1$$

and therefore

$$\frac{4 + 6\sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{3 + 3\sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{2 + \sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{1}{5 + 9\sqrt{2}}$$

as the amount of truncation at $(l)_0, (l)_1, (l)_2, (l)_3$. As these numbers

$$\frac{2}{137}(44 + 3\sqrt{2}), \frac{3}{137}(13 + 4\sqrt{2}), \frac{1}{137}(8 + 13\sqrt{2}), \frac{1}{137}(9\sqrt{2} - 5)$$

are rather impractical, we only put on record in Table IV the results relating to the cross polytope forms proper, where the denominator and the numerator of the fraction $\frac{MP}{MO}$ are both integer multiples of $\sqrt{2}$. Here the result $9 \mid 6, 3, 1$ corresponding to $[3\,3\,2\,1\,0]\sqrt{2}$ expresses that the amount of truncation at $(l)_0, (l)_1, (l)_2$ is respectively $\frac{2}{3}, \frac{1}{3}, \frac{1}{9}$.

D). *Expansion and contraction symbols.*

75. What we have to prove here is:

THEOREM XLVI „The expansion e_k, $(k = 1, 2, 3, \ldots, n-2)$, applied to the cross polytope $C_{2^n}^{(2)}$ of S_n changes the symbol of coordinates $[1\overline{00\ldots0}^{\,n-1}]\sqrt{2}$ of that regular polytope by addition of $\sqrt{2}$ to the first $k+1$ digits into $[2\overline{11\ldots1}^{\,k}\,\overline{00\ldots0}^{\,n-k-1}]\sqrt{2}$, whilst in the case of e_{n-1} where application of this rule would give a symbol without zero we have to add unity instead of $\sqrt{2}$ to all the digits, giving $[1'\,\overline{11\ldots1}^{\,n-1}]$”.

Proof. We treat the cases $k < n-1$ and $k = n-1$ separately.

Case $k < n-1$. The operation e_k acts upon the limits $(l)_k = S(k+1)$ of the cross polytope. Now the centre M of the limit $(l)_k$ represented by

$$x_1, x_2, \ldots, x_{k+1} = (1\overline{00\ldots0}^{\,k})\sqrt{2}, \quad x_{k+2} = x_{k+3} = \ldots = x_n = 0$$

has the coordinates

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{\sqrt{2}}{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_n = 0.$$

If we move this limit $(l)_k$ parallel to itself in the direction OM to a position $(l)'_k$ for which the centre M' satisfies the relation $OM' = \lambda . OM$, where λ is to be determined, we find for the coordinates of M'

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{\lambda\sqrt{2}}{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_n = 0.$$

So by this motion the coordinates $x_1, x_2, \ldots, x_{k+1}$ of any vertex A of $(l)_k$ increase by $\frac{(\lambda-1)\sqrt{2}}{k+1}$, whilst the coordinates $x_{k+2}, x_{k+3}, \ldots x_n$ of this point remain zero. So $(l)'_k$ is represented by

$$x_1, x_2, \ldots, x_{k+1} = \Big(1 + \frac{\lambda-1}{k-1}, \overbrace{\frac{\lambda-1}{k-1}, \ldots, \frac{\lambda-1}{k-1}}^{k}\Big)\sqrt{2},$$
$$x_{k+2} = x_{k+3} = \ldots = x_n = 0,$$

from which it ensues that the symbol of coordinates of the new polytope becomes

$$\Big[1 + \frac{\lambda-1}{k-1}, \overbrace{\frac{\lambda-1}{k-1}, \ldots, \frac{\lambda-1}{k-1}}^{k}, \overbrace{0, \ldots, 0}^{n-k-1}\Big]\sqrt{2}.$$

So the new polytope satisfies the law of the equality of all the edges expressed in theorem XXVIII if, and only if, we have either $\lambda = 1$ or $\lambda = k$. As $\lambda = 1$ corresponds to the cross polytope itself, we have to take $\lambda = k$ in which case we find $[2\ \overbrace{11\ldots1}^{k}\ \overbrace{00\ldots0}^{n-k-1}]$ as the theorem requires it.

Case $k = n-1$. We consider the limit $(l)_{n-1} = S(n)$ represented by

$$x_1, x_2, \ldots, x_n = (1\ \overbrace{00\ldots0}^{n-1})\sqrt{2}$$

with the centre M, the coordinates of which are

$$x_1 = x_2 = \ldots = x_n = \frac{\sqrt{2}}{n},$$

and move this $(l)_{n-1}$ parallel to itself in the direction OM to a position the centre M' of which is determined by the relation $OM' = \lambda . OM$. Then we find in the way indicated above for the symbol of coordinates of the new polytope

$$\Big[1 + \frac{\lambda-1}{n}, \overbrace{\frac{\lambda-1}{n}, \ldots, \frac{\lambda-1}{n}}^{n-1}\Big]\sqrt{2}.$$

So, if we discard immediately the supposition $\lambda = 1$ leading back to the original cross polytope, the new polytope the symbol of which contains no zero satisfies the law of theorem XXVIII, if — and only if — we have

$$\left(1 + \frac{\lambda-1}{n}\right) : \frac{\lambda-1}{n} = (1+\sqrt{2}) : 1$$

giving $\lambda - 1 = \frac{1}{2} n \sqrt{2}$. So we find the polytope with the symbol

$$[1+\tfrac{1}{2}\sqrt{2}, \overbrace{\tfrac{1}{2}\sqrt{2},\ldots,\tfrac{1}{2}\sqrt{2}}^{n-1}]\sqrt{2} = [1+\sqrt{2}, \overbrace{1, 1,\ldots, 1}^{n-1}]$$

in accordance with the statement of the theorem.

By the way we find:

THEOREM XLVII. "In the expansion e_k the limits $S(k+1)$ of $C_{2n}^{(2)}$ are moved away from the centre to a distance equal to k times the original distance for $k < n-1$ and to a distance equal to $1 + \frac{1}{2} n \sqrt{2}$ times the original distance for $k = n-1$".

This comes true, for this extension corresponds in both cases to that deduced from the sum of the digits of the symbol of coordinates of the new polytope.

As the distance OM was $\sqrt{\frac{2}{k+1}}$, it becomes $k\sqrt{\frac{2}{k+1}}$ for $k < n-1$ and $\frac{n+\sqrt{2}}{\sqrt{n}}$ for $k = n-1$.

76. THEOREM XLVIII. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $C_2^{n(2)}$ on its symbol $[1\overbrace{00..0}^{n-1}]\sqrt{2}$ is found by adding the influences of each of the expansions taken separately".

Proof. Here likewise, in the succession of two expansions the subject of the second is to be what its original subject has become under the influence of the first. So in the case of $e_2 e_1 O$ of the octahedron (fig. 17[a]) the original subject of e_2 (the triangle) is transformed by e_1 into a hexagon (fig. 17[b]) and now the hexagon is moved out, in the case $e_1 e_2 O$ the linear subject of e_1 (the edge) is transformed by e_2 into a square (fig. 17[c]) and now this square is moved out; in both cases the result (fig. 17[d]) is the same, a *tCO.* In general, for $k > l$, in the case $e_k e_l C_{2n}^{(2)}$ the subject $S(k+1)$ of e_k is transformed by e_l into the form $e_l\ S(k+1)$ of the same number of dimensions, while in the case $e_l e_k C_{2n}^{(2)}$ the subject $S(l+1)$ of e_l is transformed by e_k into an $n-1$-dimensional limit g_l of import l. Here also the geometrical condition: "that the two new positions of any vertex shall be separated by the length of an edge" leads to the ordinary composition of the motions of the centre according to the rule of the parallelogram in the case of two expansions, etc.

By the way we find:

THEOREM XLIX. "The operation e_k can still be applied to any polytope deduced from $C_{2n}^{(2)}$ in the symbol of coordinates of which the $k+1^{st}$ and the $k+2^{nd}$ digit are equal."

We indicate by means of this theorem the expansion symbol of the example $[5'4'4'3'3'2'2'2'1'1]$ of art. 55, considered as a descendent of $[1\,\overbrace{0\,0\,..\,0}^{9}]$. Of the five intervals $\sqrt{2}$, indicated by (d_1, d_2), (d_3, d_4), (d_5, d_6), (d_8, d_9), (d_9, d_{10}) the first corresponds to the original interval of the symbol of coordinates of $C_{210}^{(2)}$ whilst according to the theorem the others result from the four operations e_2, e_4, e_7, e_8. But as the symbol winds up in a unit instead of a zero we have to add e_9. So we find $e_2\, e_4\, e_7\, e_8\, e_9\; C_{210}^{(2)}$.

77. By means of the operations e_k we can deduce from $C_{2n}^{(2)}$ all the possible polytopes the square bracketed symbols of coordinates of which are characterized by the fact that there is an interval $\sqrt{2}$ between the first and the second digits. If we wish to deduce from $C_{2n}^{(2)}$ also polytopes with square bracketed symbols the two digits d_1, d_2 of which are equal we have to follow Mrs. STOTT by introducing the operation c of contraction, the subject of which is the group of limits $(l)_{n-1}$ of vertex import. With respect to this operation we can prove the theorem:

THEOREM L. "By applying the contraction c to any expansion form deduced from $C_{2n}^{(2)}$ the largest digit of the symbol of coordinates of this form is diminished by $\sqrt{2}$."

Proof. Here we have to consider the two cases of the symbol of coordinates, winding up either in 1 or in 0.

Case $[1+(a+1)\sqrt{2},\ 1+a\sqrt{2},\ 1+b\sqrt{2}, \ldots, 1]$. — If we replace $1+(a+1)\sqrt{2}$ by $1+a\sqrt{2}$ the limit g_0 represented by

$$x_1 = 1+(a+1)\sqrt{2}, \quad x_2, x_3, \ldots, x_n = (1+a\sqrt{2}, 1+b\sqrt{2}, \ldots, 1)$$

passes into

$$x_1 = 1+a\sqrt{2}, \qquad x_2, x_3, \ldots, x_n = (1+a\sqrt{2}, 1+b\sqrt{2}, \ldots, 1),$$

i.e. that limit $(l)_{n-1}$ moves parallel to the axis OX_1 towards the centre O over a distance $\sqrt{2}$. Evidently application of this process to *all* the limits g_0 corresponds to a substitution of $1+a\sqrt{2}$ for the digit $1+(a+1)\sqrt{2}$ within the square brackets. Evidently any two adjacent limits represented originally by

$$x_1 = 1+(a+1)\sqrt{2}, \quad x_2, x_3, \ldots x_n = (1+a\sqrt{2}, 1+b\sqrt{2}, \ldots, 1),$$
$$x_2 = 1+(a+1)\sqrt{2}, \quad x_1, x_3, \ldots x_n = (1+a\sqrt{2}, 1+b\sqrt{2}, \ldots, 1),$$

which were separated by the right prism

$$x_1, x_2 = (1+(a+1)\sqrt{2}, 1+a\sqrt{2}), \quad x_3, \ldots, x_n = (1+b\sqrt{2}, \ldots, 1),$$

pass into the two limits

$$x_1 = 1 + a\sqrt{2}, \quad x_2, x_3, \ldots, x_n = (1 + a\sqrt{2}, 1 + b\sqrt{2}, \ldots, 1),$$
$$x_2 = 1 + a\sqrt{2}, \quad x_1, x_3, \ldots, x_n = (1 + a\sqrt{2}, 1 + b\sqrt{2}, \ldots, 1),$$

which are in contact with each other by the $n - 2$-dimensional polytope

$$x_1 = 1 + a\sqrt{2}, x_2 = 1 + a\sqrt{2}, x_3, x_4, \ldots, x_n = (1 + b\sqrt{2}, \ldots, 1).$$

Case $[a+1, a, b, \ldots, 0]\sqrt{2}$. — Here we have to consider the influence of the replacing of $a+1$ by a. The proof runs exactly in the same lines.

Remark. By combining the theorems XLVIII and XLIX we can find the symbols in c and e_k of *any* form deduced from $C_{2n}^{(2)}$. But this process can be simplified by introducing the operation e_0 which transforms the centre O of $C_{2n}^{(2)}$ considered as an infinitesimal cross polytope $C_{2n}^{(0)}$ into $C_{2n}^{(2)}$. Then the contraction symbol c can be shunted out by substituting $e_k e_l \ldots e_m C_{2n}^{(0)}$ for $ce_k e_l \ldots e_m C_n^{(2)}$, but this implies that we replace $e_k e_l \ldots e_m C_{2n}^{(2)}$ by $e_0 e_k e_l \ldots e_m C_{2n}^{(0)}$. This remark — corresponding literally to that of art. 60 — will also be useful in part F of this section.

Meanwhile we *have* shown now that any coordinate symbol between square brackets satisfying the laws of the first part of theorem XXVIII (art. 47) can be interpreted both ways, either as a form deduced from the measure polytope or as a descendent from the cross polytope. So we have proved the following theorem already stated implicitly in art. 48:

THEOREM LI. "The families of polytopes deduced from the two patriarchs, measure polytope and cross polytope, are identical."

E. *Nets of polytopes.*

78. In accordance with the last theorem the net of measure polytopes $N(M_n^{(2)})$ can also be considered as a net $N(ce_{n-1} C_{2n}^{(2)})$ of polytopes $ce_{n-1} C_{2n}^{(2)}$. So the nets put on record for $n = 3, 4, 5$ can be transcribed as nets of cross polytope descent.

But instead of doing this we point out a particularity of the case $n = 4$. For $n = 4$ both the half measure polytopes $\pm \frac{1}{2}[1, 1, 1, 1]$ are cells C_{16} and in relation with this fact we find a new fourdimensional net of regular polytopes, i. e. S_4 possesses besides the measure polytope net exceptionally a cross polytope net too. If we suppose that the net $N(M_4^{(2)})$ be composed of alternate white and black polytopes, so that two $M_4^{(2)}$ with a common $M_3^{(2)}$ differ in colour, and that each white $M_4^{(2)}$ is truncated at one set of eight vertices, so as to retain a $+\frac{1}{2}[1, 1, 1, 1]$, whilst each black $M_4^{(2)}$ is truncated in the same way so as to retain a $-\frac{1}{2}[1, 1, 1, 1]$, the interstitial spaces between these two sets of *inclined* $C_{16}(2\sqrt{2})$ can be filled up by a third set of *erect* $C_{16}(2\sqrt{2})$, and we obtain a fourdimensional net formed by three equally numerous groups of cells $C_{16}(2\sqrt{2})$ with the property that all the polytopes of the same group are equipollent. Moreover we can transform the net $N(M_4^{(2)})$ of alternate white and black polytopes into a net of regular cells $C_{24}^{(2)}$ by decomposing each white $M_4^{(2)}$ into eight mutually congruent pyramids with the centre of the polytope as common vertex and the eight limiting cubes of the polytope as bases, and uniting each of these white pyramids to the black measure polytope with which it is in contact by its base [1]). Now what concerns us here is that by treating the new regular net $N(C_{16})$ in the same way in which the net $N(M_4)$ has been treated we find several new fourdimensional nets; for these nets the reader may compare Table II of the memoir of M^rs^. STOTT quoted several times [2]).

Remark. In art. 64 we have seen that with respect to measure polytope nets any net (c, e) is also a net (e, c). This particularity does not present itself for the nets deduced from $N(C_{16})$. So here we will have to distinguish four cases [3]), i. e. (e, c), (e, e), (c, c) and moreover (c, e).

79. We have seen that the vertices of the net $N(M_4^{(2)})$ can be represented by the symbol $[2a_1 + 1, 2a_2 + 1, 2a_3 + 1, 2a_4 + 1]$ where the a_i are arbitrary integers. By considering the point $x_i = 1$, $(i = 1, 2, 3, 4)$, as the new origin of parallel axes and omitting the square brackets we get for the coordinates of these vertices

$$2a_1,\ 2a_2,\ 2a_3,\ 2a_4.$$

From this we deduce that the vertices of the net $N(C_{16}(2\sqrt{2}))$ can be represented by the same coordinate values under addition of the condition that $\sum_1^4 a_i$ has a defined character of parity. If we choose the condition "$\sum_1^4 a_i$ is even" we get for the three sets of $C_{16}(2\sqrt{2})$ the coordinate symbols:

1) Compare p. 242 of vol. II of my textbook "Mehrdimensionale Geometrie" or *Proceedings* of the Academy of Amsterdam, vol. X, p. 536, 537.

2) In the part of that Table concerned with the nets deduced from $N(C_{16})$ the P_T of the line with the number 28 ought to find a place in the same column in the line with the number 27. Moreover we can add in the last column of the line 29 that this net is equal to that of line 47.

The fact that several nets of this part are equal to nets deduced from cell C_{24} will be explained in part F of this section.

3) In (e, c), etc. the first letter is related to C_{16}, the second to C_{24}.

$$\begin{array}{lll}
\text{I}\ldots\ldots & [2a_1+2,\ 2a_2+0,\ 2a_3+0,\ 2a_4+0], & \Sigma a_i \text{ even},\\
\text{II}\ldots\ldots & \tfrac{1}{2}[2a_1+1+1,\ 2a_2+1+1,\ 2a_3+1+1,\ 2a_4+1+1], & \text{,, odd},\\
\text{III}\ldots\ldots & -\tfrac{1}{2}[2a_1+1+1,\ 2a_2+1+1,\ 2a_3+1+1,\ 2a_4+1+1], & \text{,, even}.
\end{array}$$

Of these three sets I represents the erect group, while II and III form the two inclined groups.

If we wish to represent analytically the fourdimensional nets derived from $N(C_{16})$ we have to start from the three symbols I, II, III, and to study the influence of the operations e_k, c. As to the representation of all the vertices of these new nets by coordinate symbols these influences can be split up into two inadequate parts; of these the first deals with the variation in form of any C_{16} of each of the three groups, whilst the second is concerned with the variation of the distance of any two C_{16}. We treat each of these two parts for itself.

a) *Variation in shape.* We know the influence of the operations e_k, c on the coordinate symbol [2000] of the central $C_{16}(2\sqrt{2})$ of the erect group and from this we can deduce the corresponding influences on the $C_{16}(2\sqrt{2})$ of each of the inclined groups by means of the transformations of coordinates by which [2 0 0 0] passes into $\tfrac{1}{2}$[1 1 1 1] and $-\tfrac{1}{2}$[1 1 1 1].

The formulae corresponding to the first transformation are

$$\left.\begin{array}{l}
2y_1 = x_1 + x_2 + x_3 + x_4\\
2y_2 = x_1 + x_2 - x_3 - x_4\\
2y_3 = x_1 - x_2 + x_3 - x_4\\
2y_4 = x_1 - x_2 - x_3 + x_4
\end{array}\right\};$$

by changing the sign of y_1 we get formulae corresponding to the second transformation. In the following small table we put on record the result of the first transformation:

$$1\,[2\ 0\ 0\ 0] \ldots\ldots \tfrac{1}{2}[1\ 1\ 1\ 1]$$
$$e_1\,[4\ 2\ 0\ 0] \ldots\ldots \tfrac{1}{2}[3\ 3\ 1\ 1]$$
$$e_2\,[4\ 2\ 2\ 0] \ldots\ldots [4\ 2\ 2\ 0]$$
$$e_3\,[1'1\ 1\ 1]\sqrt{2} \ldots\ldots \tfrac{1}{2}[2'1\ 1\ 1] \text{ and } -\tfrac{1}{2}[1'1'1',\ \sqrt{2}-1]$$
$$e_1 e_2\,[6\ 4\ 2\ 0] \ldots\ldots [6\ 4\ 2\ 0]$$
$$e_1 e_3\,[2'1'1\ 1]\sqrt{2} \ldots\ldots \tfrac{1}{2}[3+2\sqrt{2},\ 3\ 1\ 1] \text{ and } -\tfrac{1}{2}[3+\sqrt{2},\ 3+\sqrt{2},\ 1',\ \sqrt{2}-1]$$
$$e_2 e_3\,[2'1'1'1]\sqrt{2} \ldots\ldots [4+2\sqrt{2},\ 2\ 2\ 0] \text{ and } -\tfrac{1}{2}[4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}]$$
$$e_1 e_2 e_3\,[3'2'1'1]\sqrt{2} \ldots\ldots [6+2\sqrt{2},\ 4\ 2\ 0] \text{ and } -\tfrac{1}{2}[6+\sqrt{2},\ 4+\sqrt{2},\ 2+\sqrt{2}\ \ \sqrt{2}]$$
$$ce_1\,[2\ 2\ 0\ 0] = C_{24}^{(2\sqrt{2})} \ldots [2\ 2\ 0\ 0]$$
$$ce_2\,[2\ 2\ 2\ 0] \ldots\ldots -\tfrac{1}{2}[3\ 1\ 1\ 1]$$
$$ce_3\,[1\ 1\ 1\ 1]\sqrt{2} = C_8^{(2\sqrt{2})\ldots} \quad [2\ 0\ 0\ 0]\sqrt{2} \text{ and } -\tfrac{1}{2}[1\ 1\ 1\ 1]\sqrt{2}$$
$$ce_1 e_2\,[4\ 4\ 2\ 0] \ldots\ldots -\tfrac{1}{2}[5\ 3\ 1\ 1]$$
$$ce_1 e_3\,[1'1'1\ 1]\sqrt{2} \ldots\ldots [2+2\sqrt{2},\ 2\ 0\ 0] \text{ and } -\tfrac{1}{2}[1'1'1\ 1]\sqrt{2}$$
$$ce_2 e_3\,[1'1'1'1]\sqrt{2} \ldots\ldots -\tfrac{1}{2}[3+2\sqrt{2},\ 1\ 1\ 1] \text{ and } -\tfrac{1}{2}[3+\sqrt{2},\ 1'1'1']$$
$$ce_1 e_2 e_3\,[2'2'1'1]\sqrt{2} \ldots\ldots -\tfrac{1}{2}[5+2\sqrt{2},\ 3\ 1\ 1] \text{ and } -\tfrac{1}{2}[5+\sqrt{2},\ 3+\sqrt{2},\ 1+\sqrt{2},\ 1+\sqrt{2}]$$

b) Variation in distance. We account for the variation of the distance of any two sixteencells due to the extension of these cells by multiplying the immovable parts of the digits of the three symbols of coordinates given above for the three groups of sixteencells by a certain constant. This constant is the extension number itself when the operation e_4 is lacking, i. e. in the two general cases (e, c) and (c, c) of nets deduced from $N(C_{16})$; in the remaining general cases (e, e) and (c, e) we have to add $\sqrt{2}$ to that multiplier in order to create room for the intermediate prisms with $2\sqrt{2}$ as height.

As we start from [2 0 0 0] the extension number is half the sum of the digits. So we find for the multiplier the values given in the following table

(e, c)	(e, e)	(c, c)	(c, e)
	$e_4 \ldots 1+\sqrt{2}$		$ce_4 \ldots \sqrt{2}$
$e_1 \ldots 3$	$e_1e_4 \ldots 3+\sqrt{2}$	$ce_1 \ldots 2$	$ce_1e_4 \ldots 2+\sqrt{2}$
$e_2 \ldots 4$	$e_2e_4 \ldots 4+\sqrt{2}$	$ce_2 \ldots 3$	$ce_2e_4 \ldots 3+\sqrt{2}$
$e_3 \ldots 1+2\sqrt{2}$	$e_3e_4 \ldots 1+3\sqrt{2}$	$ce_3 \ldots 2\sqrt{2}$	$ce_3e_4 \ldots 3\sqrt{2}$
$e_1e_2 \ldots 6$	$e_1e_2e_4 \ldots 6+\sqrt{2}$	$ce_1e_2 \ldots 5$	$ce_1e_2e_4 \ldots 5+\sqrt{2}$
$e_1e_3 \ldots 3+2\sqrt{2}$	$e_1e_3e_4 \ldots 3+3\sqrt{2}$	$ce_1e_3 \ldots 2+2\sqrt{2}$	$ce_1e_3e_4 \ldots 2+3\sqrt{2}$
$e_2e_3 \ldots 4+2\sqrt{2}$	$e_2e_3e_4 \ldots 4+3\sqrt{2}$	$ce_2e_3 \ldots 3+2\sqrt{2}$	$ce_2e_3e_4 \ldots 3+3\sqrt{2}$
$e_1e_2e_3 \ldots 6+2\sqrt{2}$	$e_1e_2e_3e_4 \ldots 6+3\sqrt{2}$	$ce_1e_2e_3 \ldots 5+2\sqrt{2}$	$ce_1e_2e_3e_4 \ldots 5+3\sqrt{2}$

80. By means of the preceding developments we can find the three net symbols for all the different nets deduced from $N(C_{16})$. But this work can be reduced by the remark that it will do to use only the net symbol of the erect group in the cases of the seven nets $1, e_1, e_2, e_1e_2, ce_1, ce_2, ce_1e_2$, while we want these of two groups only for the eight nets $e_3, e_1e_3, e_2e_3, e_1e_2e_3, ce_3, ce_1e_3, ce_2e_3, ce_1e_2e_3$, and all the three symbols in the remaining cases where e_4 occurs. The proof of this assertion is based on the following theorem, where we distinguish the three sets of cases just indicated as the set without e_3 and e_4, the set with e_3 and without e_4, and the set with e_4:

THEOREM LII. "Any of the three net symbols represents all the vertices of the net in the set without e_3 and e_4, two thirds of all the vertices in the set with e_3 and without e_4, one third of all the vertices in the set with e_4".

This theorem is an immediate consequence of the following lemma: "Any limiting tetrahedron of the net $N(C_{16})$ is common to two C_{16} belonging to different groups, any limiting triangle is common to three C_{16} no two of which belong to the same group".

The first part of this lemma is evident by itself. As to the second part related to a face we state that the angle formed by the two spaces of adjacent tetrahedra $ABCD$, $ABCD'$ of C_{16} at the common face ABC is 120° (see my paper: "On the angles of the regular polytopes, etc.", *Amer. Journ. of Math.*, vol. XXXI, p. 307), from which it ensues that any face is common to three C_{16}; as any two of these three C_{16} have a limiting tetrahedron in common they belong to different groups, etc.

The lemma just proved immediately shows the truth of the theorem. If, after having driven asunder the cells $C_{16}(2\sqrt{2})$ of the net $N(C_{16})$ so as to create room for the extension recorded above, the extended C_{16} receive the shape exacted by the character of the net under consideration by means of a regular truncation, the contact of the cells — belonging to different groups — by faces will remain uninfluenced if the operations e_3, e_4 do not yet present themselves, the truncations being then restricted either to the vertices alone or to vertices and edges; so, as any vertex of the net belongs at least to one face and each face belongs to three polytopes of the set without e_3, e_4, one of each group, each vertex of the net must be contained in each of the three net symbols of any case of that set.

So in this case the net itself can be represented by any of the three symbols, which includes that the constituents furnished by one symbol are identical with those furnished by each of the two others, though constituents of polytope and body import of one symbol may become under certain circumstances constituents respectively of vertex and edge import of an other.

Now the state of affairs changes as soon as e_3 makes its appearance. This operation still *preserves* the contact by limiting bodies of body import between cells belonging to different groups, but it *annihilates* at the same time face contact between limiting bodies of body import of the same cell. So here the limiting bodies of body import of any constituent have been split up into two sets P and Q dividing the vertices equally between them, in such a way that any two of these limits which were in face contact before belong to different sets. So here the arrangement of the three groups A, B, C of constituents is such that any constituent of group A is in body contact by its set of limits P with constituents of group B, by its set of limits Q with constituents of group C. So each of the three net symbols contains all the vertices of one group and only half the number of vertices of each of the two other groups, i.e. $\frac{2}{3}$ of the total amount.

Finally, in the set with e_4, two cells — belonging to different groups — cannot have a vertex in common; so here each net symbol represents only $\frac{1}{3}$ of the system of vertices.

We now indicate schematically how we can determine all the constituents of the different nets of C_{16}. To that end we have

1°. to deduce from the preceding developments the net symbols necessary in every case,

2°. to calculate the coordinates of the centres of the different constituents, by multiplying the coordinates of a vertex, of the midpoint of an edge, of the centre of a face and of the centre of a limiting body of [2, 0, 0, 0] by the extension number,

3°. to determine the vertices contained in the net symbols, lying at the same minimum distance from these centres.

As we shall have to consider the "extended" vertex, midpoint of edge, centre of face, or centre of limiting body mentioned sub 2° as new origin of parallel axes of coordinates in order to be able to obtain the simplest representation of the sets of vertices mentioned sub 3° we will denote this extended point henceforth by O'.

Of each of the three sets we will treat some examples, of the first $e_1 e_2 N(C_{16})$ and $ce_1 e_2 N(C_{16})$, of the second $e_2 e_3 N(C_{16})$ and $ce_1 e_3 N(C_{16})$, of the third $e_1 e_4 N(C_{16})$, $e_1 e_2 e_3 e_4 N(C_{16})$ and $ce_1 e_2 e_3 e_4 N C_{16}$. Afterwards we will put on record the coordinate symbols of all the constituents in Table VII.

81. *Case* $\boldsymbol{e_1 e_2 N(C_{16})}$. Net symbol

$$[12\, a_1 + \mathbf{6},\ 12\, a_2 + \mathbf{4},\ 12\, a_3 + \mathbf{2},\ 12\, a_4 + \mathbf{0}],\ \sum_{1}^{4} a_i \text{ even}.$$

Here the constituent of polytope import is $[6, 4, 2, 0] = e_1 e_2 C_{16}$. There are no constituents of body and face import as the operations e_4 and e_3 do not present themselves. So we have only to determine the polytopes of edge and vertex import.

Edge gap prism. By extension the centre 1, 1, 0, 0 of the edge (2, 0) 0 0 of [2, 0, 0, 0] becomes 6, 6, 0, 0. By putting in the net symbol $a_i = 0$, $(i = 1, 2, 3, 4)$, we find among others the vertices (6, 4) [2, 0] and by putting $a_1 = a_2 = 1$, $a_3 = a_4 = 0$, and taking the movable digits 6, 4 with the negative sign we find also the vertices (6, 8) [2, 0]; with respect to the new axes with the point 6, 6, 0, 0 as new origin O' these two groups of vertices can be represented together by the symbol [2, 0] [2, 0]. So we find a measure polytope C_3 which is to be interpreted here as a prism on a cube, P_C.

Vertex gap polytope. By extension of the vertex 2,0,0,0 of [2,0,0,0] we get 12,0,0,0 as new origin O'. By substituting $a_i = 0, (i = 1,2,3,4)$, n the first place and $a_1 = 2$, $a_i = 0, (i = 2,3,4)$, in the second (with he movable digit 6 taken negatively) we put in evidence the two sets of vertices 6 [4, 2, 0] and 18 [4, 2, 0], i.e. with respect to O' the vertices [6] [4, 2, 0] contained in the net symbol. But this symbol still contains other vertices lying at the same minimum distance $2\sqrt{14}$ from O', i.e. all the vertices represented with respect to that point by [6, 4, 2, 0] and no other. So we find e.g. the point 4, 6, 2, 0, with the coordinates 16, 6, 2, 0 with respect to the original axes, by considering the vertices $12a_1 + 4$, $12a_2 - 6$, $12a_3 + 2$, $12a_4$ and putting $a_1 = a_2 = 1$ and $a_3 = a_4 = 0$, etc. So the result is that the constituent of vertex import is a $[6, 4, 2, 0] = e_1 e_2 C_{16}$ and therefore identical with the constituents of polytope import.

Case $\boldsymbol{c e_1 e_2 N(C_{16})}$. Net symbol

$$[10a_1 + \mathbf{4},\ 10a_2 + \mathbf{4},\ 10a_3 + \mathbf{2},\ 10a_4 + \mathbf{0}],\ \sum_1^4 a_i \text{ even.}$$

Here the constituent of polytope import is $[4, 4, 2, 0] = ce_1 e_2 C_{16}$. As in the preceding case of $e_1 e_2 N(C_{16})$ the constituents of body and of face import are lacking. Moreover by the contraction the original edge and therefore also the constituent P_C of edge import is annihilated, i.e. P_C is reduced to its base C. We verify this analytically as follows. By extension of the midpoint 1, 1, 0, 0 of the edge (2, 0) 0, 0 of [2, 0, 0, 0] we get 5, 5, 0, 0 as new origin O'. Now the vertices at minimum distance from O' contained in the net symbol are found by putting $a_i = 0, (i = 1, 2, 3, 4)$, giving 4, 4 [2, 0], and $a_1 = a_2 = 1$, $a_3 = a_4 = 0$ (with the two digits 4 taken negatively) giving 6, 6 [2, 0], i.e. with respect to O' the two squares 1, 1 [2, 0] and $-1, -1$ [2, 0] forming two opposite faces of a cube with O' as centre.

Finally we remark that the contraction c does not affect the constituent of vertex import. This is easily verified by determining the vertices at minimum distance from the point O' with the coordinates 10, 0, 0, 0 presenting itself here.

82. *Case* $\boldsymbol{e_2 e_3 N(C_{16})}$. As the operation e_3 presents itself here we have to find besides the constituent $[2'\,1'\,1'\,1]\sqrt{2} = e_2 e_3 C_{16}$ of polytope import those of face, of edge and of vertex import, and in order to be able to gather all the vertices of these constituents we have to use two of the three net symbols. But we prefer to

investigate how far we can proceed in this way by using the first net symbol only. This much more complicated symbol is

$$[4(2+\sqrt{2})a_1+4+\sqrt{2},\ 4(2+\sqrt{2})a_2+2+\sqrt{2},\ 4(2+\sqrt{2})a_3+2+\sqrt{2},\ 4(2+\sqrt{2})a_4+\sqrt{2}],$$

Σa_i being even. We abridge it into the following form, clear by itself:

$$[4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],\ (8+4\sqrt{2})\,\overline{a_1, a_2, a_3, a_4},\ \sum_1^4 a_i \text{ even},$$

where $\overline{a_1, a_2, a_3, a_4}$ preceded by the common factor $8+4\sqrt{2}$ represents the immovable part.

Face gap prismotope. By extension the centre $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0$ of the face $(2, 0, 0)\ 0$ of $[2, 0, 0, 0]$ passes into the new origin O' with the coordinates $\frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\ 0$. By supposing the four a_i of the net symbol to disappear we get inter alia the set of vertices $(4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2})[\sqrt{2}]$, i. e. a P_3. These are the only vertices contained in the net symbol above mentioned lying at minimum distance $\frac{4}{3}\sqrt{3}$ from O', but as we shall see immediately the two other net symbols contain other vertices partaking of this property. However, in order to sharpen our analytic tools, we leave these other net symbols alone for a moment and try to deduce these lacking vertices from the simple properties of the prismotope with two regular generating polygons in planes perfectly normal to each other. By means of the P_3 just found we know that one of these polygons is a triangle, and the character of the other polygon can be deduced from its circumradius. For the relation $\rho_1^2+\rho_2^2=\rho^2$ between the circumradii ρ_1, ρ_2, ρ of the two generating polygons and the prismotope itself gives, as we have $\rho=\frac{4}{3}\sqrt{3}$ and $\rho_1=\frac{2}{3}\sqrt{6}$, $\rho_2=\frac{2}{3}\sqrt{6}$, i. e. the second polygon is also a triangle and the prismotope a $(3;3)$. We have therefore only to find a third position of the first triangle, the two end planes of P_3 containing already two positions, and this third position can be found by remarking that the centres of these three equipollent triangles are the vertices of an equilateral triangle with O' as centre. So, if p, q, r, s are the coordinates of the centre of this third position we have that the triangle with the three vertices

$$\begin{matrix} \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \sqrt{2} \\ \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & -\sqrt{2} \\ p & , \quad q & , \quad r & , \quad s \end{matrix}$$

must admit

$$\frac{4}{3}(2+\sqrt{2}),\quad \frac{4}{3}(2+\sqrt{2}),\quad \frac{4}{3}(2+\sqrt{2}),\qquad 0$$

as centre. From this it ensues that we have

$$p=q=r=\tfrac{8}{3}+2\sqrt{2},\ s=0,$$

furnishing $(4+2\sqrt{2},\ 2+2\sqrt{2},\ 2+2\sqrt{2}),\ 0$ for the third position of the first triangle [1]). Indeed the part of the second net symbol corresponding to $[4+2\sqrt{2}, 2, 2, 0]$, i. e.

$$[4+2\sqrt{2}, 2, 2, 0],\ (4+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1},\ \sum_4^1 a_i \text{ odd},$$

gives for $a_1 = a_2 = a_3 = 0$ and $a_4 = -1$ the set of vertices represented by

$$\mathbf{(4+2\sqrt{2}-0,\ 4+2\sqrt{2}-2,\ 4+2\sqrt{2}-2)\ 4+2\sqrt{2}-4-2\sqrt{2}},$$

i. e. $(4+2\sqrt{2},\ 2+2\sqrt{2},\ 2+2\sqrt{2})\ 0$.

Edge gap prism. By extension the centre $1, 1, 0, 0$ of the edge $(2, 0)\ 0, 0$ of $[2, 0, 0, 0]$ gives $2(2+\sqrt{2}),\ 2(2+\sqrt{2}),\ 0,\ 0$ for the coordinates of O'. By reducing the first net symbol to this point as new origin we get

$$[4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],\ (4+2\sqrt{2})\ \overline{2a_1-1,\ 2a_2-1,\ 2a_3,\ 2a_4},\ \sum_1^4 a_i \text{ even}.$$

By putting $a_i = 0$, $(i = 1, 2, 3, 4)$, and taking the permutable digits in the indicated order and with the positive sign we find the vertex $-\sqrt{2},\ -(2+\sqrt{2}),\ 2+\sqrt{2},\ \sqrt{2}$ lying at minimum distance $2\sqrt{(4+2\sqrt{2})}$ from O'. As this distance is smaller than $4+\sqrt{2}$ we are obliged, in order to find all the vertices contained in that symbol lying at that distance from O', to put $a_3 = a_4 = 0$ and to take either $a_1 = a_2 = 0$ or $a_1 = a_2 = 1$. So we find the 32 vertices $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$, where the $\frac{1}{2}$ refers to the first syllable corresponding to the coordinates x_1, x_2 only. Now we have furthermore to examine the other two net symbols. For O' as origin the second net symbol is

$$\left.\begin{matrix}[4+2\sqrt{2}, & 2 & , & 2 & , & 0 &] \\ -\frac{1}{2}[4+\sqrt{2}, & 2+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2} &]\end{matrix}\right\},\ (4+2\sqrt{2})\ \overline{2a_1,\ 2a_2,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ odd},$$

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[1]) Until now we have only used implicitly the condition that the planes of the generating polygons are perfectly normal to each other, in the equation $\rho_1^2+\rho_2^2=\rho^2$. As the plane $x_1+x_2+x_3=0$, $x_4=0$ is parallel to those of the first triangle, the plane $x_1=x_2=x_3$ perfectly normal to it must be parallel to those of the second. We verify this by the following table of the nine vertices of the prismotope

$$\left|\begin{matrix} 4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2} & 2+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2} & 2+\sqrt{2}, 2+\sqrt{2}, 4+\sqrt{2}, \sqrt{2} \\ 4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, -\sqrt{2} & 2+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, -\sqrt{2} & 2+\sqrt{2}, 2+\sqrt{2}, 4+\sqrt{2}, -\sqrt{2} \\ 4+2\sqrt{2}, 2+2\sqrt{2}, 2+2\sqrt{2}, 0 & 2+2\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 0 & 2+2\sqrt{2}, 2+2\sqrt{2}, 4+2\sqrt{2}, 0 \end{matrix}\right|,$$

the three rows forming the positions of the first triangle and the three columns (of sets of coordinates) those of the second. So for the triangle of the first column we have $x_1 - x_2 = 2$, $x_2 = x_3$, etc.

By continuing this research it can be verified, that each of the three net symbols contains the six vertices of a P_3 with two positions of the first triangle, i. e. two rows of the table of the nine vertices, as end planes.

the two sets of permutable digits having to be combined with the same set of immovable ones. Here we find only vertices lying at a greater distance from O', unless we take $a_1 = a_2 = 0$. So we get for $a_3, a_4 = (0, -1)$ by means of the upper half of the symbol the 16 new vertices $[2, 0][2(1+\sqrt{2}), 0]$, by means of the lower the 16 vertices $x_1, x_2 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}], x_3, x_4 = -\frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$ already contained in the set $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ deduced from the first symbol. From this may be deduced that the two halves of the third symbol will furnish the two sets $[2, 0][2(1+\sqrt{2}), 0]$ and $x_1, x_2 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$, $x_3, x_4 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$.

So the result is a polytope with 48 vertices represented by the combination of the two symbols $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ and $[2, 0][2(1+\sqrt{2}), 0]$. It proves to be a P_{tC}. For, by applying on the tC represented by the symbol $[\sqrt{2}][2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$ the transformation

$$\left.\begin{array}{l} x_1 + x_2 = y_1\sqrt{2} \\ x_1 - x_2 = y_2\sqrt{2} \end{array}\right\}, \quad \left.\begin{array}{l} x_3 + x_4 = y_3\sqrt{2} \\ x_3 - x_4 = y_4\sqrt{2} \end{array}\right\}$$

we get $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ for the 32 vertices $[\sqrt{2}][2+\sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ and $[2, 0][2(1+\sqrt{2}), 0]$ for the remaing 16 vertices $[\sqrt{2}][\sqrt{2}][2+\sqrt{2}, 2+\sqrt{2}]$.

Vertex gap polytope. By extension the vertex $2, 0, 0, 0$ of $[2, 0, 0, 0]$ gives $4(2+\sqrt{2}), 0, 0, 0$ for O'. With respect to this origin the first net symbol is

$$[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (8+4\sqrt{2})\overline{a_1-1, a_2, a_3, a_4}, \sum_1^4 a_i \text{even},$$

which can be reduced to

$$[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (8+4\sqrt{2})\ \overline{a_1, a_2, a_3, a_4}, \sum_1^4 a_i \text{odd}.$$

By taking in this last symbol $a_1, a_2, a_3, a_4 = [1, 0, 0, 0]$ and putting the digit $4+\sqrt{2}$ always where the 1 stands with the opposite sign of it, we get the 192 vertices $[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$ lying at the minimum distance $4(1+\sqrt{2})$ from O'.

With respect to the same origin O' the second net symbol is

$$\left.\begin{array}{l} [4+2\sqrt{2}, \quad 2 \quad , \quad 2 \quad , \ 0\ ] \\ -\frac{1}{2}[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}] \end{array}\right\}, (4+2\sqrt{2})\ \overline{2a_1-1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

the immovable part of which can be reduced to

$$(4+2\sqrt{2})\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{even}.$$

By considering the three groups of cases

$$a_i = 0, (i = 1, 2, 3, 4) —, a_1, a_2, a_3, a_4 = (-1, -1, 0, 0,) —,$$
$$a_i = -1, (i = 1, 2, 3, 4),$$

and adding to the immovable parts the permutable ones taken in any order, generally affected by the sign which tends to *decrease* the absolute value of the coordinate but — in connection with the negative sign before the lower half of the symbol which exacts an odd number of negative permutable digits — with exception of the smallest of these digits $\sqrt{2}$ the sign of which is to be chosen inversely so as to *increase* the absolute value of the coordinate, we get by the upper half the 96 new vertices $[4 + 2\sqrt{2}, 2 + 2\sqrt{2}, 2 + 2\sqrt{2}, 0]$ and by the lower the 96 vertices $\frac{1}{2}[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, obtained above. So the result is a polytope with 288 vertices represented by the combination of the symbols

$$[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], [4+2\sqrt{2}, 2+2\sqrt{2}, 2+2\sqrt{2}, 0].$$

As we will prove in section V this polytope with the characteristic numbers (288, 576, 336, 48) limited by $48\, tC$ is $ce_1e_2C_{24}$.

Case $ce_1e_3N(C_{16})$. Besides $[1'1'1\,1]\sqrt{2} = ce_1e_3C_{16}$ we have to look out for the face gap filling and the polytope of vertex import, the edge gap filling being reduced by contraction to the base polyhedron of the prism occurring in the case of $e_1e_3N(C_{16})$.

Face gap prismotope. Here we get for the new origin O' the coordinates $\frac{2}{3}(2 + 2\sqrt{2}), \frac{2}{3}(2 + 2\sqrt{2}), \frac{2}{3}(2 + 2\sqrt{2}), 0$, as $2 + 2\sqrt{2}$ is the extension number.

So the first and the second net symbol are

$$[2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}], (4+4\sqrt{2})\ \overline{a_1 - \tfrac{1}{3}, a_2 - \tfrac{1}{3}, a_3 - \tfrac{1}{3}, a_4}\ , \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix} [2+2\sqrt{2}, & 2 & , 0 & , 0\ ] \\ -\tfrac{1}{2}[2+\sqrt{2}, & 2+\sqrt{2}, & \sqrt{2}, & \sqrt{2}] \end{matrix}\right\}, (2+2\sqrt{2})\ \overline{2a_1 + \tfrac{1}{3}, 2a_2 + \tfrac{1}{3}, 2a_3 + \tfrac{1}{3}, 2a_4 + 1}, \sum_1^4 a_i \text{ odd}.$$

By taking in the first symbol $a_i = 0, (i = 1, 2, 3, 4)$, we find the vertices $\left(\frac{2-\sqrt{2}}{3}, \frac{2-\sqrt{2}}{3}, \frac{-4-\sqrt{2}}{3}\right)[2]$ lying at minimum distance $\frac{4}{3}\sqrt{3}$ from O', i. e. a P_3; by substituting in the upper half of the second symbol $a_i = 0, (i = 1, 2, 3), a_4 = -1$ we get moreover $\left(\frac{2+2\sqrt{2}}{3}, \frac{2+2\sqrt{2}}{3}, \frac{-4+2\sqrt{2}}{3}\right) 0$, the third triangle of the prismotope $[3; 3]$ to be found.

Vertex gap polytope. The new origin is $2(2+2\sqrt{2}), 0, 0, 0$ and the first and second net symbol become, in the shortest form possible,

$$[2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}], (4+4\sqrt{2})\ \overline{a_1\ ,\ a_2\ ,\ a_3\ ,\ a_4}\ , \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[2+2\sqrt{2}, & 2 & , 0 , & 0\ ]\\ -\tfrac{1}{2}[2+\sqrt{2}, & 2+\sqrt{2}, & \sqrt{2}, & \sqrt{2}]\end{matrix}\right\}, (2+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1}, \sum_1^4 a_i \text{ even}.$$

Putting into the first symbol $a_1, a_2, a_3, a_4 = [1, 0, 0, 0]$ and combining with the a_i differing from zero one of the two digits $2+\sqrt{2}$ taken with the sign tending to decrease the absolute value of the coordinate we get the 192 vertices $[2+3\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}]$. Putting into the upper half of the second symbol $a_i = 0, (i=1,2,3,4)$, we find moreover the 96 vertices $[2+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}, 0]$. So the result is a polytope with 288 vertices which will prove later on to admit the characteristic numbers (288, 864, 720, 144) and to be $e_2\, C_{24}$.

83. *Case* $e_1 e_4 N(C_{16})$. Here the extension number is $3+\sqrt{2}$. So we have to reduce the three net symbols

$$[4,2,0,0],(6+2\sqrt{2})\ \overline{a_1\ ,\ a_2\ ,\ a_3\ ,\ a_4}\ ,\sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even}$$

for the constituents of body, face, edge, vertex import to the new origins $(3+\sqrt{2})\ \overline{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}}$, $(3+\sqrt{2})\ \overline{\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0}$, $(3+\sqrt{2})\ \overline{1, 1, 0, 0}$, $(3+\sqrt{2})\ \overline{2, 0, 0, 0}$ respectively, the constituent of polytope import being $[4, 2, 0, 0] = e_1\, C_{16}$.

Body gap prism. The three net symbols become

$$[4,2,0,0],(3+\sqrt{2})\overline{2a_1-\tfrac{1}{2}, 2a_2-\tfrac{1}{2}, 2a_3-\tfrac{1}{2}, 2a_4-\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}, 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}}, \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}, 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}}, \sum_1^4 a_i \text{ even}.$$

By making the a_i to disappear the first and the third [1]) symbol give the sets of vertices $\left(\frac{5-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}, \frac{-3-\sqrt{2}}{2}, \frac{-3-\sqrt{2}}{2}\right)$, $\left(\frac{5+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{3}, \frac{-3+\sqrt{2}}{2}, \frac{-3+\sqrt{2}}{2}\right)$ each of which corresponds to a (2100), i. e. to a tT. So the result is a P_{tT}, all the vertices of the second symbol lying at larger distance from O' than the circumradius $\sqrt{13}$ of this P_{tT}.

Face gap prismotope. Here the three net symbols are

$$[4,2,0,0],\ (3+\sqrt{2})\ \overline{2a_1-\tfrac{2}{3},\ 2a_2-\tfrac{2}{3},\ 2a_3-\tfrac{2}{3},\ 2a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],\ (3+\sqrt{2})\ \overline{2a_1+\tfrac{1}{3},\ 2a_2+\tfrac{1}{3},\ 2a_3+\tfrac{1}{3},\ 2a_4+1},\ \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],\ (3+\sqrt{2})\ \overline{2a_1+\tfrac{1}{3},\ 2a_2+\tfrac{1}{3},\ 2a_3+\tfrac{1}{3},\ 2a_4+1},\ \sum_1^4 a_i \text{ even}.$$

By taking in the first symbol $a_i = 0$, $(i = 1, 2, 3, 4)$, in the second $a_i = 0$, $(i = 1, 2, 3)$, $a_4 = -1$, in the third $a_i = 0$, $(i = 1, 2, 3, 4)$, we get the three hexagons

$$\left.\begin{array}{ll} (2-\frac{2}{3}\sqrt{2},\ -\frac{2}{3}\sqrt{2},\ -2-\frac{2}{3}\sqrt{2}) & 0 \\ (2+\frac{1}{3}\sqrt{2},\ \ \frac{1}{3}\sqrt{2},\ -2+\frac{1}{3}\sqrt{2}) & -\sqrt{2} \\ (2+\frac{1}{3}\sqrt{2},\ \ \frac{1}{3}\sqrt{2},\ -2+\frac{1}{3}\sqrt{2}) & \sqrt{2} \end{array}\right\}.$$

So the result is a [6; 3].

Edge gap prism. Now the three net symbols become

$$[4,2,0,0],\ (3+\sqrt{2})\ \overline{2a_1-1,\ 2a_2-1,\ 2a_3\ ,\ 2a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],\ (3+\sqrt{2})\ \overline{2a_1\ ,\ 2a_2\ ,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],\ (3+\sqrt{2})\ \overline{2a_1\ ,\ 2a_2\ ,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ even}.$$

By taking in the first symbol $a_3 = a_4 = 0$ and either $a_1 = a_2 = 0$ or $a_1 = a_2 = 1$, in the second $a_1 = a_2 = 0$ and $a_3, a_4 = (-1, 0)$, in the third $a_1 = a_2 = 0$ and either $a_3 = a_4 = 0$ or $a_3 = a_4 = -1$ and by combining with the not disappearing immovable digits the greater permutable ones, generally affected by the sign tending to decrease the absolute value of the coordinate but — on account of the sign before $\frac{1}{2}$ [3, 3, 1, 1] of the second and the third symbol —

[1]) That one of the three symbols must remain inactive in the generation of the body gap prism is an immediate consequence of the lemma of art. 80.

with exception of one of the permutable units, we get successively the three quadruples of vertices

$$\tfrac{1}{2}[1+\sqrt{2},-1+\sqrt{2}]\,0,0 \text{ —}, (1,-1)\,\tfrac{1}{2}[\sqrt{2},\sqrt{2}] \text{ —}, (1,-1)(\sqrt{2},-\sqrt{2})$$

lying at minimum distance $\sqrt{6}$ from O'. These 12 points form the vertices of a prism P_O with octahedral base; each of the three quadruples just found lies in a plane passing through the axis of the prism and consists of a pair of opposite vertices of each of the two limiting octahedra. The equations of the three planes are

$$x_3=0, x_4=0 \text{ —}, x_1+x_2=0, x_3=x_4 \text{ —}, x+x_2=0, x_3+x_4=0.$$

So the axis of the prism is represented by $x_3=0$, $x_4=0$, $x_1+x_2=0$.

Moreover it is easily verified that the three quadrangles are rectangles with sides $2\sqrt{2}$ and 4. As we can unite the second and third symbols the P_O can be represented by the two symbols $\tfrac{1}{2}[1+\sqrt{2},-1+\sqrt{2}]\,0,0$ and $(1,-1)[\sqrt{2},\sqrt{2}]$.

Vertex gap polytope. Finally the three net symbols are, in the simplest form,

$$[4,2,0,0], (6+2\sqrt{2})\,\overline{a_1\quad, a_2\quad, a_3\quad, a_4}\quad, \sum_1^4 a_i \text{ odd},$$

$$\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

$$-\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd}.$$

By taking for a_1, a_2, a_3, a_4 in the first symbol $[1,0,0,0]$, in the second either $0,0,0,0$ or $(-1,-1,0,0)$ or $-1,-1,-1,-1$, in the third either $(-1,0,0,0)$ or $(-1,-1,-1,0)$, and by assigning to the permutable digits the sign which decreases the absolute value of the coordinate, we find the three sets of 48 points represented by the symbols

$$[2+\sqrt{2},2,0,0],\ \tfrac{1}{2}[2+\sqrt{2},2+\sqrt{2},\sqrt{2},\sqrt{2}],\ -\tfrac{1}{2}[2+\sqrt{2},2+\sqrt{2},\sqrt{2},\sqrt{2}],$$

which can be reduced to

$$[2+2\sqrt{2},2,0,0], [2+\sqrt{2},2+\sqrt{2},\sqrt{2},\sqrt{2}].$$

These 144 points prove to be the vertices of the polytope $e_3 C_{24}$ with the characteristic numbers (144, 576, 672, 240).

$e_1 e_2 e_3 e_4 N(C_{16})$. Extension number $6+3\sqrt{2}$, three net symbols

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\,,\ (12+6\sqrt{2})\ \overline{a_1\ ,\ a_2\ ,\ a_3\ ,\ a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

which are to be reduced to the new origins, indicated in the preceding example $e_1 e_4 N(C_{16})$. But in the case of the body gap we will mention only the first net symbol and the lower part of the third, which lead to the desired result.

Body gap prism. We find

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (6+3\sqrt{2})\ \overline{2a_1-\tfrac{1}{2}, 2a_2-\tfrac{1}{2}, 2a_3-\tfrac{1}{2}, 2a_4-\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (6+3\sqrt{2})\ 2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}. 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}, \sum_1^4 a_i \text{ even},$$

giving by means of the suppositions of the preceding example the prism P_{tO}, the two bases of which are

$$(3-\tfrac{1}{2}\sqrt{2}, 1-\tfrac{1}{2}\sqrt{2}, -1-\tfrac{1}{2}\sqrt{2}, -3-\tfrac{1}{2}\sqrt{2}),$$
$$(3+\tfrac{1}{2}\sqrt{2}, 1+\tfrac{1}{2}\sqrt{2}, -1+\tfrac{1}{2}\sqrt{2}, -3+\tfrac{1}{2}\sqrt{2}).$$

Face gap prismotope. Here we have

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\,,\ (6+3\sqrt{2})\ \overline{2a_1-\tfrac{2}{3}, 2a_2-\tfrac{2}{3}, 2a_3-\tfrac{2}{3}, 2a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions easily found successively

$$\begin{matrix}(2-\sqrt{2}, & -\sqrt{2}, & -2-\sqrt{2}) & [\sqrt{2}], \\ (2+\sqrt{2}, & \sqrt{2}, & -2+\sqrt{2}) & -\sqrt{2}, \\ (\ 2\ , & 0\ , & 2\ ) & -2\sqrt{2}, \\ (2+\sqrt{2}, & \sqrt{2}, & -2+\sqrt{2}) & \sqrt{2}, \\ (\ 2\ , & 0\ , & 2\ ) & 2\sqrt{2},\end{matrix}$$

which can be combined to

$$(2-\sqrt{2}, -\sqrt{2}, -2-\sqrt{2})[\sqrt{2}]-, (2+\sqrt{2}, \sqrt{2}, -2+\sqrt{2})[\sqrt{2}]-, (2, 0, -2)[2\sqrt{2}],$$

representing together a prismotope $[6\,;6]$.

Edge gap prism. We get

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \ (6+3\sqrt{2})\ \overline{2a_1-1, 2a_2-1, 2a_3 \quad , 2a_4 \quad}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & \sqrt{2}]\end{matrix}\right\}, (6+3\sqrt{2})\ \overline{2a_1 \quad , 2a_2 \quad , 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & \sqrt{2}]\end{matrix}\right\}, (6+3\sqrt{2})\ \overline{2a_1 \quad , 2a_2 \quad , 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions

$$\begin{matrix} \tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}] & [2+\sqrt{2}, \sqrt{2}], \\ [\quad 2 \quad , \quad 0] & -\tfrac{1}{2}[2+3\sqrt{2}, \sqrt{2}], \\ \tfrac{1}{2}[2+\sqrt{2}, \sqrt{2}] & -\tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}], \\ [\quad 2 \quad , \quad 0] & \tfrac{1}{2}[2+3\sqrt{2}, \sqrt{2}], \\ \tfrac{1}{2}[2+\sqrt{2}, \sqrt{2}] & \tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}], \end{matrix}$$

which can be combined into

$$\tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}][2+\sqrt{2}, \sqrt{2}]-, \ [2, 0][2+3\sqrt{2}, \sqrt{2}]-, \ \tfrac{1}{2}[2+\sqrt{2}, \sqrt{2},][2+2\sqrt{2}, 2\sqrt{2}],$$

representing together the 96 vertices of a P_{tCO}. For the transformation

$$\left.\begin{matrix} x_1+x_2=y_1\sqrt{2} \\ x_1-x_2=y_2\sqrt{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x_3+x_4=y_3\sqrt{2} \\ x_3-x_4=y_4\sqrt{2} \end{matrix}\right\}$$

gives immediately

$$y_2=[\sqrt{2}] \quad , \quad y_1, y_3, y_4=[4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}].$$

Vertex gap polytope. Finally we have to deal with

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \ (12+6\sqrt{2})\ \overline{a_1 \quad , a_2 \quad , a_3 \quad , a_4 \quad}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & \sqrt{2}]\end{matrix}\right\}, (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & \sqrt{2}]\end{matrix}\right\}, (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

giving by adequate substitutions

$$\begin{matrix} [6+5\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \\ \tfrac{1}{2}[6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}], \\ \tfrac{1}{2}[6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}], \\ -\tfrac{1}{2}[6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}], \\ -\tfrac{1}{2}[6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}], \end{matrix}$$

i. e.

$$[6+5\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]-, [6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}]-, [6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}],$$

representing together the 1152 vertices of the polytope $e_1 e_2 e_3 C_{24}$.

Case $ce_1e_2e_3e_4N(C_{16})$. Extension number $5+3\sqrt{2}$, three net symbols

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (10+6\sqrt{2})\, \overline{a_1, a_2, a_3, a_4}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l} -\frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ -\frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{array}\right\}, (5+3\sqrt{2})\, \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l} \frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ \frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{array}\right\}, (5+3\sqrt{2})\, \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

which are to be reduced to the new origins, to be formed according to the indications of the preceding example. Here the polytope of edge import is lacking. In the case of the body gap we mention only the first net symbol and the lower part of the third, which lead to the desired result.

Body gap prism. We find

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (5+3\sqrt{2})\, \overline{2a_1-\tfrac{1}{2}, 2a_2-\tfrac{1}{2}, 2a_3-\tfrac{1}{2}, 2a_4-\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}], (5+3\sqrt{2})\, \overline{2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}, 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

giving by means of the substitutions $a_i = 0, (i = 1, 2, 3, 4)$, the prism P_{tT}, the two bases of which are

$$\left(\frac{3-\sqrt{2}}{2}, \frac{3-\sqrt{2}}{2}, \frac{-1-\sqrt{2}}{2}, \frac{-5-\sqrt{2}}{2}\right),$$
$$\left(\frac{3+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}, \frac{-5+\sqrt{2}}{2}\right).$$

Face gap prismotope. Here we have

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (5+3\sqrt{2})\, \overline{2a_1-\tfrac{2}{3}, 2a_2-\tfrac{2}{3}, 2a_3-\tfrac{2}{3}, 2a_4}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l} -\frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ -\frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{array}\right\}, (5+3\sqrt{2})\, \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l} \frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ \frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}. 1+\sqrt{2}] \end{array}\right\}, (5+3\sqrt{2})\, \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions easily found

$$\begin{array}{ll} (\tfrac{2}{3}-\sqrt{2}, \tfrac{2}{3}-\sqrt{2}, -\tfrac{4}{3}-\sqrt{2}) & [\sqrt{2}], \\ (\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2}) - & \sqrt{2}, \\ (\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}) - & 2\sqrt{2}, \\ (\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2}) & \sqrt{2}, \\ (\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}) & 2\sqrt{2}, \end{array}$$

which can be telescoped into

$$(\tfrac{2}{3}-\sqrt{2}, \tfrac{2}{3}-\sqrt{2}, -\tfrac{4}{3}-\sqrt{2})[\sqrt{2}]-,$$
$$(\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2})[\sqrt{2}]-, (\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3})[2\sqrt{2}],$$

representing together the vertices of a prismotope [6; 3].

Vertex gap polytope. Here we find finally

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (10+6\sqrt{2})\,\overline{a_1, a_2, a_3, a_4}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix} -\tfrac{1}{2}[5+2\sqrt{2}, & 3, & 1, & 1] \\ -\tfrac{1}{2}[5+\sqrt{2}, & 3+\sqrt{2}, & 1+\sqrt{2}, & 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix} \tfrac{1}{2}[5+2\sqrt{2}, & 3, & 1, & 1] \\ \tfrac{1}{2}[5+\sqrt{2}, & 3+\sqrt{2}, & 1+\sqrt{2}, & 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

giving — as it ought to do — quite the same result as in the preceding example.

84. Probably after all the indications contained in the treatment of several special cases Table VII would be quite clear by itself but for the first column of the part corresponding to the second extreme polytope and the last column but one; so we have to add a few words about these two columns. [1])

In the two special cases treated in art. 81 the vertex polytopes proved to be polytopes all the vertices of which can be represented by one symbol, i. e. polytopes of measure polytope extraction, viz. $ce_1 e_2 e_3 C_8 = e_1 e_2 C_{16}$. But in the five cases studied in the art[s]. 82, 83 we had to deal with vertex polytopes the vertices of which cannot be represented by one symbol only, i. e. with forms which do not belong to the measure polytope family. These forms were said to be derivable from the cell C_{24} by applying respectively the sets of operations $ce_1 e_2$, e_2, e_3, $e_1 e_1 e_3$. Now in part F of this section will be shown, not only that *all* the forms appearing here as vertex polytopes — whether their vertices are represented by one, two or three symbols — can be deduced from cell C_{24} by applying the operations e_k and c, but also by which set of operations any required result is to be obtained. This set of operations is indicated for all possible cases in the first column of the part of Table VII corresponding to g_0. So in the second case of art. 82 we found $e_2 C_{24}$; but as the general theory (compare Theorem LV) demands $ce_1 e_3 C_{24}$ which is equal to $e_2 C_{24}$, we have inscribed $ce_1 e_3 C_{24}$. [2])

The remark of the second foot note of art. 78 — that several nets deduced from $N(C_{16})$ are equal to nets deduced from $N(C_{24})$ —

[1]) The very last column will be explained later on.

[2]) The deduction of the symbols contained in the Table by applying the operations e_k and c to the cell C_{24}, i. e. to $[1, 1, 0, 0]\sqrt{2}$, will be given in the last section of this memoir.

must now be generalized to this: "Every net deduced from $N(C_{16})$ can at the same time be deduced from $N(C_{24})$." Now the last column but one indicates the name of the corresponding C_{24}-net. So we have $e_3\, e_4\, N(C_{16}) = e_1\, e_4\, N(C_{24})$, etc.

We must remember that the symbols given in Table VII have to be completed by applying the transformations indicated in art. 79. Moreover we fix our attention on the particular form in which the symbols of each constituent appear. Every prismotope g_2 is decomposed as to its vertices into two or three fourdimensional prisms, one of which degenerates in some cases into a regular polygon; of the fourdimensional prisms g_3, g_1 the first is determined by its two bases, whilst the latter appears as prismotope (4; 4) or as a combination of prismotopes, etc.

F. *Polarity.*

85. In art. 67 we remarked that in S_n any polytope derived by means of the operations e_k with or without c from the measure polytope M_n can also be derived from the cross polytope C_{2^n}. In art. 77 we stated this result in the form of theorem LI after having demonstrated it by showing that the *total* set of symbols of coordinates of the group derived from C_{2^n} is equal to that of the group derived from M_n. We have to come back to this result once more here, in order to indicate how it depends on the laws of reciprocity and what is the general relation between the two symbols of expansion operations of the *same* polytope deduced from M_n on one hand and from C_{2^n} on the other, which couples of symbols have been given for $n = 3, 4, 5$ (compare the foot note in art. 72) in the first and the second column of Table IV.

It goes without saying that the dependence between theorem LI and the laws of reciprocity merely consists in this that the polar reciprocal polytope of a regular polytope A of S_n with respect to a concentric spherical space is an other regular polytope A' and that in this polarity the vertices, edges, faces, etc. of the one correspond to the limits $(l)_{n-1}, (l)_{n-2}, (l)_{n-3}$, etc. of the other. So we have still only to deduce the relation between the two symbols of the same polytope. This task can be performed by comparing the first two columns of Table IV with each other and by generalizing for an arbitrary n the outcome of this comparison. So for $a < b < \ldots < s < t < n-1$ we immediately deduce from Table IV the following general laws:

$$\left.\begin{array}{ll} e_a\, e_b \ldots e_s\, e_t\, e_{n-1}\, M_n = & e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, e_{n-1}\; C_{2^n} \\ c\, e_a\, e_b \ldots e_s\, e_t\, e_{n-1}\, M_n = & e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\; C_{2^n} \\ e_a\, e_b \ldots e_s\, e_t\, M_n & = c\, e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, e_{n-1}\; C_{2^n} \\ c\, e_a\, e_b \ldots e_s\, e_t\, M_n & = c\, e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\; C_{2^n} \end{array}\right\}.$$

The proof of these general laws can be based on the remark that each pair of polytopes forming the two members of any of these four equations admits the same symbol of coordinates; if k is the number of the symbols $e_a, e_b, \ldots, e_s, e_t$ these symbols of coordinates are successively:

$$\left.\begin{array}{r} [1 + (k+1)\sqrt{2}, \overline{1 + k\sqrt{2}}^{\,n-t-1}, \overline{1 + (k-1)\sqrt{2}, \ldots}^{\,t-s}, \overline{1 + \sqrt{2},}^{\,b-a}\ \overline{1}^{\,a}\ ] \\ [k+1, \overline{k, \ldots, k}^{\,n-t-1}, \overline{k-1, \ldots, k-1}^{\,t-s}, \ldots, \overline{1, \ldots, 1}^{\,b-a}, \overline{0, \ldots, 0}^{\,a}] \\ [\overline{1 + k\sqrt{2}}^{\,n-t}, \overline{1 + (k-1)\sqrt{2}, \ldots}^{\,t-s}, \overline{1 + \sqrt{2},}^{\,b-a}\ \overline{1}^{\,a}\ ] \\ [k\ , \overline{k-1, \ldots, k-1}^{\,t-s}, \ldots, \overline{1, \ldots, 1}^{\,b-a}, \overline{0, \ldots, 0}^{\,a}] \end{array}\right\}.$$

By introducing the operation e_0 corresponding to the generation of the regular polytopes starting from a point and representing this point for M_n by P_0, for C_{2^n} by P_0' we can unite these four general laws in:

THEOREM LIII. "The two polytopes

$$e_a\, e_b\, e_c \ldots e_r\, e_s\, e_t\, P_0\ ,\ e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'}\, P_0'$$

are equal [1]) if and only if we have generally

$$a + t' = b + s' = c + r' = \ldots = r + c' = s + b' = t + a' = n - 1."$$

86. The influence of theorem LIII on the results laid down in art.^{s} 65 and 66 is evident.

By polarizing an expansion or contraction form derived from the cross polytope C_{2^n} of S_n with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles[2]), to which corresponds the inverted symbol of characteristic numbers of the original polytope, etc.

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[1]) This theorem gives for M_n and C_{2^n} what theorem XXIII contains about the two differently orientated positions of the simplex; it holds not only for M_n and C_{2^n}, n being general, but also for the polytopes C_{120} and C_{600} of S_4 and in the same way there exists a theorem analogous to theorem XXIII for the cell C_{24} of S_4 in its two different positions with respect to the system of coordinates. We shall have to come back to this point in the fifth section of this memoir.

[2]) Compare for S_4 the foot note of art. 65.

THEOREM LIV. "Any polytope $(P)_n$ of cross polytope descent in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain concentric spherical space. But in the case of the cross polytope itself these spaces pass through the centre."

This theorem is a mere transcription of theorem XL.

87. If we apply the general relations of polarity, which have led us in art. 67 to theorem XLI, to the particular case of the polarly related nets $N(C_{16})$ and $N(C_{24})$ of S_4 we get:

THEOREM LV. "If the sets of operations E and E' are complementary to each other, i. e. if E' contains the operations e_{4-k} complementary to the operations e_k of E and no other, we have

$$EN(C_{16}) = cE'e_4 N(C_{24}),\ Ee_4 N(C_{16}) = E'e_4 N(C_{24}),\ cEN(C_{16}) = cE'N(C_{24}),$$
$$cEe_4 N(C_{16}) = E'N(C_{24})\text{"}$$

An analytical proof of this theorem would require a more ample knowledge of the net symbols of the nets deduced from $N(C_{24})$ than we have at our disposal, after having nearly finished the third part of this memoir. We will therefore invert the order of ideas, i. e. we will content ourselves here by giving the analytical form of the geometric facts and use theorem LV and the last column but one of Table VII based on it in the last section of this memoir dealing with the extra regular polytopes, to facilitate and control the deduction of the polytopes and nets, deduced from C_{24}. There we shall have occasion to apply the same principle to the polarly related polytopes C_{600} and C_{120}.[1])

88. The connection between C_8, C_{16}, C_{24} according to which the $C_{24}^{(2)}$ can be split up with respect to its vertices into a $C_8^{(2)}$ and a $C_{16}(2\sqrt{2})$ and with repect to its limiting spaces into a $C_8(2\sqrt{2})$ and a $C_{16}^{(4)}$ leads to connections between the polytopes and the nets which cannot be deduced from polarity only. So we find:

$$C_{24} = ce_2\, C_8\ (= ce_1\, C_{16}),\ e_1\, C_{24} = e_1\, e_2\, C_{16}$$

and

$$N(C_{24}) = ce_1\, N(C_{16}) = ce_4\, N(C_{16}) = ce_3\, N(C_{24}).$$

But there is still a more striking coincidence to be indicated, viz. that the nets $e_2\, N(C_{16})$ and $e_1\, e_2\, N(C_{16})$ are respectively equal to the nets

[1]) We defer the investigation of the reciprocal nets of those given in Table VII to the paper announced in the foot note of art. 68.

$ce_1 e_3 N(C_8)$ and $ce_1 e_2 e_3 N(C_8)$, the constituent C_8 forming at the same time the g_1 of the former couple and the g_2 of the latter. We shall have occasion to profit by this coincidence in the next article.

G. *Symmetry, considerations of the theory of groups, regularity.*

89. On account of the fact that the offspring of the cross polytope is identical with that of the measure polytope, the theorems XLII and XLIII may be applied to any form of cross polytope descent.

So we have only to add a few lines with respect to the regularity and for the same reason this task has to be performed with respect to the nets deduced from $N(C_{16})$ only.

If we individualize the 31 nets of Table VII by an N bearing the rank number as subscript we can say that the nets N_1, N_{17}, N_{24} are regular and that the degree of regularity of the nets N_3, N_5 with two equal extreme constituents is known, as these nets are at the same time measure polytope nets. As moreover each of the 26 remaining nets admits faces of at least two different shapes, the degree of regularity of each of these nets is either $\frac{4}{10}$ or $\frac{3}{10}$, according to its having only one kind or more than one kind of edge. But now it is immediately clear that any net admitting a constituent g_3 has at least two different kinds of edges, as the erect edges of the fourdimensional g_3, characterized by the property that the same coordinates of the two end points differ by unity, cannot be at the same time edges of the groundform in any of its three orientations. So we have still to treat the twelve cases N_2, N_4, N_6, N_7, N_8, N_{18}, N_{19}, N_{20}, N_{21}, N_{22}, N_{23}, N_{27} forming two different groups, one of the nets N_{18}, N_{19}, N_{27} with groundforms admitting only one kind of edge and one containing the other nine not characterized by this property. Now we can decide the question with respect to any of the nets of these two groups with the least amount of trouble by means of the following general problem, where G is the groundform given in Table VII, P the pattern vertex obtained by omitting the square brackets of G, whilst Q_i and Q'_i represent the vertices of the net adjacent to P, of which Q_i are and Q'_i are not vertices of G:

"Determine the repetitions r of G (in its three orientations) with P as vertex and examine whether or not all the vertices Q_i and Q'_i are vertices of the same number of these repetitions (G included)".

The first case must present itself for the three nets N_{18}, N_{19}, N_{27}. For the groundform of each of these nets admits one kind of edge and its repetitions containing P are grouped regularly round P;

so these repetitions must be arranged in the same manner round every edge. But this decides that the arrangement of *all* the constituents round every edge is the same, as there is only one other constituent, viz g_0. So the degree of regularity of N_{18}, N_{19}, N_{27} is $\frac{2}{5}$.

In all the nine cases of the second group there are two or more different kinds of edge and the degree of regularity is $\frac{3}{10}$. From these cases we treat a couple of examples.

Example $e_1 N(C_{16})$. All the repetitions of the groundform are represented by the system of the three symbols

$$\left.\begin{array}{l} [6a_1 \quad +\mathbf{4},\ 6a_2 \quad +\mathbf{2},\ 6a_3 \quad +\mathbf{0},\ 6a_4 \quad +\mathbf{0}],\ \Sigma a_i \text{ even} \\ \frac{1}{2}[6a_1+3+\mathbf{3},\ 6a_2+3+\mathbf{3},\ 6a_3+3+\mathbf{1},\ 6a_4+3+\mathbf{1}],\ \Sigma a_i \text{ odd} \\ -\frac{1}{2}[6a_1+3+\mathbf{3},\ 6a_2+3+\mathbf{3},\ 6a_3+3+\mathbf{1},\ 6a_4+3+\mathbf{1}],\ \Sigma a_i \text{ even} \end{array}\right\}.$$

So the repetitions r with 4, 2, 0, 0 as vertex are:

$$\left.\begin{array}{lllll} [& \mathbf{4}, & \mathbf{2}, & \mathbf{0}, & \mathbf{0}] \ldots r_1 \\ [6 & -\mathbf{2},\ 6 & -\mathbf{4}, & \mathbf{0}, & \mathbf{0}] \ldots r_2 \\ \frac{1}{2}[& 3+\mathbf{1}, & 3-\mathbf{1}, & -6+3+\mathbf{3}, & 3-\mathbf{3}] \ldots s_1 \\ \frac{1}{2}[& 3+\mathbf{1}, & 3-\mathbf{1}, & 3-\mathbf{3}, & -6+3+\mathbf{3}] \ldots s_2 \\ -\frac{1}{2}[& 3+\mathbf{1}, & 3-\mathbf{1}, & 3-\mathbf{3}, & 3-\mathbf{3}] \ldots t_1 \\ -\frac{1}{2}[& 3+\mathbf{1}, & 3-\mathbf{1}, & -6+3+\mathbf{3}, & -6+3+\mathbf{3}] \ldots t_2 \end{array}\right\}$$

which may be indicated by the symbols $r_1, r_2, \ldots, t_2$. Now the adjacent vertex 2, 4, 0, 0 is vertex of the six repetitions and 4, 0, 2, 0 of r_1, s_2, t_1 only. So there are two kinds of edges and the degree of regularity is $\frac{3}{10}$.

Example $e_3 N(C_{16})$. If we telescope $[pp_1+Q_1, pp_2+Q_2, pp_3+Q_3, pp_4+Q_4]$ into $[q_1, q_2, q_3, q_4]\ (p)\ \overline{p_1, p_2, p_3, p_4}$ the repetitions of the groundform $[2+\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}]$ can be represented by

$$[2+\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2}],\ (1+2\sqrt{2})\ \overline{2a_1,\ 2a_2,\ 2a_3,\ 2a_4},\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l} \frac{1}{2}[1+2\sqrt{2},\ 1,\ 1,\ 1] \\ \frac{1}{2}[1+\sqrt{2},\ 1+\sqrt{2},\ 1+\sqrt{2},\ 1-\sqrt{2}] \end{array}\right\},\ (1+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l} -\frac{1}{2}[1+2\sqrt{2},\ 1,\ 1,\ 1] \\ -\frac{1}{2}[1+\sqrt{2},\ 1+\sqrt{2},\ 1+\sqrt{2},\ 1-\sqrt{2}] \end{array}\right\},\ (1+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ even}.$$

So the repetitions r with $2+\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}$ as vertex are only

$$\begin{array}{l} [\quad \mathbf{2}+\sqrt{\mathbf{2}},\quad \sqrt{\mathbf{2}},\quad \sqrt{\mathbf{2}},\quad \sqrt{\mathbf{2}}], \\ [1+2\sqrt{2}+\mathbf{1}-\sqrt{\mathbf{2}},\ 1+2\sqrt{2}-\mathbf{1}+\sqrt{\mathbf{2}},\ 1+2\sqrt{2}-(\mathbf{1}+\sqrt{\mathbf{2}}),\ 1+2\sqrt{2}-(\mathbf{1}+\sqrt{\mathbf{2}})]. \end{array}$$

Now $\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}$ is vertex of both, whilst on the other hand $2+\sqrt{2}, \sqrt{2}, \sqrt{2}, -\sqrt{2}$ is vertex of the first only. So two kinds of edges, degree of regularity $\frac{3}{10}$.

The very last column of Table VII contains the results.

Section IV:

Polytopes and nets derived from the half measure polytope.

A. *The symbol of coordinates.*

90. Several times we have commemorated the fact that the eight vertices of a cube can be split up into two groups of four points, the vertices of two regular tetrahedra, and that with respect to the cube the vertices of each group may be said to be non adjacent, i. e. not connected by an edge of the cube – see e. g. the introduction of section I and the foot note of art. 4. [1]) Also that the sixteen vertices of an eightcell can be split up into two groups of eight non adjacent points, the vertices of two regular sixteencells (compare e. g. art. 78). So in general the 2^n vertices of the measure polytope M_n of space S_n can be split up into two groups of 2^{n-1} non adjacent points, but the polytopes of which these groups of points are the vertices are not regular for $n > 4$. So in the case $n = 5$ there are two different kinds of limits $(l)_4$, viz. cells C_{16} forming what remains of the limiting eightcells of M_5 and simplexes $S(5)$ replacing the vanished vertices of M_5. In relation with their generation we call these new polytopes *half measure polytopes* and we investigate in this section these polytopes and the nets which can be derived from them.

In the cases [111] and [1111] of cube and eightcell we have represented the two half measure polytopes by the symbols $\pm\frac{1}{2}[111]$ and $\pm\frac{1}{2}[1111]$ respectively. Likewise we indicate by $\pm\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ the two half measure polytopes into which M_n can be decomposed according to the vertices, where $+\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ includes all the vertices of which an *even* number of coordinates is negative and $-\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ all the vertices of which an *odd* number of coordinates is negative. These symbols immediately reveal a difference in character between the half measure polytopes of S_{2n} and S_{2n+1} which we will represent for short by HM_{2n} and HM_{2n+1}. In the case of HM_{2n} the polytope admits a centre of symmetry, as the reversion of the signs of all the coordinates of any vertex furnishes an other vertex of the same group; on the contrary in the case of HM_{2n+1} every vertex is

---

[1]) The result mentioned contains a numerical error; it ought to be replaced by

$$(\tfrac{1}{4}(7+\sqrt{2}),\ \tfrac{1}{4}(3+\sqrt{2}),\ \tfrac{1}{4}(\sqrt{2}-1),\ \tfrac{1}{4}(5+3\sqrt{2})),$$
$$(\tfrac{1}{4}(7+3\sqrt{2}),\ \tfrac{1}{4}(3-\sqrt{2}),\ \tfrac{1}{4}(1+\sqrt{2}),\ \tfrac{1}{4}(5+\sqrt{2})),$$

see „Wiskundige Opgaven", vol. XI, problem 96.

opposite to the simplex replacing the opposite vertex of the measure polytope. So in this respect HM_{2n} presents analogy to measure polytope and cross polytope, whilst HM_{2n+1} imitates the simplex.

We still remark that the case $n=2$ is exceptional in this sense that the corresponding HM_2 is a line, i. e. a diagonal of the square, instead of a form of two dimensions; as we shall see this remark is essential in the theory of the nets derived from the half measure polytopes.

91. It is easy to prove that the half measure polytopes partake of the two properties characterizing the semiregular polytopes considered in the preceding sections, i. e. that all the vertices are of the same kind and all the edges of the same length, here $2\sqrt{2}$. Indeed we already solved incidentally in art. 47 the more general question:

"Under what circumstances will the symbols

$$\pm \tfrac{1}{2}[a_1, a_2, \ldots, a_n]$$

represent the vertices of polytopes in S_n, all the edges of which have the same length, say $2\sqrt{2}$"?

For the length $2\sqrt{2}$ of the edges the solution takes the form of

THEOREM LVI. "The symbol $\pm\tfrac{1}{2}[a_1, a_2, \ldots, a_n]$ for which

$$a_1 \geqq a_2 \geqq \ldots\ldots \geqq a_n$$

represents a polytope admitting the required properties under the conditions: $a_{n-1}=a_n=1$ and the difference between any two unequal adjacent digits equal to 2".

So we find

in S_3 the two forms $\tfrac{1}{2}[111]$, $\tfrac{1}{2}[311]$,
„ S_4 „ four „ $\tfrac{1}{2}[1111]$, $\tfrac{1}{2}[3311]$, $\tfrac{1}{2}[3111]$, $\tfrac{1}{2}[5311]$,
„ S_5 „ eight „ $\tfrac{1}{2}[11111]$, $\tfrac{1}{2}[33311]$, $\tfrac{1}{2}[33111]$, $\tfrac{1}{2}[31111]$, $\tfrac{1}{2}[55311]$, $\tfrac{1}{2}[53311]$, $\tfrac{1}{2}[53111]$, $\tfrac{1}{2}[75311]$,

etc., which are represented in the following table by other symbols referring to T, C_{16} and HM; these symbols will be explained later on. [1])

$$n=3$$

$$\tfrac{1}{2}[1\,1\,1] = T = HM_3, \qquad \tfrac{1}{2}[3\,1\,1] = tT = e_2\,HM_3.$$

$$n=4$$

$$\left.\begin{array}{l} \tfrac{1}{2}[1111] = \quad C_{16} = \quad HM_4 \\ \tfrac{1}{2}[3311] = e_1\,C_{16} = e_2\,HM_4 \end{array}\right\}, \quad \left.\begin{array}{l} \tfrac{1}{2}[3111] = \quad ce_2\,C_{16} = \quad e_3\,HM_4 \\ \tfrac{1}{2}[5311] = ce_1\,e_2\,C_{16} = e_2\,e_3\,HM_4 \end{array}\right\}.$$

[1]) We remark here that the symbols e before HM_n are related to the limits of M_n.

$$n = 5$$

$$\left.\begin{array}{l} \frac{1}{2}[1\,1\,1\,1\,1] = HM_5 \\ \frac{1}{2}[3\,3\,3\,1\,1] = e_2\,HM_5 \\ \frac{1}{2}[3\,3\,1\,1\,1] = e_3\,HM_5 \\ \frac{1}{2}[3\,1\,1\,1\,1] = e_4\,HM_5 \end{array}\right\} , \quad \left.\begin{array}{l} \frac{1}{2}[5\,5\,3\,1\,1] = e_2\,e_3\,HM_5 \\ \frac{1}{2}[5\,3\,3\,1\,1] = e_2\,e_4\,HM_5 \\ \frac{1}{2}[5\,3\,1\,1\,1] = e_3\,e_4\,HM_5 \\ \frac{1}{2}[7\,5\,3\,1\,1] = e_2\,e_3\,e_4\,HM_5 \end{array}\right\} .$$

We introduce for these forms and for the corresponding ones in spaces of a higher number of dimensions the collective "half measure polytope descendent", which we abreviate to *hmpd.*

B. *The characteristic numbers.*

92. It is not difficult to determine the characteristic numbers of HM_n for a general n. For, if a_p and a'_p denote the numbers of limits $(l)_p$ of M_n and HM_n respectively, we have the relations

$$\left.\begin{array}{llll} a'_0 & = \frac{1}{2}a_0 & & \\ a'_1 & = & & a_2 \\ a'_2 & = & & 4\,a_3 \\ a'_3 & = a_3 & + \frac{1}{2}(n)_4 & a_0 \\ a'_4 & = a_4 & + \frac{1}{2}(n)_5 & a_0 \\ \dots & & & \\ a'_p & = a_p & + \frac{1}{2}(n)_{p+1} & a_0 \\ \dots & & & \\ a'_{n-1} & = a_{n-1} & + \frac{1}{2}(n)_n & a_0 \end{array}\right\} ,$$

where at the right the numbers are arranged in two columns of which the first contains old, the second new limits. Indeed the process transforming M_n into HM_n — which may be called an alternate truncation — destroys half the number of vertices, all the edges, all the faces, and maintains all the other limits $(l)_3$, $(l)_4, \dots, (l)_{n-1}$ of M_n but in an altered shape, bringing new sets of edges, faces, limiting bodies, etc. into existence. Now each face of M_n produces an edge of HM_n, each limiting body of M_n — becoming a T — produces four triangular faces of HM_n and finally in general any set of $p+1$ vertices of M_n adjacent to a vertex destroyed produces a regular simplex $S(p+1)$ forming a limit $(l)_p$ of HM_n, for $p = 4, 5, \dots, n-1$. This accounts for all the relations given above. Now, as the characteristic numbers of M_n are given by the equation

$$a_p = (n)_p\, 2^{n-p} \ , \ (p = 0, 1, 2, \dots, n-1),$$

we find for HM_n:

$$a'_0 = 2^{n-1}, \quad a'_1 = (n)_2\, 2^{n-2}, \quad a'_2 = (n)_3\, 2^{n-1},$$
$$a'_p = (n)_p\, 2^{n-p} + (n)_{p+1}\, 2^{n-1}, \quad p = 3, 4, \ldots, n-1.$$

Neither is it difficult to prove that the characteristic numbers a'_p satisfy the law of Euler. To that end we go back to the relations given above and transform the Eulerian expression $a'_0 - a'_1 + a'_2 - \ldots + (-1)^{n-1} a'_{n-1}$ into

$$[\tfrac{1}{2} a_0 - a_3 + a_4 - \ldots + (-1)^{n-1} a_n]$$
$$- [a_2 - 4a_3 + \tfrac{1}{2} a_0 \{(n)_4 - (n)_5 + \ldots + (-1)^n (n)_n\}],$$

of which two sums between square brackets the first contains the contributions of the first column (old elements) and the second of the second (new elements). Now we add to each of these two sums between square brackets $\frac{1}{2} a_0 - a_1 + a_2$. So we get

$$[a_0 - a_1 + a_2 - \ldots + (-1)^{n-1} a_n]$$
$$- [\tfrac{1}{2} a_0 - a_1 + 2a_2 - 4a_3 + \tfrac{1}{2} a_0 \{(n)_4 - (n)_5 + \ldots + (-1)^n (n)_n\}].$$

But as we have

$$- a_1 + 2a_2 - 4a_3 = \tfrac{1}{2} a_0 \{- (n)_1 + (n)_2 - (n)_3\}$$

the second sum disappears, as it is equal to

$$\tfrac{1}{2} a_0 \{1 - (n)_1 + (n)_2 - (n)_3 + \ldots + (-1)^n (n)_n\} = \tfrac{1}{2} a_0 (1 - 1)^n.$$

So we find that the Eulerian expression of HM_n is equal to that of M_n and has therefore the value 2 for n odd and the value 0 for n even, etc.

We give here the results up to $n = 8$. They are

$n = 5$ ... (16, 80, 160, 120, 26),
$n = 6$ ... (32, 240, 640, 640, 252, 44),
$n = 7$ ... (64, 672, 2240, 2800, 1624, 532, 78),
$n = 8$ ... (128, 1792, 7168, 10752, 8288, 4032, 1136, 144).

In the outset we remarked that HM_5 admits two kinds of limits $(l)_4$, viz. cells C_{16} and simplexes $S(5)$. Here we remember that in general for $n > 4$ the HM_n is bounded by two kinds of limits $(l)_{n-1}$, viz. limits HM_{n-1} forming what remains of the limits M_{n-1} of M_n and limits $S(n)$ replacing the vanished vertices of M_n. It will be useful to call the HM_{n-1} the "original", the $S(n)$ the "truncation" limits.

93. In the cases of the offspring of simplex, measure polytope, and cross polytope we have used two different methods for the determination of the characteristic numbers, one fulfilling the exigencies for $n < 6$ as far as these numbers only are concerned, an

other giving for $n > 5$ not only the characteristic numbers but also the numbers of any limit of any kind; here we will do likewise.

So in the case of the polytopes connected with HM_5 in the manner indicated in theorem LVI we have to determine:

1°. the number of vertices according to general principles,

2°. the number of edges concurring in any vertex and thereby the total number of edges,

3°. the limiting polytopes $(l)_4$, which limits reveal at the same time the limiting bodies $(l)_3$,

4°. the number of faces (by means of Euler's rule).

But before applying this method to a definite example we give some further explanation with respect to the equations of the four-dimensional spaces containing the limits $(l)_{n-1}$ of the *hmpd.* deduced from HM_n in S_n, as this will save us trouble in the exposition of the direct method.

If $\frac{1}{2}[a_1 a_2 \ldots a_n]$ is the symbol of coordinates, where the digits have been arranged in diminishing order, we consider the vertices represented by

$$\begin{array}{cc} (a_1\ a_2\ \ldots\ a_p) & \frac{1}{2}[a_{p+1}\ a_{p+2}\ \ldots\ a_n] \\ x_1, x_2, \ldots, x_p & x_{p+1}, x_{p+2}, \ldots, x_n \end{array}$$

lying in the space S_{n-1} represented by the equation

$$x_1 + x_2 + \ldots + x_p = a_1 + a_2 + \ldots + a_p.$$

Evidently these vertices will determine a limit $(l)_{n-1}$ of the polytope, if $(a_1 a_2 \ldots a_p)$ and $\frac{1}{2}[a_{p+1} a_{p+2} \ldots a_n]$ represent polytopes $(P)_{p-1}$ and $(P)_{n-p}$ respectively, this $(l)_{n-1}$ being then a prismotope which may be denoted by $(P_{p-1}; P_{n-p})$. Now $(a_1 a_2 \ldots a_p)$ always represents a $(P)_{p-1}$, unless all the digits $a_1 a_1, \ldots, a_p$ are equal, in which case $(a_1 a_2 \ldots a_p)$ is a petrified syllable. On the other hand $\frac{1}{2}[a_{p+1} a_{p+2} \ldots a_n]$ always represents a $(P)_{n-p}$, unless we have either $p = n - 2$, or $p = n - 1$; for, as we remarked already $p = n - 2$ gives the syllable $\frac{1}{2}[11]$, i. e. a line segment instead of a square, and $p = n - 1$ gives a vertex only instead of an edge.

To this we have to add a few words only about the extreme cases $p = 1$ and $p = n$. For $p = 1$ we find the polytope with the coordinate symbol $\frac{1}{2}[a_2 a_3 \ldots a_n]$ lying in a space S_{n-1} represented by $\pm x_i = a_1$; it can be deduced from HM_{n-1}. For $p = n$ the result is different for n even and n odd, the polytope having as HM_n itself a centre of symmetry in the first case and two different limits, either a vertex and an $(l)_{n-1}$ or two differently shaped $(l)_{n-1}$, opposite to each other in the second. Or otherwise, as follows. For n even the diagonals of M_n split up into two groups of non adjacent ones, of those bearing vertices belonging also to HM_n and of those bearing vertices cut off by the alternate truncation leading from M_n to HM_n; the 2^{n-2} diagonals of the first group are normal to two limits of vertex import [1]) in the considered polytope, whilst the 2^{n-2} diagonals of the second are normal to two limits which may be called of truncation import as they are derived from truncation limits of HM_n in passing to the polytope under consideration. For n odd there is only one group of diagonals of M_n, each of which bears only one vertex of HM_n; so each of these diagonals is normal to two differently shaped limits of the polytope, to one limit of vertex and to one limit of truncation import. But in the two cases, of n even and n odd, we have to deal with the two equations $\Sigma \pm x_i = \Sigma a_i$ and $\Sigma \pm x_i = \Sigma a_i - 2$, the last digit $a_n = 1$ having to be taken with the positive sign for limits of vertex import, with the negative sign for limits of truncation import.

After this introduction we treat a definite example.

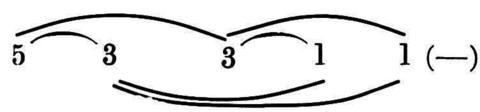
94. *Case* $\frac{1}{2}$ **[5 3 3 1 1].**

The number of vertices is 2^4 times 5! divided by 2^2, i. e. 480.

The vertices adjacent to the pattern vertex 5 3 3 1 1 are

	5 1 3 3 1	
3 3 5 1 1	5 3 1 3 1	5 3 3 —1 —1
3 5 3 1 1	5 1 3 1 3	
	5 3 1 1 3	

which may be indicated by the brackets and the negative sign after the two units in the symbol



So seven edges concur in any vertex, i. e. the total number of edges is half the product of 480 and 7, i. e. 1680.

Now we have to pass to the limiting polytopes.

The spaces S_4 represented by $\pm x_i = 5$ give $2.(5)_1 = 10$ limits $\frac{1}{2}[3\ 3\ 1\ 1]$ of polytope import.

[1]) Also the import of the different limits $(l)_{n-1}$ of HM_n will be considered in relation with the limits $(l)_{n-1}$ of M_n. So the equations $\pm x_i = a_1$ will give limits of $(l)_{n-1}$ import, the equations $\pm x_i \pm x_j = a_1 + a_2$ will give limits of $(l)_{n-2}$ import, etc., this series ending in general in limits of body and limits of vertex import, as no edge or face of M_n partakes in the limitation of HM_n.

The spaces S_4 represented by $\pm x_i \pm x_j = 5 + 3$ give $2^2.(5)_2 =$ 40 limits (53) $\frac{1}{2}[3\,1\,1]$ of body import.

The spaces S_4 represented by $\Sigma \pm x_i = 13$ give 2^4 limits (5 3 3 1 1) of vertex import, where (5 3 3 1 1) = (4 2 2 0 0). [1]

The spaces S_4 represented by $\Sigma \pm x_i = 11$ give 2^4 limits (5 3 3 1 — 1) introduced by the alternate truncation.

So the limiting polytopes are

$$10\, e_1\, C_{16} + 40\, P_{tT} + 16\, e_2\, S(5) + 16\, e_1 e_3\, S(5),$$

i. e. 82 in toto.

Now from the list of limiting bodies

$$\begin{array}{rl} e_1\, C_{16} & \ldots\ 8\ O\ ,\ 16\ tT \\ P_{tT} & \ldots\ 2\ tT\ ,\ \ 4\ P_3,\ \ 4\ P_6 \\ e_2\, S(5) & \ldots\ 5\ CO,\ 10\ P_3,\ \ 5\ O \\ e_1\, e_3\, S(5) & \ldots\ 5\ tT\ ,\ 10\ P_6,\ 10\ P_3,\ 5\ CO \end{array}$$

of the four different limiting polytopes we can deduce that our polytope is limited by

$$\tfrac{1}{2}(10 \times 8 + 16 \times 5)\, O,\ \tfrac{1}{2}(10 \times 16 + 40 \times 2 + 16 \times 5)\, tT,$$
$$\tfrac{1}{2}(16 \times 5 + 16 \times 5)\, CO,\ \tfrac{1}{2}(40 \times 4 + 16 \times 10 + 16 \times 10)\, P_3,$$
$$\tfrac{1}{2}(40 \times 4 + 16 \times 10)\, P_6$$

i. e. by 720 polyhedra, viz.

$$80\ O,\ 160\ tT,\ 80\ CO,\ 240\ P_3,\ 160\ P_6.$$

Now finally, according to Euler's rule, the number of faces is 1840. So the result is

$$(480,\ 1680,\ 1840,\ 720,\ 82).$$

This example shows that the method explained *is* sufficient for S_5, as far as the characteristic numbers themselves are concerned. But if we want to extend our knowledge of these *lmpd.* — in relation with the difficulty of realising their lopsided form — by determining the numbers of the *different kinds* of limits the method is insufficient even in S_5 and has to be completed, in one sense or other, with respect to the different kinds of edges and of faces. We shall see that the direct method, which will be explained in the next article, furnishes this complement at least expense.

95. Here once more the direct method in view is based on the

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[1]) This (42200) with edges $2\sqrt{2}$ is similar to (21100) with edges $\sqrt{2}$, i.e. to $e_2\, S(5)$. Likewise (5331—1) leads by (64420) to (32210) or (32110), i. e. to $e_1\, e_3\ S(5)$.

distinction of the different kinds of limits $(l)_p$ by what we have called formerly "unextended" symbols. If we take care to exclude always the petrified syllables we can formulate the method in:

THEOREM LVII. "We obtain the unextended symbol of a polytope $(P)_d$ the vertices of which are vertices of the given *hmpd.* of S_n by applying to the n digits of the symbol of coordinates $\frac{1}{2}[a_1\, a_2 \ldots a_{n-1}\, a_n]$ of this polytope one of the three following processes:

1°. Take the last digit a_n, first with the *positive* and afterwards with the *negative* sign, and place for both cases between pairs of round brackets either one group of $d+1$ digits, or two groups containing together $d+2$ digits, or three groups containing together $d+3$ digits, etc., omitting the digits not included.

2°. Place before $\frac{1}{2}[11]$ of the remaining digits $a_1, a_2, \ldots, a_{n-2}$ between pairs of round brackets either one group of d digits, or two groups containing together $d+1$ digits, etc., omitting the digits not included — and the syllable with one digit for $d=1$.

3°. Place before $\frac{1}{2}[a_{n-k+1}\, a_{n-k+2} \ldots a_{n-1}\, a_n]$, where $k=3, 4, \ldots, d$ successively, between pairs of round brackets either one group of $d-k+1$ of the remaining $n-k$ digits, or two groups containing together $d-k+2$ of these digits, etc., omitting the digits not included — and the syllable with one digit for $d=k$."

"In each of these cases the $(P)_d$ obtained will be a *limiting* polytope of *hmpd.*, if the syllables between round brackets satisfy the two following conditions:

a) each syllable with middle digits exhausts these digits of the symbol of the given *hmpd.*,

b) no two syllables without middle digits have the same end digits."

The proof of this theorem, forming an adaption of theorem XXX to the special character of the *hmpd.*, embodied in the $\frac{1}{2}$ before the square brackets of their symbol, can be copied from that of theorem XXX and theorem XXX'.

We apply it to two definite examples, one in $S(5)$, the other in S_6.

Case $\frac{1}{2}$ **[5 5 3 1 1]**. —

If we place before a vertical stroke the limits deduced from 55311 and after it the *different* ones furnished by 5531 — 1, we get

$(l)_1 \ldots (53)_2, (31)_2, \frac{1}{2}[11]_1$	
$(l)_2 \ldots (553)_1, (531)_4, (311)_1, (53)\frac{1}{2}[11]_2$	$(31-1)_2$
$(l)_3 \ldots (5531)_2, (5311)_2, (553)\frac{1}{2}[11]_1, \frac{1}{2}[311]_1$	$(531-1)_4$
$(l)_4 \ldots (55311)_1, \frac{1}{2}[5311]_2$	$(5531-1)_2$

where the small subscripts at the right indicate the number of limits concurring in any vertex [1]). So we find through any vertex

five edges,

two p_3, two p_4, six p_6,

one P_3, five tT, four tO,

one $(55311) = ce_1\, e_2\, S(5)$, two $(5531-1) = e_1\, e_2\, S(5)$,

two $\frac{1}{2}[5311] = ce_1\, e_2\, C_{16}$,

and this gives in a transparent way in toto

$$\frac{5.480}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ i. e. } 1200\ (l)_1,$$

$$\frac{2.480}{3} = 320\, p_3 \quad , \frac{2.480}{4} = 240\, p_4 \quad , \frac{6.480}{6} = 480\, p_6 \ldots. \quad ,, \quad 1040\ (l)_2,$$

$$\frac{480}{6} = 80\, P_3 \quad , \frac{5.480}{12} = 200\, tT \quad , \frac{4.480}{24} = 80\, tO \ldots. \quad ,, \quad 360\ (l)_3,$$

$$\frac{480}{30} = 16\, ce_1\, e_2\, S(5), \frac{2.480}{60} = 16\, e_1\, e_2\, S(5), \frac{2.480}{96} = 10\, ce_1\, e_2\, C_{16} \quad ,, \quad 42\ (l)_4.$$

So the result is

(480, 1200, 1040, 360, 42)

in accordance with the law of Euler.

Case $\frac{1}{2}$ **[755311]**. —

In the same way we find here the table:

$(l)_1$	$(75)_2, (53)_2, (31)_2, \frac{1}{2}[11]_1$	
$(l)_2$	$(755)_1, (75)(53)_2, (75)(31)_4, (553)_1, (531)_4, (311)_1, (75)\frac{1}{2}[11]_2, (53)\frac{1}{2}[11]_2$	$(31-1)_2$
$(l)_3$	$(7553)_1, (755)(31)_2, (75)(531)_4, (75)(311)_2, (5531)_2, (5311)_2, (755)\frac{1}{2}[11]_1, (75)(53)\frac{1}{2}[11]_2, (553)\frac{1}{2}[11]_1, \frac{1}{2}[311]_1$	$(75)(31-1)_4, (531-1)_4$
$(l)_4$	$(75531)_2, (755)(311)_1, (75)(5311)_2, (55311)_1, (7553)\frac{1}{2}[11]_1, (75)\frac{1}{2}[311]_2, \frac{1}{2}[5311]_2$	$(755)(31-1)_2, (75)(531-1)_4, (5531-1)_2$
$(l)_5$	$(755311)_1, (755)\frac{1}{2}[311]_1, (75)\frac{1}{2}[5311]_2, \frac{1}{2}[55311]_1$	$(75531-1)_2$

So we find through any vertex

seven edges,

three p_3, ten p_4, six p_6,

one CO, five tT, six P_3, eight P_6, two C, four tO,

two (32110), one (22100), two (32100), four P_{tT}, four P_{tO},

one P_{CO}, one (3 ; 3), two (3 ; 6), two $\frac{1}{2}[5311]$,

one (322100), one $(p_3 ; tT)$, two $P_{\frac{1}{2}[5311]}$, one $\frac{1}{2}[55311]$,

two (432110);

[1]) So (531) is to bear the subscript 4, as the 5 may be related either to x_2 or to x_3 and the 1 either to x_4 or to x_5; so (31—1) is to admit the subscript 2, as the three digits may apply either to $+x_3, +x_4, -x_5$ or to $+x_3, -x_4, +x_5$, etc.

as the number of vertices is $2^5 . 6!$ divided by 2^2, i. e. 5760, we get in toto

$\frac{7.5760}{2}$ .... i. e. $20160 (l)_1$,

$\frac{3.5760}{3} = 5760 p_3, \frac{10.5760}{4} = 14400 p_4, \frac{6.5760}{6} = 5760 p_6$ „ $25920 (l)_2$,

$\frac{5760}{12} = 480 CO, \frac{5.5760}{12} = 2400 tT, \frac{6.5760}{6} = 5760 P_3,$

$\frac{8.5760}{12} = 3840 P_6, \frac{2.5760}{8} = 1440 C,$

$\frac{4.5760}{24} = 960 tO$ .... „ $14880 (l)_3$,

$\frac{2.5760}{60} = 192 e_1 e_3 S(5), \frac{5760}{30} = 192\, ce_1 e_2 S(5),$

$\frac{2.5760}{60} = 192 e_1 e_2 S(5), \frac{4.5760}{24} = 960 P_{tT},$

$\frac{4.5760}{48} = 480 P_{tO}, \frac{5760}{24} = 240 P_{CO},$

$\frac{5760}{9} = 640 (3;3), \frac{2.5760}{18} = 640 (3;6),$

$\frac{2.5760}{96} = 120\, ce_1 e_2 C_{16}$ .... „ $3656 (l)_4$,

$\frac{5760}{180} = 32 e_2 e_3 S(6), \frac{5760}{36} = 160 (p_3; tT),$

$\frac{2.5760}{192} = 60 P_{ce_1 e_2 C_{16}}, \frac{5760}{480} = 12 e_2 e_3 HM_5,$

$\frac{2.5760}{360} = 32 e_1 e_2 e_4 S(6)$ .... „ $296 (l)_5$.

So the result is, in accordance with the law of Euler,

(5760, 20160, 25920, 14880, 3656, 296).

The results obtained in this way are tabulated in Tables VIII and IX.

Table VIII, concerned with the *hmpd.* in S_3, S_4, S_5, has been divided vertically into six main parts, giving respectively the expansion symbol, the symbol of coordinates, the symbol of characteristic numbers, the faces, the limiting polyhedra, the limiting polytopes. The part of the faces is split up into three columns successively related to triangular, square, hexagonal faces; likewise that of the limiting bodies is split up into seven columns corresponding to the seven possibilities T, O, P_3, tT, CO, P_6, tO. Of the two numbers given in any case the first always indicates the total number of the limits, the second that of the limits concurring in any vertex. But in the sixth part, making its appearance for $n = 5$, the arrangement is an other one, the character of the limiting polytopes and their total number having interchanged places; so in any case the total number appears at the head of the column and the character at the first of the two horizontal places in the column. So the polytope $e_4 HM_5 = \frac{1}{2}[31111]$ with the characteristic numbers 80, 400, 720, 480, 82 is limited by $480 p_3$ and $240 p_4$ of which 18 and 12 respectively meet in any vertex, by $240T$ and $240P_3$ of which 12 and 18 respectively meet in any vertex, and by $10C_{16}$, $40P_T$, $16S(5)$ and $16e_3 S(5)$ of which 1, 4, 1 and 4 respectively meet in any vertex.

Table IX, concerned with the *hmpd.* in S_6, has been divided in the same way into seven main parts. It will be clear without farther explanation; only we are bound to add that in the first column of the sixth part 2 „ means that $2C_{16}$ is to be taken 60 times and that in this part and the next the numbers of limits concurring at any vertex have been omitted.

96. We insert a few remarks about the character of the limits.

Faces. We find only p_3, p_4, p_6.

Limiting bodies. The set of limiting bodies obtained for $n = 5$ is completed by the addition of C for $n = 6$.

Limiting polytopes. In general the limiting polytopes are

1°. Simplex forms, deduced from $S(n)$, $S(n-1)$, ..., $S(3)$,

2°. Half measure polytope forms, deduced from HM_{n-1}, HM_{n-2}, ..., HM_5,

3°. Prismotopes the constituents of which are simplex forms, deduced from $S(n-1)$, ..., $S(3)$, and at most one half measure polytope form, deduced from HM_{n-2}, ..., HM_5.

This general result shows that the list of limiting bodies is complete for $n = 6$. Moreover that the list of fourdimensional limits will be complete for $n = 8$, as the case $n = 8$ brings C_8 for the first time, etc.

In order to show how theorem LVII works we give the list of the limits $(P)_6$ of the tendimensional form $\frac{1}{2}$ **[9775533311]**:

(9775533), — (9775) (5333), (9775) (533) (31), — (977) (55333), (977)(5533)(31),(977)(553)(331),—(97)(755333),(97)(75533)(31), (97)(7553)(331),(97)(755)(3331),(97)(75)(53331),(97)(75)(533)(311), (97)(75)(533)(31—1),(97)(75)(53)(3311),(97)(75)(53)(331—1),— (7755333), — (775533) (31), — (77553) (331), — (7755) (3331), — (775)(53331),(775)(533)(311),(775)(533)(31—1),(775)(53)(3311), (775)(53)(331—1),—(7553331),—(75533)(311),(75533)(31—1),—(7533) (3311), (7553) (331—1), (755) (33311), (755) (3331—1), (75)(533311), (75)(53331—1),—(5533311),—(553331—1),——(977553) $\frac{1}{2}$[11], — (9775) (533) $\frac{1}{2}$[11], — (977) (5533) $\frac{1}{2}$[11], — (97) (75533) $\frac{1}{2}$[11], (97) (75) (5333) $\frac{1}{2}$[11], — (775533) $\frac{1}{2}$[11], — (775) (5333) $\frac{1}{2}$[11], — (755333) $\frac{1}{2}$[11], —— (9775) $\frac{1}{2}$[311], — (97) (755) $\frac{1}{2}$[311], (97) (75) (53) $\frac{1}{2}$[311], — (7755) $\frac{1}{2}$[311], — (775) (53) $\frac{1}{2}$[311], — (7553) $\frac{1}{2}$[311], — (75) (533) $\frac{1}{2}$[311], —— (977)$\frac{1}{2}$[3311],—(97)(75)$\frac{1}{2}$[3311],—(775)$\frac{1}{2}$[3311],—(755)$\frac{1}{2}$[3311],— (75)(53)$\frac{1}{2}$[3311],—(553)$\frac{1}{2}$[3311],——(97)$\frac{1}{2}$[33311],—(75)$\frac{1}{2}$[33311], —— $\frac{1}{2}$[533311].

C. *Extension number and truncation fractions.*

97. THEOREM LVIII. "The polytopes $\frac{1}{2}[a_1 a_2 \ldots a_{n-1} a_n]$ of S_n, all with edges $2\sqrt{2}$, can be found by means of a regular extension of the measure polytope $M_n^{(2)}$ followed by a regular truncation at the two groups of non adjacent vertices of M_n, either with or without truncation at the limiting $(l)_3$, or at the limiting $(l)_3$ and $(l)_4$, or at the limiting $(l)_3$, $(l)_4$ and $(l)_5$, etc. or at the limiting $(l)_3$, $(l)_4$, $(l)_5$, etc. and $(l)_{n-2}$."

This theorem is an immediate consequence of the character of the equations of the spaces S_{n-1} bearing the limits $(l)_{n-1}$ of the *hmpd.*

The extension number is once more the largest digit of the symbol of coordinates, i. e. a_1; so here it is always *odd.*

On account of the lopsidedness of the *hmpd.* we measure the amount of truncation on the corresponding *half* diameter limited at the centre O of the polytope. So in the case $\frac{1}{2}[775533311]$ the truncation corresponding to the space S_8 with the equation $\sum_1^5 x_i = 27$, i. e. the truncation at the limits M_4 of M_9 extended, is $\frac{PQ}{PO}$, if P is the centre of the M_4 and Q the point of intersection of OP and the indicated space S_8. As $\sum_1^5 x_i$ is 35 for M_9 extended we find $\frac{PQ}{PO} = \frac{PO - QO}{PO} = \frac{35-27}{35} = \frac{8}{35}$. So the truncation fraction is $\frac{8}{35}$ in this case.

This case shows clearly that *in general* the fraction number admits as denominator the product of the extension number by the number of coordinates figuring in the equation of the truncating space. So reducing this denominator to the extension number the numerator itself becomes in general a fraction. Therefore it is impossible to introduce here the notion of truncation integer.

The following list contains the truncation fractions for the *hmpd.* in S_3, S_4, S_5, S_6; here τ_0, τ_0', τ_3, τ_4 represent successively the two truncations at the vertices and the truncations at the limits $(l)_3$, $(l)_4$.

		τ_0	τ_0'	τ_3
S_3	$\frac{1}{2}[311]$	$\frac{4}{9}$	$\frac{2}{3}$	
S_4	$\frac{1}{2}[3311]$	$\frac{1}{3}$	$\frac{1}{2}$	
	$\frac{1}{2}[3111]$	$\frac{1}{2}$	$\frac{2}{3}$	
	$\frac{1}{2}[5311]$	$\frac{1}{2}$	$\frac{3}{5}$	
S_5	$\frac{1}{2}[33311]$	$\frac{4}{15}$	$\frac{2}{5}$	
	$\frac{1}{2}[33111]$	$\frac{2}{5}$	$\frac{8}{15}$	
	$\frac{1}{2}[31111]$	$\frac{8}{15}$	$\frac{2}{3}$	$\frac{1}{3}$
	$\frac{1}{2}[55311]$	$\frac{2}{5}$	$\frac{12}{25}$	
	$\frac{1}{2}[53311]$	$\frac{12}{25}$	$\frac{14}{25}$	$\frac{1}{5}$
	$\frac{1}{2}[53111]$	$\frac{14}{25}$	$\frac{16}{25}$	$\frac{1}{5}$
	$\frac{1}{2}[75311]$	$\frac{18}{35}$	$\frac{4}{7}$	$\frac{1}{7}$

		τ_0	τ_0'	τ_3	τ_4
S_6	$\frac{1}{2}[333311]$	$\frac{2}{9}$	$\frac{1}{3}$		
	$\frac{1}{2}[333111]$	$\frac{1}{3}$	$\frac{4}{9}$		
	$\frac{1}{2}[331111]$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{9}$	
	$\frac{1}{2}[311111]$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{1}{3}$
	$\frac{1}{2}[555311]$	$\frac{1}{3}$	$\frac{2}{5}$		
	$\frac{1}{2}[553311]$	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{2}{15}$	
	$\frac{1}{2}[533311]$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{4}{15}$	$\frac{1}{5}$
	$\frac{1}{2}[553111]$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{2}{15}$	
	$\frac{1}{2}[533111]$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{4}{15}$	$\frac{1}{5}$
	$\frac{1}{2}[531111]$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{1}{5}$
	$\frac{1}{2}[775311]$	$\frac{3}{7}$	$\frac{10}{21}$	$\frac{2}{21}$	
	$\frac{1}{2}[755311]$	$\frac{10}{21}$	$\frac{11}{21}$	$\frac{4}{21}$	$\frac{1}{7}$
	$\frac{1}{2}[753311]$	$\frac{11}{21}$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
	$\frac{1}{2}[753111]$	$\frac{4}{7}$	$\frac{13}{21}$	$\frac{2}{7}$	$\frac{1}{7}$
	$\frac{1}{2}[975311]$	$\frac{14}{27}$	$\frac{5}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

D. *Expansion and contraction symbols.*

98. For $k = 2, 3, \ldots, n-2, n-1$ any limit $M_k^{(2)}$ of the $M_n^{(2)}$ from which the $HM_n(2\sqrt{2})$ has been deduced bears a limit of $HM_n(2\sqrt{2})$, this limit $\frac{1}{2}[\overbrace{11\ldots1}^{k}]$ being a $HM_k(2\sqrt{2})$ and therefore an $(l)_k$ for $k = 3, 4, \ldots, n-1$, but an edge for $k = 2$. Now we will define the expansion e_k of $HM_n(2\sqrt{2})$ — for $k = 2, 3, \ldots, n-2$ — as the influence of the motion of the limits $\frac{1}{2}[\overbrace{11\ldots1}^{k}]$ contained in the limits $M_k^{(2)}$ of $M_n^{(2)}$ caused by a translational motion of these limits $M_k^{(2)}$ and what they contain, to equal distances away from the centre O of $M_n^{(2)}$, each $M_k^{(2)}$ moving in the direction of the line OM joining O to its centre M, these $M_k^{(2)}$ remaining equipollent to their original position, the motion being extended over such a distance that the two new positions of any vertex which was common to two $M_k^{(2)}$ shall be separated by the length $2\sqrt{2}$. In order to justify this definition we have to show for what reasons we deviate here from the custom followed until now: to bring the operation e_k in relation with the limits $(l)_k$ of the polytope itself.

For the deviation indicated we have two reasons. The first is of a didactic cast: it is easier to imagine the motions of the limits of $M_n^{(2)}$ than those of $HM_n^{(2\sqrt{2})}$. But the second is of more importance: "if the limits of $HM_n^{(2\sqrt{2})}$ are carried away by the limits of the circumscribed $M_n^{(2)}$ which contain them, these latter limits being moved out in the ordinary way, we get precisely those expansion operations which lead to the whole set of polytopes *hmpd.* of S_n." This advantage is twofold. In the first place: the only expansion of $HM_n^{(2\sqrt{2})}$ which has no equivalent under the e_k applied to $M_n^{(2)}$, i. e. the expansion according to the *faces*, is excluded, and this is right, for we will show afterwards that this expansion is either impossible or it leads to a polytope which can be derived from $M_n^{(4)}$. But, what is still more, by adhering to the limits of $M_n^{(2)}$ we are never at a loss with respect to the question to which group of limits of $HM_n^{(2\sqrt{2})}$ the expansion is to be applied. So in the case of S_5 the HM_5 admits as limiting bodies tetrahedra only, but they are of two different kinds, i. e. we must distinguish between a T common to two C_{16} and a T common to a cell C_{16} and a cell Cr_5; so, of these two groups the first must undergo the operation e_3, if we wish to apply it, as a T common to two C_{16} is contained in the cube common to the two adjacent eightcells bearing the two C_{16}. Moreover we will prove afterwards that the contraction operation always leads to forms deducible from $M_n^{(4)}$; so we have to consider here the operations e_k only.

On the other hand we do not deny that the new definition has a drawback with respect to the operation of expansion according to the edges of $HM_n^{(2\sqrt{2})}$, a difference in the notation making its appearance there. According to HM_n itself this operation ought to be called $e_1 HM_n$; nevertheless we propose to indicate it by the symbol $e_2 HM_n$. This is still more annoying in S_3 and S_4 where we have $e_2 HM_3 = e_1 T = tT$ and $e_2 HM_4 = e_1 C_{16}$ (see the small table at the end of art. 91). But still we reckon the advantages so prevailing that we do not mind of accepting this small disadvantage into the bargain, the more so as it is easily held under control.

Starting from the new definition we prove:

THEOREM LIX. "The expansion e_k, $(k = 2, 3, \ldots, n-2)$, applied to $HM_n^{(2\sqrt{2})}$ changes the symbol of coordinates $\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ of that polytope into $\frac{1}{2}[\overline{33\ldots3}^{\,n-k}\,\overline{11\ldots1}^{\,k}]$."

Proof. If we move the limit $HM_k^{(2\sqrt{2})}$ represented by

$$x_1 = x_2 = \ldots = x_{n-k} = 1,\ x_{n-k+1}, x_{n-k+2}, \ldots, x_n = \tfrac{1}{2}[\overline{11\ldots1}^{\,k}]$$

in the direction of the line joining O to its centre M, for which

$$x_1 = x_2 = \ldots = x_{n-k} = 1,\ x_{n-k+1} = x_{n-k+2} = \ldots = x_n = 0,$$

to a λ times larger distance from O we get a new position of this $HM_k(2\sqrt{2})$ characterized by

$$x_1 = x_2 = \ldots = x_{n-k} = \lambda,\ x_{n-k+1}, x_{n-k+2}, \ldots, x_n = \tfrac{1}{2}[\overbrace{11\ldots1}^{k}],$$

in which it is a limit $HM_k(2\sqrt{2})$ of the new polytope $\frac{1}{2}[\overbrace{\lambda\lambda\ldots\lambda}^{n-k}\overbrace{11\ldots1}^{k}]$. According to theorem LVI this new polytope has edges of the same length if and only if we put $\lambda = 3$. This proves the theorem and leads moreover to the result:

THEOREM LX. "In the expansion e_k the limits $HM_k(2\sqrt{2})$ of $HM_n(2\sqrt{2})$ are moved away from the centre to a distance always three times the original distance."

This result is also an immediate consequence of the fact that the largest digit 3 of the symbol of the new polytope is the extension number.

Remark. We may express the influence of the operation e_k on the symbol $\frac{1}{2}[\overbrace{11\ldots1}^{n}]$ by saying that it creates an interval 2 between the $n - k^{th}$ and the $n - k + 1^{st}$ digit. This is in accordance with the remark inserted at the end of art. 58. In moving out the limits M_k of M_n the distance to be described in order to give the new edges a length $2\sqrt{2}$ is $\sqrt{2}$ times the distance to be described in order to give these edges a length 2; so the interval created which was $\sqrt{2}$ in the case of $M_n^{(2)}$ must be $\sqrt{2}$ times $\sqrt{2}$, i. e. 2 in the case of $HM_n(2\sqrt{2})$.

THEOREM LXI. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $HM_n(2\sqrt{2})$ on its symbol $\frac{1}{2}[\overbrace{11\ldots1}^{n}]$ is found by adding together the influences of each of the expansions taken separately.

The proof of this theorem can be copied from art. 59. It leads immediately to:

THEOREM LXII. "The operation e_k can still be applied to any expansion form deduced from $HM(2\sqrt{2})$ in the symbol of coordinates of which the $n - k^{th}$ and the $n - k + 1^{st}$ digits, i. e. the k^{th} and the $k + 1^{st}$ digits counted from the end, are equal."

So in the case $\frac{1}{2}[9775533311]$ we have an $e_2\, e_5\, e_7\, e_9\, HM^2_{10}$.

99. We have to come back to the face expansion of the *hmpd.* and to their contraction.

The faces of the polytope $\frac{1}{2}[a_1\, a_2 \ldots a_{n-2}\, 1\ 1]$ replacing the faces

(11—1) of $HM_n^{(2\sqrt{2})}$ are represented by (31—1) for $a_{n-2}=3$ and by (11—1) for $a_{n-2}=1$. So we treat these two cases together by considering the face.

$$x_i = a_i\ (i=1, 2, \ldots, n-3)\quad , \qquad x_{n-2}, x_{n-1}, x_n = (a_{n-2}\ 1{-}1)$$

with the centre

$$x_i = a_i\ (i=1, 2, \ldots, n-3)\quad , \quad 3x_{n-2} = 3x_{n-1} = 3x_n = a_{n-2}.$$

By moving this face away from the centre O to a distance λ times as large its centre is transported to the point

$$x_i = \lambda a_i\ (i=1, 2, \ldots, n-3)\ , \ 3x_{n-2} = 3x_{n-1} = 3x_n = \lambda a_{n-2}.$$

So the new position of the face leads to a new polytope $\frac{1}{2}[\lambda a_1, \lambda a_2, \ldots \lambda a_{n-3}, \ldots]$. As the length $2\sqrt{2}$ of the sides of this face is maintained and the length of the edge $(\lambda a_k, \lambda a_{k+1})$ is $2\lambda\sqrt{2}$ if a_k and a_{k+1} are unequal, we only can arrive for $\lambda \neq 1$ at a polytope all the edges of which have the same length $2\sqrt{2}$ if all the digits $a_1, a_2, \ldots, a_{n-3}$ are equal, i. e. in the four cases

$$\tfrac{1}{2}[\overset{n}{\overline{11\ldots1}}],\ \tfrac{1}{2}[\overset{n-3}{\overline{33\ldots3}}111],\ \tfrac{1}{2}[\overset{n-2}{\overline{33\ldots3}}11],\ \tfrac{1}{2}[\overset{n-3}{\overline{55\ldots5}}311].$$

In these cases the face becomes

$$\left.\begin{array}{llll} x_i = \lambda, (i=1, 2, \ldots, n-3), & x_{n-2}, x_{n-1}, x_n & = & \left(\frac{\lambda+2}{3}, \frac{\lambda+2}{3}, \frac{\lambda-4}{3}\right) \\ x_i = 3\lambda, \quad ,, & ,, & = & ,, \\ x_i = 3\lambda, \quad ,, & ,, & = & (\lambda+2, \ \lambda\ , \lambda-2) \\ x_i = 5\lambda, \quad ,, & ,, & = & ,, \end{array}\right\},$$

furnishing for the edge (a_{n-3}, a_{n-2}) of the new polytope the four symbols

$$\left(\lambda, \frac{\lambda+2}{3}\right),\ \left(3\lambda, \frac{\lambda+2}{3}\right),\ (3\lambda, \lambda+2),\ (5\lambda, \lambda+2).$$

So, if ε represents either 2 or 0 we have in these four cases

$$\lambda = \tfrac{3}{2}\varepsilon + 1,\ 3\lambda = 3\varepsilon + 2,\ 2\lambda = \varepsilon + 2,\ 4\lambda = \varepsilon + 2;$$

so the values of λ different from unity are respectively

$$4 \quad , \quad \tfrac{1}{4} \quad , \quad 2 \quad , \quad \tfrac{1}{2} \quad ,$$

of which the integer values are the only available ones. So the face expansion can be applied to $\frac{1}{2}[\overset{n}{\overline{11\ldots1}}]$ giving $[\overset{n-3}{\overline{44\ldots4}}220]$ and to $\frac{1}{2}[\overset{n-2}{\overline{33\ldots3}}11]$ giving $[\overset{n-3}{\overline{66\ldots6}}420]$, i. e. in both available cases measure polytope forms deducible from $M_n^{(4)}$. Therefore we can disregard altogether the expansion of the *hmpd.* according to their own faces and take into consideration the expansions e_k, $(k=2,3,\ldots n-2)$, of the $M_n^{(2)}$ only.

We now pass to the contraction. A motion of the limits of vertex import of $\frac{1}{2}[a_1\, a_2 \ldots a_{n-1}\, a_n]$, i. e. of $(a_1\, a_2 \ldots a_{n-1}\, a_n)$ towards the centre gives $(a_1-\lambda, a_2-\lambda, \ldots, a_{n-1}-\lambda, a_n-\lambda)$. So the only new form we can get is $(a_1-1, a_2-1, \ldots, a_{n-2}-1, 0, 0)$, i. e. a form deducible from $M_n^{(4)}$, etc.

100. We conclude this part by proving the following theorems, which will be useful in the next:

THEOREM LXIII. "The limits of truncation import of $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n^{(2\sqrt{2})}$ are $e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)^{2(\sqrt{2})}$."

According to the preceding theorem we have

$$e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n = \tfrac{1}{2}[\overbrace{2p+1}^{n-k_p}, \overbrace{2p-1}^{k_p-k_{p-1}}, \ldots, \overbrace{33\ldots3}^{k_2-k_1}, \overbrace{11\ldots1}^{k_1}].$$

So the limits of truncation import are

$$(\overbrace{2p+1}^{n-k_p}, \overbrace{2p-1}^{k_p-k_{p-1}}, \ldots, \overbrace{33\ldots3}^{k_2-k_1}, \overbrace{11\ldots1}^{k_1-1}, -1),$$

i. e.

$$(\overbrace{2p+2}^{n-k_p}, \overbrace{2p}^{k_p-k_{p-1}}, \ldots, \overbrace{44\ldots4}^{k_2-k_1}, \overbrace{22\ldots2}^{k_1-1}, 0),$$

or reversed

$$-(2p+2, \overbrace{2p}^{k_1-1}, \overbrace{2p-2}^{k_2-k_1}, \ldots\ldots, \overbrace{22\ldots2}^{k_p-k_{p-1}}, \overbrace{00\ldots0}^{n-k_p}),$$

i. e. $-\; e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)^{(2\sqrt{2})}$.

THEOREM LXIV. "The number k_1 of the units figuring in the symbol of coordinates $\frac{1}{2}[a_n\, a_{n-1} \ldots\ldots \overbrace{11\ldots1}^{k_1}]$ of an *hmpd.* in S_n indicates how many limits of truncation import pass through any vertex."

The number of vertices of $\frac{1}{2}[\overbrace{2p+1}^{n-k_p}, \overbrace{2p-1}^{k_p-k_{p-1}}, \ldots, \overbrace{33\ldots3}^{k_2-k_1}, \overbrace{11\ldots1}^{k_1}]$, respectively of its limits $(\overbrace{2p+1}^{n-k_p}, \overbrace{2p-1}^{k_p-k_{p-1}}, \ldots, \overbrace{33\ldots3}^{k_2-k_1}, \overbrace{11\ldots1}^{k_1-1}, -1)$ of truncation import is represented by

$$\frac{2^{n-1}.\, n!}{(n-k_p)!\,(k_p-k_{p-1})!\ldots(k_2-k_1)!\,k_1!}, \quad \text{resp.} \quad \frac{n!}{(n-k_p)!\,(k_p-k_{p-1})!\ldots(k_2-k_1)!\,(k_1-1)!}.$$

So the 2^{n-1} limits of truncation import admit together a number of vertices equal to k_1 times that of the *hmpd.* itself.

E. *Nets of polytopes.*

101. Let us consider the net $N(M_n^2)$ and suppose that it is composed of alternate white and black $M_n^{(2)}$, so that any two $M_n^{(2)}$ with a common limiting $M_{n-1}^{(2)}$ differ in colour. Let us imagine that each white $M_n^{(2)}$ is split up into an inscribed positive HM_n $(= + \frac{1}{2}[\overline{11\,..\,1}^{\,n}])$ and 2^{n-1} pyramids on regular simplexes $S(n)(2\sqrt{2})$ the vertex edges of which have a length 2 and meet at right angles, and that in the same way each black M_n is split up into an inscribed negative HM_n $(= - \frac{1}{2}[\overline{11\,..\,1}^{\,n}])$ and 2^{n-1} pyramids. Then it is clear that a space filling of S_n is formed by three groups of polytopes, two groups of HM_n, i. e. a group of positive ones and a group of negative ones, and one group of cross polytopes $[2\overline{00\,..\,0}^{\,n-1}]$, each of which has for centre a vertex of the net $N(M_n^2)$ not belonging to an HM_n and is generated by the addition of 2^n of the equal pyramids. This net, which may be represented by the symbol $N(\pm HM_n, Cr_n)$, forms our starting point here. It is our aim to deduce from this simple net several other ones the constituents of which are forms derived from the regular polytopes and *hmpd.*, partaking with each other of the properties of admitting one kind of vertices and one length of edge, by considering in the application of the expansion operations either the two sets of half measure polytopes as independent and the set of cross polytopes as dependent variables, or reversely.

Any HM_n of the original net $N(\pm HM_n, Cr_n)$ is limited by HM_{n-1} of $(l)_{n-1}$ import and by simplexes $S(n)$ of truncation import; by each HM_{n-1} it is in contact with an HM_n of the other kind, by each $S(n)$ with a Cr_n. We now follow two polytopes HM_n, Cr_n in $S(n)$ contact through any group of expansion operations leading to a new net, by which operations HM_n and its $S(n)$ pass into $(P)_n$ and $(Q)_{n-1}$ and likewise Cr_n and its $S(n)$ into $(P)_n$ and $(Q)'_{n-1}$. Then it is evident that $(Q)_{n-1}$ and $(Q)'_{n-1}$ must coincide, as the application of the operation e_n with respect to the group of Cr_n origin on one hand and the group of HM_n origin on the other would lead to a net with two different kinds of vertices, those of the group of Cr_n origin and those of the group of HM_n origin. This coincidence dominates the *hmpd.* nets, as it creates a very close relation between the two chief constituents. If we denote by the symbol $e_n\ HM_n$ the separation of the two groups of HM_n from each other by the intercalation of prisms on their original limits, the relation between the two chief constituents of an *hmpd.* net can be thrown into the following form:

THEOREM LXV. "In the *hmpd.* nets the constituent of HM_n origin unequivocally determines that of Cr_n origin and vice versa. If the former is $e_{k_1}\, e_{k_2} \ldots e_{k_{p-1}}\, e_{k_p}\, HM_n$, the latter is represented by $e_{k_1-1}\, e_{k_2-1} \ldots e_{k_{p-1}-1}\, e_{k_p-1}\, Cr_n$."

We divide the proof of this theorem in two parts. In the first part we suppose k_p different from n, in the second we trace the influence of the occurrence of $e_n\, HM_n$.

Let the set of operations to be applied to the Cr_n, in order to obtain a polytope able to form an *hmpd.* net with $e_{k_1}\, e_k \ldots e_{k_{p-1}}\, e_{k_p}\, HM_n$, be represented by $e_{k'_1}\, e_{k'_2} \ldots e_{k'_{q-1}}\, e_{k'_q}$. Then according to the results obtained in the preceding section the limiting $S(n)^{(2\sqrt{2})}$ of Cr_n is transformed into $e_{k'_1}\, e_{k'_2} \ldots e_{k'_{q-1}}\, e_{k'_q}\, S(n)^{(2\sqrt{2})}$, whilst on the other hand the $S(n)^{(2\sqrt{2})}$ of HM_n is transformed into

$$- e_{k_1-1}\, e_{k_2-1} \ldots e_{k_{p-1}-1}\, e_{k_p-1}\, S(n)^{(2\sqrt{2})}.$$

As the negative sign of the second symbol is accounted for by the position of the two polytopes at different sides of the common limit deduced from $S(n)$ the coincidence requires that we have

$$k'_1 = k_1 - 1,\ k'_2 = k_2 - 1,\ \ldots,\ k'_{q-1} = k_{p-1} - 1,\ k'_q = k_p - 1,$$

as the theorem states.

We now suppose that the operation e_n is added to the set of e_k expansions to be applied to the HM_n, i. e. that we drive the two groups of HM_n apart by prisms. Then the enlargement of the side $H_+ H_-$ of the triangle $CH_+ H_-$ (fig. 18), formed by the centres C, H_+, H_- of any triplet of constituents of different kind in mutually $(l)_{n-1}$ contact, caused by the intercalation of the prism implies enlargement of the two other sides, as the triangle must remain similar to itself. This enlargement of CH_+ and CH_- cannot be effected by the application of the operation e_n between the two constituents of different form (see pag. 90); so it must be caused by application of the operation e_{n-1} to the polytopes of Cr_n origin. In other words: the theorem to be proved also holds for the case that e_n occurs under the operations e_k to be applied to the HM_n groups.

Moreover from theorem LXIV we deduce:

THEOREM LXVI. "The totality of the vertices of any *hmpd.* net can always be represented by means of one net symbol, viz. that corresponding to the constituent of Cr_n origin."

We still remark that the number of *hmpd.* nets in S_n is 2^{n-1}.

For we can start either from HM_n as it is, or from one of the $(n-1)_1$ forms $e_{k_1} HM_n$, or from one of the $(n-1)_2$ forms $e_{k_1} e_{k_2} HM_n$, etc., giving altogether

$$1 + (n-1)_1 + (n-1)_2 + \ldots\ldots + (n-1)_{n-1} = (1+1)^{n-1} = 2^{n-1}$$

possibilities. These nets must all be new for $n > 4$, if they prove to exist. On the other hand a preparatory study of the cases $n = 3$ and $n = 4$ will show that $n = 3$ furnishes nothing new, whilst $n = 4$ produces four new cases only.

102. *Hmpd. nets in S_3.* — If we interprete the net of T and O as $N(\pm HM_3, Cr_3)$ the four cases we meet here are

1 .... HM_3, Cr_3	3 .... $e_3 HM_3$, $e_2 Cr_3$
2 .... $e_2 HM_3$, $e_1 Cr_3$	4 .... $e_2 e_3 HM_3$, $e_1 e_2 Cr_3$

or in other form

1 .... T, O 12	3 .... T, RCO, .. C 19
2 .... tT, tO, .. CO 24	4 .... tT, tCO, .. tC 23

Here the third constituents CO, C, tC are polyhedra filling gaps, whilst the numbers 12, 24, 19, 23 refer to the stereoscopic diagrams of ANDREINI. Compare also Table III of M[rs]. STOTT's memoir.

Let us pass now to the deduction of the coordinate symbols of these four nets. To that end we have to start in the first case from a T and an O in face contact — and in the other cases from what these polyhedra have become — and to calculate by means of the distance of their centres the periodic term which is to figure in the symbol. We therefore elucidate the mutual position of the two polyhedra in face contact in fig. 19, in projection on to a plane normal to one of the three diameters of the O group. But for clearness' sake we have represented in each of the four cases the T and the O — or what they have become — lying apart; in order to re-establish the real state we have to move the T parallel to itself so as to bring the invisible shadowed face of T indicated by dotted lines in contact with the visible shadowed face of O, i. e. $A'B'$ into coincidence with AB. As we want only the net symbol with respect to the group of O, the origin of the system $O(XYZ)$ of coordinates has been chosen in the centre of the O of the diagram.

The simple diagrams of fig. 19 show an easier way leading to the knowledge of the periodic term of the net symbol. Indeed, in each of the four cases the O — or what it has become — is in contact by the edge AB with an other polyhedron congruent to it. In other words: if the coordinates x, y of the centre M of AB are p, the centres of the O group are represented by the frame $[2a_1 p, 2a_2 p, 2a_3 p]$ under the conditions a_1, a_2, a_3 integer and $\sum_1^3 a_i$ even, i. e. $2p$ is the *period* of the net. So, as the p has in the four cases successively the values $1, 3, 1+\sqrt{2}, 3+\sqrt{2}$ we find for the four net symbols under the stated conditions

$$1 \ldots [2a_1 + \mathbf{2}, \quad 2a_2 + \mathbf{0}, \quad 2a_3 + \mathbf{0}],$$
$$2 \ldots [6a_1 + \mathbf{4}, \quad 6a_2 + \mathbf{2}, \quad 6a_3 + \mathbf{0}],$$
$$3 \ldots [2(1+\sqrt{2})a_1 + \mathbf{2} + \sqrt{\mathbf{2}}, \; 2(1+\sqrt{2})a_2 + \sqrt{\mathbf{2}}, \; 2(1+\sqrt{2})a_3 + \sqrt{\mathbf{2}}],$$
$$4 \ldots [2(3+\sqrt{2})a_1 + \mathbf{4} + \sqrt{\mathbf{2}}, \; 2(3+\sqrt{2})a_2 + \mathbf{2} + \sqrt{\mathbf{2}}, \; 2(3+\sqrt{2})a_3 + \sqrt{\mathbf{2}}].$$

Though we pursue the study of these threedimensional nets merely from a didactic point of view it is not necessary to deduce from these net symbols of the O group the net symbols of the two T groups. All we want is to show how the third constituents CO, C, tC can be found. Therefore we give here the net symbols of the two T groups in the form:

$$1 \ldots \pm\tfrac{1}{2}[2a_1 \pm 1 + \mathbf{1}, \quad 2a_2 \pm 1 + \mathbf{1}, \quad 2a_3 \pm 1 + \mathbf{1}],$$
$$2 \ldots \pm\tfrac{1}{2}[3(2a_1 \pm 1) + \mathbf{3}, \quad 3(2a_2 \pm 1) + \mathbf{1}, \quad 3(2a_3 \pm 1) + \mathbf{1}],$$
$$3 \ldots \pm\tfrac{1}{2}[(2a_1 \pm 1)(1+\sqrt{2}) + \mathbf{1}, (2a_2 \pm 1)(1+\sqrt{2}) + \mathbf{1}, (2a_3 \pm 1)(1+\sqrt{2}) + \mathbf{1}],$$
$$4 \ldots \pm\tfrac{1}{2}[(2a_1 \pm 1)(3+\sqrt{2}) + \mathbf{3}, (2a_2 \pm 1)(3+\sqrt{2}) + \mathbf{1}, (2a_3 \pm 1)(3+\sqrt{2}) + \mathbf{1}],$$

where the double sign refers to the two groups $\pm HM_3$ and the conditions about the a_i and their sum remain the same.

As the polyhedra of the O group remain in contact by faces with those of the two T groups and by edges with each other we have only to look out for new polyhedra filling vertex gaps which make their appearance in the second, third and fourth cases on account of the truncation of the polyhedra of the O group at the vertices. Though all the *vertices* of these new constituents are contained in the net, the second and the fourth cases show that it may happen that some of the *faces* of these new bodies have to be furnished by the polyhedra of the T groups. At any rate we have to determine the new constituent by starting from an octahedron vertex and deducing from the net symbol the vertices at minimum distance from that point.

We treat further each of the four cases by itself.

Case (O, T). — In this case there is no third constituent. Nevertheless we deduce from the net symbol of group O given above that the vertices of all the CO represented by $[2a_1 + \mathbf{2}, 2a_2 + \mathbf{2}, 2a_3 + \mathbf{0}]$, $\sum_1^3 a_i$ odd, are vertices of the net. But these CO are no constituents of the net; for the centre of the CO corresponding to any set of integers a_i satisfying the condition $\sum_1^3 a_i$ odd is the point $2a_1, 2a_2, 2a_3$, and for $\sum_1^3 a_i$ odd this centre itself is a vertex of the net, i. e. these CO overlap.

Case (tO, tT). — As we have $p = 3$ the point 2, 0, 0, originally common to the central O and an other in vertex contact with it, is carried away from the origin to thrice the distance and arrives at 6, 0, 0. So with respect to this centre of a new constituent as new origin the original net symbol becomes $[6(a_1 - 1) + \mathbf{4}, 6a_2 + \mathbf{2}, 6a_3 + \mathbf{0}]$, Σa_i even, i. e. $[6a_1 + \mathbf{4}, 6a_2 + \mathbf{2}, 6a_3 + \mathbf{0}]$, Σa_i odd. Now the supposition $a_1 = -1, a_2 = a_3 = 0$ gives the square $-2[2, 0]$ and so the six suppositions $a_1, a_2, a_3 = [1\ 0\ 0]$ give the six squares of the $[2, 2, 0]$, i. e. of the CO. The eight triangles of this CO are furnished by tT, four of each group. So by putting $a_1 = 0$, $a_2 = -1$, $a_3 = -1$ in the net symbol of the group of positive tT we get $\frac{1}{2}[3 + \mathbf{3}, -3 + \mathbf{1}, -3 + \mathbf{1}]$, i. e. reduced to the new origin 6, 0, 0 the symmetrical form $\frac{1}{2}[-3 + \mathbf{3}, -3 + \mathbf{1}, -3 + \mathbf{1}]$, the triangle $(0, -2, -2)$ of which is a face of the CO found above.

Case (RCO, T). — Here we have $p = 1 + \sqrt{2}$ and the centre of the new constituent becomes $2(1 + \sqrt{2})$, 0, 0. So the net symbol with respect to that new origin is

$$[2(1+\sqrt{2})a_1 + \mathbf{2} + \sqrt{\mathbf{2}},\ 2(1+\sqrt{2})a_2 + \sqrt{\mathbf{2}},\ 2(1+\sqrt{2})a_3 + \sqrt{\mathbf{2}}],\ \sum_1^3 a_1 \text{ odd}.$$

Here the six suppositions $a_1, a_2, a_3 = [100]$ give the six limiting squares of the cube $[\sqrt{2}, \sqrt{2}, \sqrt{2}]$.

Case (tCO, tT). — Here $p = 3 + \sqrt{2}$ and therefore $2(3 + \sqrt{2})$, 0, 0 is the new origin, leading to the new form

$$[2(3+\sqrt{2})a_1 + \mathbf{4} + \sqrt{\mathbf{2}},\ 2(3+\sqrt{2})a_2 + \mathbf{2} + \sqrt{\mathbf{2}},\ 2(3+\sqrt{2})a_3 + \sqrt{\mathbf{2}}],\ \sum_1^3 a_i \text{ odd}$$

of the net symbol. Here the same suppositions give the six limiting octagons of the tC represented by $[2 + \sqrt{2}, 2 + \sqrt{2}, \sqrt{2}]$. By putting $a_1 = 0$, $a_2 = -1$, $a_3 = -1$ in the net symbol of the group of positive tT we get here

$$\tfrac{1}{2}[3 + \sqrt{2} + \mathbf{3}, -(3 + \sqrt{2}) + \mathbf{1}, -(3 + \sqrt{2}) + \mathbf{1}],$$

or with respect to the new origin

$$\tfrac{1}{2}[-(3 + \sqrt{2}) + \mathbf{3}, -(3 + \sqrt{2}) + \mathbf{1}, -(3 + \sqrt{2}) + \mathbf{1}],$$

the triangle $(-\sqrt{2}, -2-\sqrt{2}, -2-\sqrt{2})$ of which is a face of the tC.

Remark. The p introduced above is not to be confounded with the extension number of the octahedron group which according to the rule connected with the sum of the digits would be $1, 3, 1 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}$ in the four cases.

103. The four cases of *lmpd.* nets in S_3 considered above agree in this that the third constituent is the contraction form of the constituent of octahedron origin. Indeed the contraction forms of O, tO, RCO, tCO are respectively a vertex, CO, C, tC. This fact is too general to be accidental, we will show why it *must* be so.

Therefore we recur to theorem LXVI. As all the vertices of the net figure in the net symbol of the octahedron group — which implies as we already remarked that all the vertices of the new constituent are contained in the net symbol —, the faces which that new constituent has in common with the adjacent polyhedra of the octahedron group must define that new polyhedron. Now in the original net (O, T) any vertex V is a point of concurrence of six O, the centres of which are the opposite vertices V'_i of the six edges of the net of cubes from which (O, T) has been deduced. So the six faces of contact of the new constituent with the six polyhedra of octahedron origin lie in planes normal to the lines OV'_i, in the centres of these faces, lying at equal distance from O. These simple considerations lead to three possibilities compatible with the condition that the new constituent must admit vertices of the same kind and edges of the same length: either the new constituent is equal to the constituent of octahedron origin, or the new one is the contraction form of the other, or the other is the contraction form of the new one. But the first and the last suppositions are to be rejected. For the first would bring equality between the two kinds of limits of the constituent of tetrahedron origin which have been called original limits and limits of truncation import, whilst the last is inadmissible as the constituent of octahedron origin is no contraction form.

We now prove that the preceding result holds for any *lmpd.* net in S_n. If once for all we distinguish for short the constituent of HM_n origin as the first and that of Cr_n origin as the second we can extend theorem LXV by proving:

THEOREM LXVII. "Any *lmpd.* net has three different constituents, none of which is a prism. The third is the contraction form of the second. So, if the first is $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n$ and therefore the second $e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} Cr_n$, the third is $ce_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} Cr_n$. In this form of the statement each of the three unequivocally determines the two others."

Proof. In the original $N(HM_n, Cr_n)$ any two Cr_n in contact are either in edge contact, or in vertex contact, in other words the contact of the highest order between two Cr_n of the net is edge contact. Now this contact of the highest order can only be annihilated by separation of the Cr_n i. e. by applying the expansion e_n to them. As this operation is excluded (as leading to a net with two kinds of vertices) the edge contact between the polytopes of Cr_n origin is maintained, though it is changed in character by the operations e_k, $1 < k < n$, contact by *edge* being replaced by contact of an $(l)_{n-1}$ limit of *edge import*. This proves in the first place that there is only room for one new constituent different from a prism, viz. a new polytope with respect to the Cr_n of vertex import; vertex contact being annihilated in any net deduced from $N(HM_n, Cr_n)$, this third constituent *always* makes its appearance. Now we have only to prove still that this third constituent is the contraction form of the second; we prove this in two different ways, in the first place by considering the contact with the second, in the second place by considering the contact with the first constituent.

According to theorem LXVI here also the third constituent is determined by the limits $(l)_{n-1}$ of contact with the $2n$ adjacent polytopes of Cr_n origin, the centres of which are the vertices of a cross polytope with the centre of the vertex gap as centre. So, here also, *if* the $2n$ limits $(l)_{n-1}$ are to determine a polytope with vertices of one kind and edges of one length, there are three possibilities: either the third constituent is equal to the second, or it is the contraction form of the second, or it has the second for contraction form. Here also the first and the last suppositions are inadmissible for the reasons indicated in the case $n = 3$. So the theorem is proved.

We add the following second proof, which we consider even more convincing, as a confirmation of the result obtained. In the notation of the problem the limit of vertex import of $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n$ is — compare the proof of theorem LXIII — represented by

$$(\overline{2p+1}^{\,n-k_p}, \overline{2p-1}^{\,k_p-k_{p-1}}, \ldots, \overline{33..3}^{\,k_2-k_1}, \overline{11..1}^{\,k_1}),$$

i. e.

$$(\overline{2p}^{\,n-k_p}, \overline{2p-2}^{\,k_p-k_{p-1}}, \ldots, \overline{22..2}^{\,k_2-k_1}, \overline{00..0}^{\,k_1}),$$

or reversed

$$-(\overline{2p}^{\,k_1}, \overline{2p-2}^{\,k_2-k_1}, \ldots, \overline{22..2}^{\,k_p-k_{p-1}}, \overline{00..0}^{\,n-k_p}),$$

i. e. $- ce_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)(2\sqrt{2})$. So the limit $(l)_{n-1}$ of highest import of the third constituent is the contraction form of the corresponding limit of the second constituent, i. e. the third constituent itself is the contraction form of the second. Or shorter still: by the *reversion* of the symbols the transition from $(a_1\, a_2 \ldots a_{n-2}\; 1 - 1)$ to $(a_1\, a_2 \ldots a_{n-2}\; 1\; 1)$ manifests itself by the diminution of the *first* digit by 2, i. e. by the operation of contraction, leading to the result mentioned in the theorem.

Remark. There is a characteristic difference between the three groups of nets — [a] the simplex nets, [b] the measure polytope nets, [c] the half measure polytope nets — as to the character of the constituents. As we have seen in the preceding sections the simplex nets admit exclusively principal constituents, i. e. neither prisms nor prismotopes, whilst the measure polytope nets admit only *two* principal constituents with exception of the original net of measure polytopes. Now in the case of the *hmpd.* net we always find *three* principal constituents with exception of the original net $(HM_n,\, Cr_n)$; as soon as two of the three constituents become equal to each other we fall back on a measure polytope net. This only happens for $n > 3$ in S_4, as we shall see in the next article.

104. *Hmpd. nets in* S_4. — Here we have to examine the eight cases:

1	$HM_4,$	Cr_4	5	$e_2\, e_3\, HM_4,$	$e_1\, e_2\, Cr_4$
2	$e_2\, HM_4,$	$e_1\, Cr_4$	6	$e_2\, e_4\, HM_4,$	$e_1\, e_3\, Cr_4$
3	$e_3\, HM_4,$	$e_2\, Cr_4$	7	$e_3\, e_4\, HM_4,$	$e_2\, e_3\, Cr_4$
4	$e_4\, HM_4,$	$e_3\, Cr_4$	8 ...	$e_2\, e_3\, e_4\, HM_4,$	$e_1\, e_2\, e_3\, Cr_4$

Of these eight cases only four are new. The first is $N(C_{16})$, the three equal groups of C_{16} being the groups of $+HM_4$, $-HM_4$, Cr_4. The second case is $e_1\, N(C_{16})$; as $e_2\, HM_4 = e_1\, Cr_4$ we find only two principal constituents. The third case is $ce_2\, N(C_{16})$; as $e_3\, HM_4 = ce_2\, Cr_4$, the third constituent is equal to the first. Finally the fifth case is $ce_1\, e_2\, N(C_{16})$; as $e_2\, e_3\, HM_4 = ce_1\, e_2\, Cr_4$, here also the third constituent is equal to the first. In the four remaining cases the three chief constituents are different; so these cases are new. We represent them in the following small table

$$[\;e_4\, HM_4,\quad e_3\, Cr_4,\quad ce_3\, Cr_4,\quad P_T\,],\quad 2(1+\sqrt{2})\, a_i\; [2+\sqrt{2},\; \sqrt{2},\; \sqrt{2},\; \sqrt{2}],$$
$$[\;e_2\, e_4\, HM_4,\quad e_1\, e_3\, Cr_4,\quad ce_1\, e_3\, Cr_4,\quad P_{tT}\,],\quad 2(3+\sqrt{2})\, a_i\; [4+\sqrt{2},\; 2+\sqrt{2},\; \sqrt{2},\; \sqrt{2}],$$
$$[\;e_3\, e_4\, HM_4,\quad e_2\, e_3\, Cr_4,\quad ce_2\, e_3\, Cr_4,\quad P_T\,],\quad 2(3+\sqrt{2})\, a_i\; [4+\sqrt{2},\; 2+\sqrt{2},\; 2+\sqrt{2},\; \sqrt{2}],$$
$$[\;e_2\, e_3\, e_4\, HM_4,\quad e_1\, e_2\, e_3\, Cr_4,\quad ce_1\, e_2\, e_3\, Cr_4,\quad P_{tT}\,],\quad 2(5+\sqrt{2})\, a_i\; [6+\sqrt{2},\; 4+\sqrt{2},\; 2+\sqrt{2},\; \sqrt{2}],$$

enumerating the quadruplets of constituents and in condensed form the net symbols; in latter symbols the immovable parts of the digits are placed before the square brackets, whilst the sum of the four integers a_i is always even.

In order to get a better insight into the constitution of the fourdimensional *hmpd.* nets we tabulate the contact between the different constituents. To that end we introduce first a short notation with respect to the nets themselves and to their constituents and the threedimensional limits of these. We denote the *hmpd.* nets in S_4 by the collective symbol NH_4 and distinguish them mutually from each other by putting before that symbol the system of expansion operations applied to the second constituent Cr_4; so the four nets found above are $e_3\,NH_4$, $e_1\,e_3\,NH_4$, $e_2\,e_3\,NH_4$, $e_1\,e_2\,e_3\,NH_4$. Moreover we indicate the four constituents of each net, i. e. the three principal ones taken in the order of succession assumed in theorem LXVII and the prism, by A, B, C, D and we represent their different limits $(l)_3$ by means of subscripts in connexion with their import; so A_3, A_t, A_0 will represent the limits of body, truncation, vertex import of A, whilst $B_i\,(i = 3, 2, 1, 0)$ and $C_k\,(k = 3, 2, 0)$ will represent the limits of $(l)_i$ import of B and of $(l)_k$ import of C, and D_3, D_2, D_0 will stand for the bases of D and the upright limits $(l)_3$ of that prism which correspond to the faces of face import and of vertex import of the bases. So we find the following small table, where the numbers under the columns show how many $(l)_3$ of each kind each polytope admits:

Net	A_3	A_t	A_0	B_3	B_2	B_1	B^0	C_3	C_2	C_0	D_3	D_2	D_0
$e_3\,NH_4$	T	T	—	T	P_3	P_4	C	—	—	C	T	P_3	—
$e_1\,e_3$ "	tT	tT	O	tT	P_6	P_4	RCO	O	P_3	RCO	tT	P_6	P_3
$e_2\,e_3$ "	T	CO	T	CO	P_3	P_8	tC	T	—	tC	T	P_3	—
$e_1\,e_2\,e_3$ "	tT	tO	tT	tO	P_6	P_8	tCO	tT	P_3	tCO	tT	P_6	P_3
	8	8	8	16	32	24	8	16	32	8	2	4	4

This table shows that the contact between the four different constituents is the same in the four nets, i. e. that we have in general

$$A_3 = D_3,\ A_t = B_3,\ A_0 = C_3,\ B_2 = D_2,\ B_0 = C_0,\ C_2 = D_0,$$

whilst B is in contact by its limits B_1 of edge import with other polytopes B, this transformed edge contact being preserved. So the different threedimensional limits cover each other two by two.

The contact between the different constituents can also be deduced from the following small table in which we repeat the constituents of the net in an other form:

Net	C	B	A	D
$e_3\,NH_4$	$[1\,1\,1\,1]\sqrt{2}$	$[1'1\,1\,1]\sqrt{2}$	$\frac{1}{2}[1111]$	$\frac{1}{2}[111][1]\sqrt{2}$
$e_1\,e_3$,,	$[1'1'1\,1]$,,	$[2'1'1\,1]$,,	,, $[3311]$	,, $[311][1]$,,
$e_2\,e_3$,,	$[1'1'1'1]$,,	$[2'1'1'1]$,,	,, $[3111]$	,, $[111][1]$,,
$e_1\,e_2\,e_3$,,	$[2'2'1'1]$,,	$[3'2'1'1]$,,	,, $[5311]$	,, $[311][1]$,,

So from this table we deduce $A_3 = D_3$ by remarking that the digits of the first syllable of D are the last three digits of the unique syllable of A; in order to facilitate comparison of A and D we have reversed the order of A, B, C.

So we find $A_t = B_3$ (or rather $A_t = - B_3$) as we get the same form by placing the four digits of A between round brackets after having taken the last unit with the negative sign and by placing the digits of B, multiplied by $\sqrt{2}$, between round brackets; etc.

105. Before passing to the case $n = 5$ we will put the last two small tables of the preceding article on duty as to the general results they may suggest for $n > 4$.

We begin by fixing our attention on the extreme case of the relation between the two constituents A and C, being governed in the case $e_3 NH_4$ by a vertex only. Here A_0, the limit of vertex import of A, is still a vertex; so we have to accept for C the polytope deduced from Cr_4 which admits as limit C_3 of body import a vertex and this is the eightcell $ce_3 Cr_4$. The same remark holds for $e_2 NH_3$ already, i.e. for the third of the four cases treated in art. 103.

But the first of our two tables, i. e. the table of contacts, suggests a remark of much wider scope. We deduce it from the fact that each constituent with three kinds of limits $(l)_3$ is in contact with the three others, whilst the only one with four different kinds of limits $(l)_3$ is in contact with the three others and with itself.

This fact suggests that in space S_n we will want in all n different constituents $A, B, C, \ldots,$ of which B only admits at most n different limits $(l)_{n-1}$ and all the others at most $n - 1$. We have used this suggestion as working hypothesis and found by its help the sixteen *hmpd.* nets of S_5; this was an easy task: as theorem LXVII gives the three principal constituents A, B, C and the prism D can be deduced from them, the table of contacts shows immediately which limits $(l)_4$ remain uncovered and these limits reveal the character of the fifth constituent. [1])

But there is an other method of deducing the new constituent, much more capable of being extended to S_n, viz the determination of their coordinate symbols by transformation of the net symbol to

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[1]) It may seem in accordance with this suggestion that in the cases $e_2 NH_3$ and $e_1 e_2 NH_3$ of S_3 we have found no fourth constituent i. e. no prism, though they require the operation e_2 with respect to the two groups of HM_3 of different orientation, driving these groups asunder. But this not appearing of the prism is rather due to the fact that two adjacent HM_3 of different orientation are in contact by an edge only instead of by a face, so that the separation intercalates a square instead of a prism.

new origins. We introduce this method by remarking that the addition of the second syllable $[1]\sqrt{2}$ of the symbol of the prism D in the last table of the preceding article has a deeper meaning than might be supposed: in this form the coordinate symbol of D is derived from the net symbol, and by examining how this process runs in S_4 we easily hit upon its generalization for S_n, if necessary by the assistance of the knowledge of the fifth constituent in S_5 found in the manner described above. So we indicate for any net in S_4 how the coordinate symbol of the constituents can be derived from the net symbol.

In fig. 20 we represent by $O(X_1\, X_2\, X_3\, X_4)$ the system of coordinates and by the shaded pentagon with the axis of symmetry OM a fourth part of the section of the plane $O(X_1\, X_2)$ with the central polytope B. Then OP_0 is the "period" p of the net and the point P_1 of OX_1 lying at twice that distance from O is the centre of an adjacent polytope C filling a vertex gap, whilst P_2 with the coordinates $2p, 2p, 0, 0$ is the centre of an other polytope B in contact with the central one by a polyhedron of edge import. Moreover P_3 is the point $2p, 2p, 2p, 0$ and P_4 the point all the coordinates of which are $2p$; of these P_3 corresponds in character with P_1, and P_4 with O and P_2. So the midpoint Q_4 of OP_4 must be the centre of a polytope in threedimensional contact of body import with the two polytopes B with the centres O and P_4, i.e. of a polytope A. On the other hand the midpoint Q_3 of OP_3 must be the centre of the prism interposed between the two polytopes A of different orientation with the centres Q_4, latter point being the image of Q_4 with respect to the space $x_4 = 0$ as mirror, as these polytopes are derived from the two HM_4 of the original net (HM_4, Cr_4) which were in body contact in that space $x_4 = 0$.

In this manner we find in general for all the cases in S_4 for the coordinates of the centres of the adjacent polytopes

$2p$,	0 ,	0 ,	0 ,	in the case of C ,
$2p$,	$2p$,	0 ,	0 ,	„ „ „ „ an other B ,
p ,	p ,	p ,	p ,	„ „ „ „ A ,
p ,	p ,	p ,	0 ,	„ „ „ „ D ,

whilst the upright edges of the prism D are parallel to the axis OX_4.

Now we consider the case $e_1\, e_2\, e_3\, NH_4$ in order to show how the process runs. Here we have $p = 5 + \sqrt{2}$, whilst the central B is represented by $[6 + \sqrt{2}, 4 + \sqrt{2}, 2 + \sqrt{2}, \sqrt{2}]$. So we obtain C, A, D successivey as follows:

$$\begin{array}{rcccc} & 6+\sqrt{2}, & 4+\sqrt{2}, & 2+\sqrt{2}, & \sqrt{2} \\ & 10+2\sqrt{2}, & 0, & 0, & 0 \\ \hline & -(4+\sqrt{2}), & 4+\sqrt{2}, & 2+\sqrt{2}, & \sqrt{2} \end{array} \text{ subtr.}$$

furnishing the polytope $[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, i.e. C;

$$\begin{array}{cccc} 6+\sqrt{2}, & 4+\sqrt{2}, & 2+\sqrt{2}, & \sqrt{2} \\ 5+\sqrt{2}, & 5+\sqrt{2}, & 5+\sqrt{2}, & 5+\sqrt{2} \\ \hline 1, & -1, & -3, & -5 \end{array} \text{ subtr.}$$

leading to the polytope $-\frac{1}{2}[5311]$, i. e. A;

$$\begin{array}{cccc} 6+\sqrt{2}, & 4+\sqrt{2}, & 2+\sqrt{2}, & [\sqrt{2}] \\ 5+\sqrt{2}, & 5+\sqrt{2}, & 5+\sqrt{2}, & 0 \\ \hline 1, & -1, & -3, & [\sqrt{2}] \end{array} \text{ subtr.}$$

giving finally the polytope $\frac{1}{2}[311][\sqrt{2}]$, i. e. D.

This will be clear, if we only add one word about the factor $\frac{1}{2}$ before the symbols of A and D, viz. that we want this factor in order to have symbols representing polytopes with one kind of vertex and one length of edge.

106. *Hmpd. nets in S_5.* — We have determined the sixteen *hmpd.* nets of S_5 by means of the two methods given in outline in the preceding article.

The results of the first method are put on record in Table X. This table is divided by vertical lines into eight parts; of these the first contains the symbol of the nets, the last two their constituents and the five others the limits $(l)_4$ of each of the five constituents A, B, C, D, E. In the construction of this table we started from theorem LXVII enabling us to register in the last part but one in the columns with the superscripts A, B, C the character of the three principal constituents and to add under D, in the cases where e_4 appears amongst the expansion symbols of the net, the prisms on the polytopes of polytope import of A as bases. After having finished this task we have inscribed in the columns with the headings $A_4, A_t, \ldots D_t, D_0$ the limits $(l)_4$ of these constituents A, B, C, D, taken from the tables given in the preceding sections of this memoir; this will be clear if we add the remark that the notation D_3, D_t, D_0 for the limits $(l)_4$ of D differing from the bases D_4 has been chosen in accordance with the consideration of these bases as deduced from HM_4. This second task having been performed we can formulate the contact between the constituents A, B, C, D; we find generally:

$$A_4 = A_4 \text{ (if } e_4 \text{ is \textit{absent})} \text{ and } A_4 = D_4 \text{ (if } e_4 \text{ is \textit{present}),}$$
$$A_t = B_4, A_0 = C_4, B_3 = D_3, B_1 = B_1, B_0 = C_0, C_3 = D_0.$$

So A_3, B_2, C_2, D_t remain uncovered, i. e. have still to be covered by limits $(l)_4$ of E. We represent these limits $(l)_4$ of E by E_a, E_b, E_c, E_d, indicating by the subscripts the constituents with which they are in $(l)_4$ contact and repeat these limits in the column with the headings E_a, E_b, E_c, E_d. Finally from these limits we deduce the constituent E itself, see the last column of the seventh part of the table. We remark that this fifth constituent is a prismotope, the two components of which are HM_3 (or $e_2\,HM_3$) and p_4 (or p_8); it presents itself if and only if either e_3, or e_4, or both operations are present.

In applying the second method to S_5 we have to extend the $M_4^{(2p)}$ of fig. 20 with the broken line $O\,P_1\,P_2\,P_3\,P_4$ of edges leading from O to the opposite vertex P_4 into an $M_5^{(2p)}$ with $O\,P_1\,P_2\,P_3\,P_4\,P_5$ as corresponding broken line of edges from O to the opposite vertex P. If we represent the midpoints of OP_5, OP_4, OP_3 respectively by Q_5, Q_4, Q_3 we find for the new origins leading to the constituents C, A, D, E the points P_1, Q_5, Q_4, Q_3 with the coordinates

$$\left.\begin{array}{ccccc} 2p, & 0, & 0, & 0, & 0 \\ p, & p, & p, & p, & p \\ p, & p, & p, & p, & 0 \\ p, & p, & p, & 0, & 0 \end{array}\right\} .$$

So in the case $\mathbf{e_1\,e_3\,e_4\,NM_5}$, i. e. $[3'2'1'1'1]\sqrt{2}$ with $p = 5 + \sqrt{2}$ the constituents A, D, E are obtained by the three processes

$$\begin{array}{ccccc} 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & 2+\sqrt{2}\ , & \sqrt{2} \\ 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2} \\ \hline 1\ , & -1\ , & -3\ , & -3\ , & -5 \end{array}\ \text{subtr.}$$

$$\begin{array}{ccccc} 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & 2+\sqrt{2}\ , & [\sqrt{2}] \\ 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 0 \\ \hline 1\ , & -1\ , & -3\ , & -3\ , & [\sqrt{2}] \end{array}\ \text{subtr.}$$

$$\begin{array}{ccccc} 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & [2+\sqrt{2}\ , & \sqrt{2}] \\ 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 0\ , & 0 \\ \hline 1\ , & -1\ , & -3\ , & [2+\sqrt{2}\ , & \sqrt{2}] \end{array}\ \text{subtr.}$$

giving respectively $\frac{1}{2}[53311]$, $\frac{1}{2}[3311][1]\sqrt{2}$, $\frac{1}{2}[311][1'1]\sqrt{2}$. The results obtained in this way are collected in Table XI. To this we have only to add a few remarks.

The processes used just now show clearly why the syllables $\frac{1}{2}[3311]$ and $\frac{1}{2}[311]$ of D and E must correspond in digits with the last digits of $\frac{1}{2}[53311]$, the symbol of A, and likewise why the other syllables $[1]\sqrt{2}$ and $[1'1]\sqrt{2}$ must correspond in the same manner with either of the symbols $[3'2'1'1'1]$ and $[2'2'1'1'1]$ of B and C. Also why D must be a prism and E a prismotope, in connexion with the faculty of inverting the signs of $\sqrt{2}$ in the case of D, and of $2+\sqrt{2}$ and $\sqrt{2}$ in the case of E, these inversion having no influence whatever on the distance of the vertices obtained of the new origin which is to be the centre of the gap filling polytope.

Moreover the processes themselves indicate under which circumstances the prism D and the prismotope E present themselves. If the symbol of B winds up in zero the second syllable of the symbol of D is $[0]$, i. e, the prism is lacking; but we know from theorem XXXV that the last digit of the symbol of B *is* zero, if the operation e_4 has not been applied to B. Likewise, if the last two digits of the symbol of B are zero, the second syllable of E is $[0, 0]$, i. e. there is no prismotope E, and the last two digits of the symbol of B *are* zero, if neither e_3 nor e_4 has been applied to B.

Finally it is evident why we cannot add a fourth process to the three considered ones and subtract $5+\sqrt{2}$, $5+\sqrt{2}$, 0, 0, 0. For then we would get $1, -1, [2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, leading to $\frac{1}{2}[11][1'1'1]\sqrt{2}$, i. e. — as $\frac{1}{2}[11]$ is an edge instead of a face — to a limiting body and not to a limit $(l)_4$.

107. *Hmpd. nets in* S_n. — It is easy to see how the processes of the preceding article must be extended to S_n, as the algorithm always remains the same and the number of the subtractions has to be augmented until only three digits of the subtrahend differ from zero. So, if we indicate by $A^{(k)}$ the constituent obtained by the subtraction of $\overbrace{pp..p}^{n-k}\ \overbrace{00..0}^{k}$ we can formulate the general result in the following theorem:

THEOREM LXVIII. — "In any net deduced from NH_n we find, besides the three *principal* constituents A, B, C always present, under certain circumstances one or more prismotopes $A^{(k)}$ for $k=1, 2, \ldots, n-3$, which may be called *accidental* constituents. The prismotope $A^{(k)}$ presents itself if — and only if — one or more of the expansions $e_{n-k}, e_{n-k+1}, e_{n-1}$ have contributed to the transformation of Cr_n into B; the two syllables of its symbol are the last $n-k$ digits of A between square brackets preceded by $\frac{1}{2}$ and the last k digits of B between square brackets."

So we find in the case $B = [5'4'4'3'3'2'2'2'1'1]V2$ of S_{10}:

$$
\begin{aligned}
C &= \quad [4'4'4'3'3'2'2'2'1'1]V2 \\
A &= \tfrac{1}{2}[9\,7\,5\,5\,5\,3\,3\,1\,1\,1] \\
A^{(1)} &= \text{,,}[7\,5\,5\,5\,3\,3\,1\,1\,1]\ [1]V2 \\
A^{(2)} &= \text{,,}[5\,5\,5\,3\,3\,1\,1\,1]\ [1'1]\ \text{,,} \\
A^{(3)} &= \text{,,}[5\,5\,3\,3\,1\,1\,1]\ [2'1'1]\ \text{,,} \\
A^{(4)} &= \text{,,}[5\,3\,3\,1\,1\,1]\ [2'2'1'1]\ \text{,,} \\
A^{(5)} &= \text{,,}[3\,3\,1\,1\,1]\ [2'2'2'1'1]\ \text{,,} \\
A^{(6)} &= \text{,,}[3\,1\,1\,1]\ [3'2'2'2'1'1]\ \text{,,} \\
A^{(7)} &= \text{,,}[1\,1\,1]\ [3'3'2'2'2'1'1]\ \text{,,}
\end{aligned}
$$

By applying this theorem we find immediately the sixteen nets of S_5, as they have been registered in the eighth part of Table X with the heading "constituents in an other notation". Moreover Table XI gives the corresponding results for the 32 nets of S_6.

F. *Polarity.*

108. By polarizing an n-dimensional *hmpd.* with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles, to which corresponds the inversed symbol of characteristic numbers of the original polytope. Moreover, if $\frac{1}{2}[a_1, a_2, \ldots, a_{n-1}, a_n]$ is the coordinate symbol of the original *hmpd.*, this symbol also represents the limiting spaces S_{n-1} of the new polytope in space coordinates.

The fact that there is no *hmpd.* proper in S_3 and S_4 implies the corresponding fact with respect to the new forms. So, if by the subscript s is indicated that space coordinates are meant, we have: $\frac{1}{2}[\ 111]_s = (4, 6, 4) = T'$, $\frac{1}{2}[311]_s = (8, 18, 12) = T$ with pyramids on the faces,
$\frac{1}{2}[1111]_s = (16, 32, 24, 8) = M_4$, $\frac{1}{2}[3311]_s = (24, 96, 120, 48) = M_4$ with pyramids on the cubes, etc.

109. Theorem LXIX. "Any *hmpd.* in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices of the *hmpd.* include an other polytope, the reciprocal polar of the original polytope with respect to a certain concentric spherical space, unless the chosen

hmpd. be the cross polytope HM_4 of S_4 in which case all the spaces S_3 pass through the centre."

The simple geometrical proof of this theorem can be copied from that of theorem XL (see art. 66).

110. We have to add a single word about the reciprocation of the *hmpd.* nets. The results obtained here run parallel to those of art. 68.

In general the system of vertices found by polarizing an *hmpd.* net is the combination of several groups of limits $M_k^{(2p)}$ of the measure polytopes of the net $N(M_n^{(2p)})$, p being the *period.* These groups are formed by the centres of the constituents $B, C, A, A^{(1)}, \ldots, A^{(n-3)}$, i. e.

for B	the *even* vertices of $N(M_n^{(2p)})$, represented by $[2pa_1, \ldots, 2pa_n]$,	$\sum_1^n a_i$ even,
" C	" *odd* " " " , " " " ,	$\sum_1^n a_i$ odd,
" A	" centres of the M_n of $N(M_n^{(2p)})$,	
" $A^{(1)}$	" " " " limiting M_{n-1} of the M_n of $N(M_n^{(2p)})$,	
" $A^{(2)}$	" " " " " M_{n-2} " " " " " ,	
. . .	. . . . ,	
" $A^{(n-3)}$	" " " " " M_3 " " " " " .	

In the case of the net NH_n itself only the first and the third group are present; so in S_4 we find then the net $N(C_{24})$. In all other cases we have to deal with at least three groups, the first three. As we already remarked in art. 68 an other paper, also destined to complement art. 39, will contain more ample developments about these reciprocal nets.

G. *Symmetry, considerations of the theory of groups, regularity.*

111. We first determine the spaces of symmetry Sy_{n-1} of HM_n itself and afterwards those of any *hmpd.* derived from it.

Case of HM_n. — We have to investigate here how the reasoning which led us to the spaces of symmetry of the measure polytope is affected by the alternate truncation.

In the case of M_n we found two possibilities under which the space S_{n-1} bisecting orthogonally the join $A_1 A_2$ of two vertices A_1, A_2 is a space Sy_{n-1} of the polytope, i. e. that $A_1 A_2$ is either an edge or the diagonal of a face; in the first case we got the n spaces $x_i = 0$, in the second the $n(n-1)$ spaces $x_i \pm x_k = 0$. Now on the one hand it is immediately evident that the alternate truncation behaves itself differently with respect to these two groups of spaces: it destroys the symmetry property of the first and preserves that of the second. But on the other hand we have to examine whether the alternate truncation does not enervate the force of the argument by means of which we excluded the cases that $A_1 A_2$ was a diagonal of a limiting M_k of the M_n for $k > 2$, i. e. that the projections of the two regular simplexes $S(k)$ of the vertices of M_k adjacent to A_1 and to A_2 on the space normal to $A_1 A_2$ are of opposite orientation. Indeed this argumentation has to be revised, as the two simplexes $S(k)$ disappear altogether by applying the truncation and are replaced as groups of vertices of HM_k adjacent to A_1 and to A_2 by the two sets of $\frac{1}{2} k(k-1)$ vertices of M_k lying in the following layers S_{n-1} normal to $A_1 A_2$. But the two polytopes [1]) determined by these groups of vertices are neither central symmetric and maintain the property of the differently orientated projections, unless they coincide in the space S_{n-1} normally bisecting $A_1 A_2$ for $k = 4$. So any space orthogonally bisecting a diagonal of a limiting sixteencell of HM_n is an Sy_{n-1} and therefore HM_n also admits two groups of spaces Sy_{n-1}, the spaces $x_i \pm x_k = 0$ and the spaces $x_i \pm x_k \pm x_l \pm x_m = 0$. The number of the former is always $n(n-1)$, whilst that of the latter is $\frac{1}{3} n(n-1)(n-2)(n-3)$ for $n > 4$ and four for $n = 4$.

Case of the hmpd. derived from HM_n. — From the structure of the *hmpd.* it is immediately evident that a space S_{n-1} is an Sy_{n-1} for an *hmpd.* if and only if it is an Sy_{n-1} for the HM_n from which the *hmpd.* has been dirived. So we have proved the

THEOREM LXX. "Any *hmpd.* of S_n admits two groups of spaces Sy_{n-1}, viz. the $n(n-1)$ spaces $x_i \pm x_k = 0$ and the $\frac{1}{3} n(n-1)(n-2)(n-3)$ spaces $x_i \pm x_k \pm x_l \pm x_m = 0$".

112. From theorem XLIII we deduce:

THEOREM LXXI. "The order of the group of anallagmatic displacements of HM_n and of the *hmpd.* derived from it is $2^{n-2} n!$ for $n > 4$".

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} included, is $2^{n-1} n!$. In this extended group the first group of order $2^{n-1} n!$ forms a perfect subgroup".

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[1]) Compare for these polytopes: "The sections of the measure polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal", *Proceedings* of Amsterdam, vol. X, p. 495.

The proof of this theorem is to be based on the remark that the order of the group must be half of that of theorem XLIII on account of the alternate truncation.

113. As to the application of ELTE's scale of regularity we have to use theorem XLIV. We illustrate this, sticking to the original scale, by the following examples.

a). *Example* $\frac{1}{2}[\mathbf{11111}]$. Here we find one kind of edge, one kind of face, but two kinds of limiting tetrahedra, viz. tetrahedra of body import and tetrahedra of truncation import. So the contributions to the numerator are 1 from each of the three groups of vertices, edges, faces, and $\frac{1}{2}$ from the limiting bodies. So the fraction is $\frac{3+\frac{1}{2}}{5} = \frac{7}{10}$.

b). *Example* $\frac{1}{2}[\mathbf{553111}]$. Here we find three different groups of edges $(5, 3), (3, 1), \frac{1}{2}[1, 1]$. So the fraction is $\frac{1+\frac{1}{2}}{6} = \frac{1}{4}$

c). *Example* $\mathbf{N}(\mathbf{HM}_5, \mathbf{Cr}_5)$. This simple net admits one kind of edge, one kind of face, but two kinds of limiting tetrahedra, as a tetrahedron of body import of HM_5 is common to four HM_5, a tetrahedron of truncation import to two HM_5 and one Cr_5. So we find $\frac{3+\frac{1}{2}}{6} = \frac{7}{12}$.

d). *Example* $\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{NH}_6$. Here we have to deal with three groups of constituents represented with their frames in the table

$B\,..\,[321000]\,2\,...\,(2p_1\quad, 2p_2\quad, 2p_3\quad, 2p_4\quad, 2p_5\quad, 2p_6\quad)\,5, \Sigma p$ even,

$C\,..\,[221000]\,2\,...\,(2p_1\quad, 2p_2\quad, 2p_3\quad, 2p_4\quad, 2p_5\quad, 2p_6\quad)\,5, \Sigma p$ odd,

$A\,..\,\frac{1}{2}[555311]\,....\,(2p_1+1, 2p_2+1, 2p_3+1, 2p_4+1, 2p_5+1, 2p_6+1)\,5.$

So through the vertex 6, 4, 2, 0, 0, 0 pass

$$
\begin{array}{rl}
[\;\mathbf{6},\; \mathbf{4},\; \mathbf{2},\; \mathbf{0},\; \mathbf{0},\; \mathbf{0}\,] & \ldots\ldots B_1 \\
[10+\mathbf{4}, 10+\mathbf{6},\; \mathbf{2},\; \mathbf{0},\; \mathbf{0},\; \mathbf{0}\,] & \ldots\ldots B_2 \\
[10+\mathbf{4},\; \mathbf{4},\; \mathbf{2},\; \mathbf{0},\; \mathbf{0},\; \mathbf{0}\,] & \ldots\ldots C \\
-\frac{1}{2}[\,5+\mathbf{1},\; 5-\mathbf{1}, 5-\mathbf{3},\; 5-\mathbf{5},\; 5-\mathbf{5},\; 5-\mathbf{5}\,] & \ldots\ldots A_1 \\
\frac{1}{2}[\,5+\mathbf{1},\; 5-\mathbf{1}, 5-\mathbf{3}, (-5+\mathbf{5},\; 5-\mathbf{5},\; 5-\mathbf{5})] & ..\,A_2, A_3, A_4 \\
-\frac{1}{2}[\,5+\mathbf{1},\; 5-\mathbf{1}, 5-\mathbf{3}, (-5+\mathbf{5}, -5+\mathbf{5},\; 5-\mathbf{5})] & ..\,A_5, A_6, A_7 \\
\frac{1}{2}[\,5+\mathbf{1},\; 5-\mathbf{1}, 5-\mathbf{3}, -5+\mathbf{5}, -5+\mathbf{5}, -5+\mathbf{5}\,] & \ldots\ldots A_8.
\end{array}
$$

Now the edge (64)2000 belongs to all these polytopes with exception of C, 6(42)000 belongs to all with exception of B_2, whilst 64(20)00 belongs to seven only. So we find three kinds of edges and the fraction is $\frac{3}{14}$.

Remark. Only the HM_n itself admits a regularity fraction $\frac{7}{2n}$, all the *hmpd.* derived from it a fraction $\frac{3}{2n}$.

As a rule the net NH_n admits the fraction $\frac{7}{2(n+1)}$ and a net derived from it $\frac{3}{2(n+1)}$.

Groningen, December, 1912.

Fig. 13.

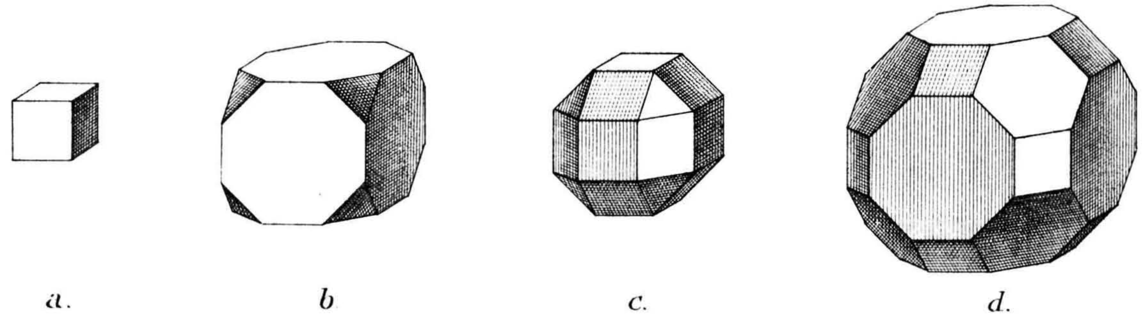


Fig. 15.

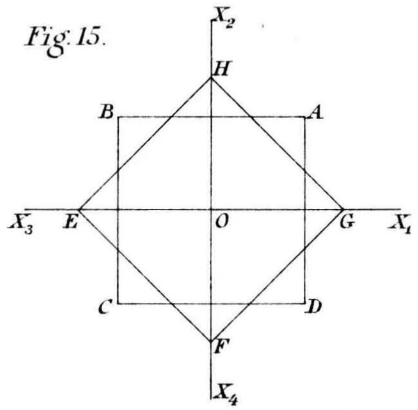


Fig. 16.

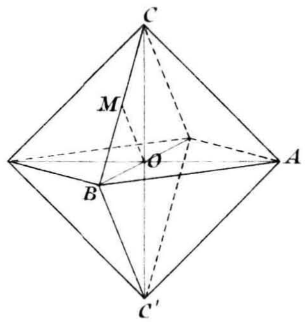


Fig. 19.

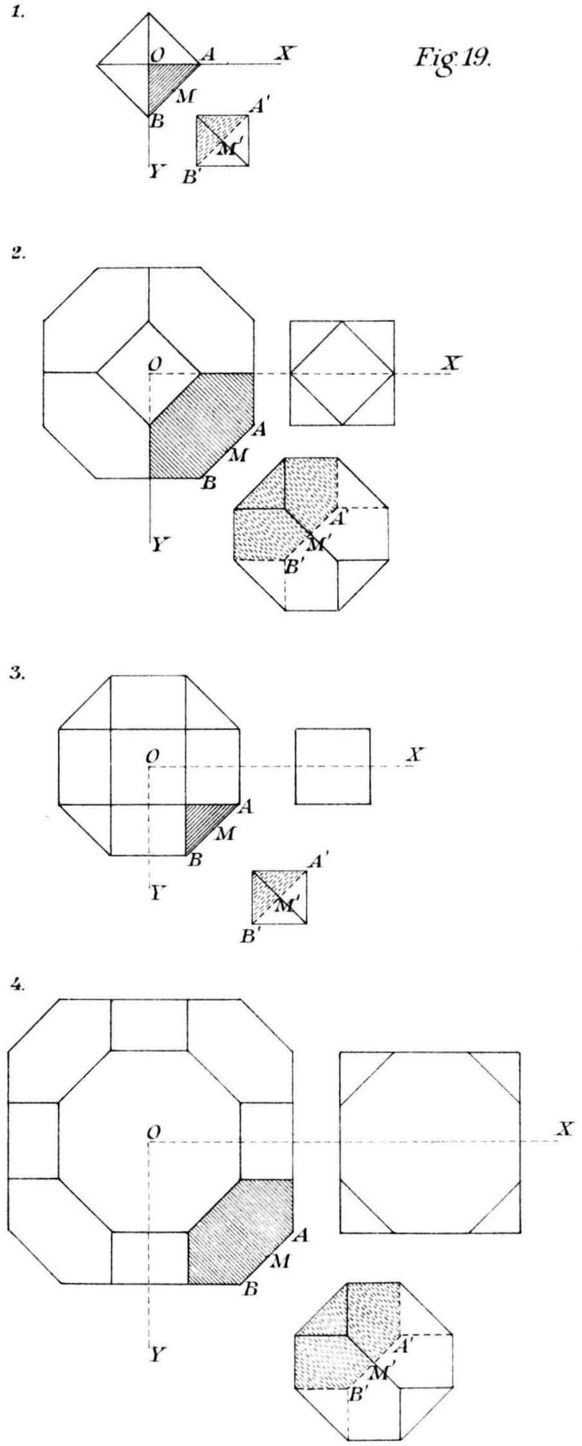


Fig. 14.

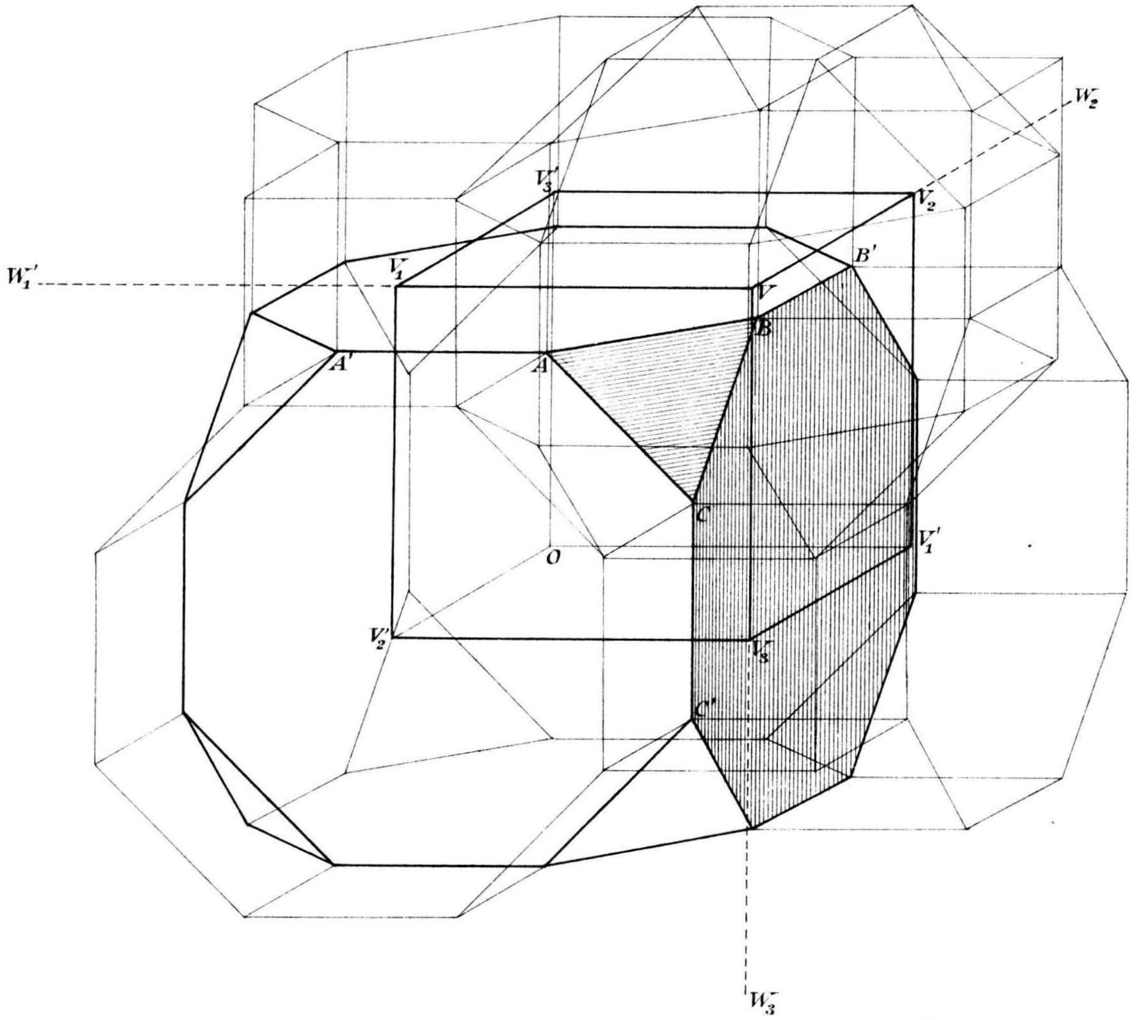


Fig. 17.

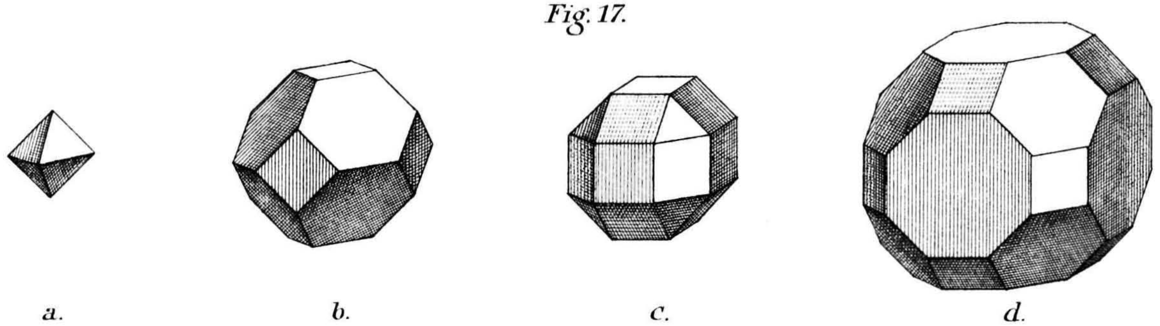


Fig. 20.

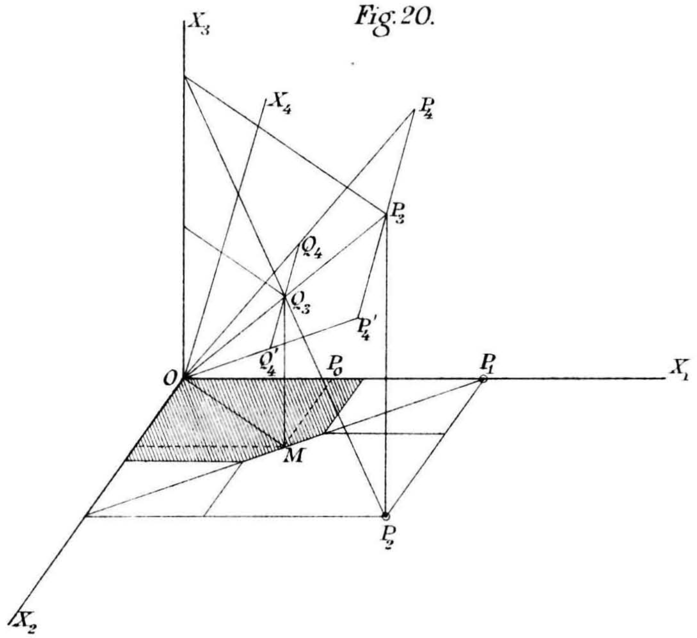
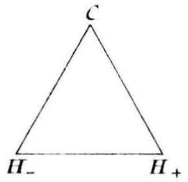


Fig. 18.



J. Bijtel lith. P. J. Mulder & Zn. impr. Leiden.

$n = 3$

						6	12	8				
C	=	$ce_2\ O$	[1 1 1]	$(8_1\ 12_1\ 6_1)$	1	p_4	—	—	1			
$e_1\ C = tC$	=	$ce_1\ e_2\ O$	[1′1′1]	$(24_1\ 36_2\ 14_2)$	$\frac{1}{2}$	p_8	—	p_3	1′	1		
$e_2\ C = RCO$	=	$e_2\ O$	[1′1 1]	$(24_1\ 48_2\ 26_3)$	$\frac{1}{2}$	p_4	p_4	p_3	1′	2, 1		
$e_1\ e_2\ C = tCO$	=	$e_1\ e_2\ O$	[2′1′1]	$(48_1\ 72_3\ 26_3)$	$\frac{1}{2}$	p_8	p_4	p_6	2′	3, 1		
$ce_1\ C = CO$	=	$ce_1\ O$	[110] $\sqrt{2}$	$(12_1\ 24_1\ 14_2)$	$\frac{2}{3}$	p_4	—	p_3	1″	1	2	1
$ce_2\ C = O$	=	O	[100] $\sqrt{2}$	$(6_1\ 12_1\ 8_1)$	1	—	—	p_3	1″	2, 1	1	
$ce_1\ e_2\ C = tO$	=	$e_1\ O$	[210] $\sqrt{2}$	$(24_1\ 36_2\ 14_2)$	$\frac{1}{2}$	p_4	—	p_6	2″	3, 1	3	1

$n = 4$

						8	24	32	16				
C_8	=	$ce_3\ C_{16}$	[1 1 1 1]	$(16_1\ 32_1\ 24_1\ 8_1)$	1	C	—	—	—	1			
$e_1\ C_8$	=	$ce_2\ e_3\ C_{16}$	[1′1′1′1]	$(64_1\ 128_2\ 88_2\ 24_2)$	$\frac{3}{8}$	tC	—	—	T	1′	1		
$e_2\ C_8$	=	$ce_1\ e_3\ C_{16}$	[1′1′1 1]	$(96_1\ 288_2\ 248_4\ 56_3)$	$\frac{3}{8}$	RCO	—	P_3	O	1′	2, 1		
$e_3\ C_8$	=	$e_3\ C_{16}$	[1′1 1 1]	$(64_1\ 192_2\ 208_3\ 80_4)$	$\frac{3}{8}$	C	P_4	P_3	T	1′	3, 2, 1		
$e_1\ e_2\ C_8$	=	$ce_1\ e_2\ e_3\ C_{16}$	[2′2′1′1]	$(192_1\ 384_3\ 248_4\ 56_3)$	$\frac{3}{8}$	tCO	—	P_3	tT	2′	3, 1		
$e_1\ e_3\ C_8$	=	$e_2\ e_3\ C_{16}$	[2′1′1′1]	$(192_1\ 480_3\ 368_5\ 80_4)$	$\frac{3}{8}$	tC	P_8	P_3	CO	2′	4, 2, 1		
$e_2\ e_3\ C_8$	=	$e_1\ e_3\ C_{16}$	[2′1′1 1]	$(192_1\ 480_3\ 368_5\ 80_4)$	$\frac{3}{8}$	RCO	P_4	P_6	tT	2′	5, 3, 1		
$e_1\ e_2\ e_3\ C_8$	=	$e_1\ e_2\ e_3\ C_{16}$	[3′2′1′1]	$(384_1\ 768_4\ 464_6\ 80_4)$	$\frac{3}{8}$	tCO	P_8	P_6	tO	3′	6, 3, 1		
$ce_1\ C_8$	=	$c\ e_2\ C_{16}$	[1110] $\sqrt{2}$	$(32_1\ 96_1\ 88_2\ 24_2)$	$\frac{1}{2}$	CO	—	—	T	1″	1	3	2, 1
$ce_2\ C_8$	$= C_{24} =$	$ce_1\ C_{16}$	[1100] $\sqrt{2}$	$(24_1\ 96_1\ 96_1\ 24_1)$	1	O	—	—	O	1″	2, 1	2	1
$ce_3\ C_8$	=	C_{16}	[1000] $\sqrt{2}$	$(8_1\ 24_1\ 32_1\ 16_1)$	1	—	—	—	T	1″	3, 2, 1	1	
$ce_1\ e_2\ C_8$	=	$ce_1\ e_2\ C_{16}$	[2210] $\sqrt{2}$	$(96_1\ 192_2\ 120_3\ 24_2)$	$\frac{3}{8}$	tO	—	—	tT	2″	3, 1	5	3, 1
$ce_1\ e_3\ C_8$	=	$e_2\ C_{16}$	[2110] $\sqrt{2}$	$(96_1\ 288_2\ 240_4\ 48_3)$	$\frac{3}{8}$	CO	P_4	—	CO	2″	4, 2, 1	4	2, 1
$ce_2\ e_3\ C_8$	=	$e_1\ C_{16}$	[2100] $\sqrt{2}$	$(48_1\ 120_2\ 96_2\ 24_2)$	$\frac{3}{8}$	O	—	—	tT	2″	5, 3, 1	3	1
$ce_1\ e_2\ e_3\ C_8$	=	$e_1\ e_2\ C_{16}$	[3210] $\sqrt{2}$	$(192_1\ 384_3\ 240_4\ 48_3)$	$\frac{3}{8}$	tO	P_4	—	tO	3″	6, 3, 1	6	3, 1

$n = 5$

						10	40	80	80	32				
C_{10}	=	$ce_4\ C_{32}$	[1 1 1 1 1]	$(32_1\ 80_1\ 80_1\ 40_1\ 10_1)$	1	C_8	—	—	—	—	1			
$e_1\ C_{10}$	=	$ce_3\ e_4\ C_{32}$	[1′1′1′1′1]	$(160_1\ 400_2\ 400_2\ 200_2\ 42_2)$	$\frac{3}{10}$	$e_1\ C_8$	—	—	—	$S(5)$	1′	1		
$e_2\ C_{10}$	=	$ce_2\ e_4\ C_{32}$	[1′1′1′1 1]	$(320_1\ 1280_2\ 1520_4\ 680_4\ 122_3)$	$\frac{3}{10}$	$e_2\ C_8$	—	—	P_T	$ce_1\ S(5)$	1′	2, 1		
$e_3\ C_{10}$	=	$ce_1\ e_4\ C_{32}$	[1′1′1 1 1]	$(320_1\ 1440_2\ 2160_4\ 1240_6\ 202_4)$	$\frac{3}{10}$	$e_3\ C_8$	—	(4 ; 3)	P_O	$ce_1\ S(5)$	1′	3, 2, 1		
$e_4\ C_{10}$	=	$e_4\ C_{32}$	[1′1 1 1 1]	$(160_1\ 640_2\ 1040_3\ 800_4\ 242_5)$	$\frac{3}{10}$	C_8	P_C	(4 ; 3)	P_T	$S(5)$	1′	4, 3, 2, 1		
$e_1\ e_2\ C_{10}$	=	$ce_2\ e_3\ e_4\ C_{32}$	[2′2′2′1′1]	$(640_1\ 1600_3\ 1520_4\ 680_4\ 122_3)$	$\frac{3}{10}$	$e_1\ e_2\ C_8$	—	—	P_T	$e_1\ S(5)$	2′	3, 1		
$e_1\ e_3\ C_{10}$	=	$ce_1\ e_3\ e_4\ C_{32}$	[2′2′1′1′1]	$(960_1\ 3360_3\ 3760_6\ 1560_7\ 202_4)$	$\frac{3}{10}$	$e_1\ e_3\ C_8$	—	(8 ; 3)	P_O	$e_2\ S(5)$	2′	4, 2, 1		
$e_1\ e_4\ C_{10}$	=	$e_3\ e_4\ C_{32}$	[2′1′1′1′1]	$(640_1\ 2240_3\ 2960_5\ 1600_7\ 242_5)$	$\frac{3}{10}$	$e_1\ C_8$	P_{tC}	(8 ; 3)	P_T	$e_3\ S(5)$	2′	5, 3, 2, 1		
$e_2\ e_3\ C_{10}$	=	$ce_1\ e_2\ e_4\ C_{32}$	[2′2′1′1 1]	$(960_1\ 2880_3\ 2960_6\ 1240_6\ 202_4)$	$\frac{3}{10}$	$e_2\ e_3\ C_8$	—	(4 ; 3)	P_{tT}	$ce_1\ e_2\ S(5)$	2′	5, 3, 1		
$e_2\ e_4\ C_{10}$	=	$e_2\ e_4\ C_{32}$	[2′1′1′1 1]	$(960_1\ 3840_3\ 4720_7\ 2080_9\ 242_5)$	$\frac{3}{10}$	$e_2\ C_8$	P_{RCO}	(4 ; 3)	P_{CO}	$e_2\ S(5)$	2′	6, 4, 2, 1		
$e_3\ e_4\ C_{10}$	=	$e_1\ e_4\ C_{32}$	[2′1′1 1 1]	$(640_1\ 2240_3\ 2880_5\ 1520_7\ 242_5)$	$\frac{3}{10}$	$e_3\ C_8$	P_C	(4 ; 6)	P_{tT}	$e_1\ S(5)$	2′	7, 5, 3, 1		
$e_1\ e_2\ e_3\ C_{10}$	=	$ce_1\ e_2\ e_3\ e_4\ C_{32}$	[3′3′2′1′1]	$(1920_1\ 4800_4\ 4240_7\ 1560_7\ 202_4)$	$\frac{3}{10}$	$e_1\ e_2\ e_3\ C_8$	—	(8 ; 3)	P_{tT}	$e_1\ e_2\ S(5)$	3′	6, 3, 1		
$e_1\ e_2\ e_4\ C_{10}$	=	$e_2\ e_3\ e_4\ C_{32}$	[3′2′2′1′1]	$(1920_1\ 5760_4\ 6000_8\ 2400_8\ 242_5)$	$\frac{3}{10}$	$e_1\ e_2\ C_8$	P_{tCO}	(8 ; 3)	P_{CO}	$e_1\ e_3\ S(5)$	3′	7, 4, 2, 1		
$e_1\ e_3\ e_4\ C_{10}$	=	$e_1\ e_3\ e_4\ C_{32}$	[3′2′1′1′1]	$(1920_1\ 5760_4\ 5760_8\ 2160_9\ 242_5)$	$\frac{3}{10}$	$e_1\ e_3\ C_8$	P_{tC}	(8 ; 6)	P_{tT}	$e_2\ e_3\ S(5)$	3′	8, 5, 3, 1		
$e_2\ e_3\ e_4\ C_{10}$	=	$e_1\ e_2\ e_4\ C_{32}$	[3′2′1′1 1]	$(1920_1\ 5760_4\ 5920_8\ 2320_9\ 242_5)$	$\frac{3}{10}$	$e_2\ e_3\ C_8$	P_{RCO}	(4 ; 6)	P_{tO}	$e_1\ e_2\ S(5)$	3′	9, 6, 3, 1		
$e_1\ e_2\ e_3\ e_4\ C_{10}$	=	$e_1\ e_2\ e_3\ e_4\ C_{32}$	[4′3′2′1′1]	$(3840_1\ 9600_5\ 8160_{10}\ 2640_{10}\ 242_5)$	$\frac{3}{10}$	$e_1\ e_2\ e_3\ C_8$	P_{tCO}	(8 ; 6)	P_{tO}	$e_1\ e_2\ e_3\ S(5)$	4′	10, 6, 3, 1		
$ce_1\ C_{10}$	=	$ce_3\ C_{32}$	[11110] $\sqrt{2}$	$(80_1\ 320_1\ 400_2\ 200_2\ 42_2)$	$\frac{2}{5}$	$ce_1\ C_8$	—	—	—	$S(5)$	1″	1	4	3, 2, 1
$ce_2\ C_{10}$	=	$ce_2\ C_{32}$	[11100] $\sqrt{2}$	$(80_1\ 480_1\ 640_2\ 280_3\ 42_2)$	$\frac{1}{2}$	$ce_2\ C_8$	—	—	—	$ce_1\ S(5)$	1″	2, 1	3	2, 1
$ce_3\ C_{10}$	=	$ce_1\ C_{32}$	[11000] $\sqrt{2}$	$(40_1\ 240_1\ 400_2\ 240_2\ 42_2)$	$\frac{1}{2}$	$ce_3\ C_8$	—	—	—	$ce_1\ S(5)$	1″	3, 2, 1	2	1
$ce_4\ C_{10}$	=	C_{32}	[10000] $\sqrt{2}$	$(10_1\ 40_1\ 80_1\ 80_1\ 32_1)$	1	—	—	—	—	$S(5)$	1″	4, 3, 2, 1	1	
$ce_1\ e_2\ C_{10}$	=	$ce_2\ e_3\ C_{32}$	[22210] $\sqrt{2}$	$(320_1\ 800_2\ 720_3\ 280_3\ 42_2)$	$\frac{3}{10}$	$ce_1\ e_2\ C_8$	—	—	—	$e_1\ S(5)$	2″	3, 1	7	5, 3, 1
$ce_1\ e_3\ C_{10}$	=	$ce_1\ e_3\ C_{32}$	[22110] $\sqrt{2}$	$(480_1\ 1920_2\ 2160_5\ 840_5\ 122_3)$	$\frac{3}{10}$	$ce_1\ e_3\ C_8$	—	(4 ; 3)	—	$e_2\ S(5)$	2″	4, 2, 1	6	4, 2, 1
$ce_1\ e_4\ C_{10}$	=	$e_3\ C_{32}$	[21110] $\sqrt{2}$	$(320_1\ 1440_2\ 2160_4\ 1200_6\ 162_4)$	$\frac{3}{10}$	$ce_1\ C_8$	P_{CO}	(4 ; 3)	—	$e_3\ S(5)$	2″	5, 3, 2, 1	5	3, 2, 1
$ce_2\ e_3\ C_{10}$	=	$ce_1\ e_2\ C_{32}$	[22100] $\sqrt{2}$	$(160_1\ 480_2\ 560_3\ 280_3\ 42_2)$	$\frac{3}{10}$	$ce_2\ e_3\ C_8$	—	—	—	$ce_1\ e_2\ S(5)$	2″	5, 3, 1	5	3, 1
$ce_2\ e_4\ C_{10}$	=	$e_2\ C_{32}$	[21100] $\sqrt{2}$	$(240_1\ 1200_2\ 1520_4\ 640_4\ 82_3)$	$\frac{3}{10}$	$ce_2\ C_8$	P_O	—	—	$e_2\ S(5)$	2″	6, 4, 2, 1	4	2, 1
$ce_3\ e_4\ C_{10}$	=	$e_1\ C_{32}$	[21000] $\sqrt{2}$	$(80_1\ 280_2\ 400_2\ 240_2\ 42_2)$	$\frac{3}{10}$	$ce_3\ C_8$	—	—	—	$e_1\ S(5)$	2″	7, 5, 3, 1	3	1
$ce_1\ e_2\ e_3\ C_{10}$	=	$ce_1\ e_2\ e_3\ C_{32}$	[33210] $\sqrt{2}$	$(960_1\ 2400_3\ 2160_5\ 840_5\ 122_3)$	$\frac{3}{10}$	$ce_1\ e_2\ e_3\ C_8$	—	(4 ; 3)	—	$e_1\ e_2\ S(5)$	3″	6, 3, 1	9	6, 3, 1
$ce_1\ e_2\ e_4\ C_{10}$	=	$e_2\ e_3\ C_{32}$	[32210] $\sqrt{2}$	$(960_1\ 2880_3\ 2960_6\ 1200_6\ 162_4)$	$\frac{3}{10}$	$ce_1\ e_2\ C_8$	P_{tO}	(4 ; 3)	—	$e_2\ e_3\ S(5)$	3″	7, 4, 2, 1	8	5, 3, 1
$ce_1\ e_3\ e_4\ C_{10}$	=	$e_1\ e_3\ C_{32}$	[32110] $\sqrt{2}$	$(960_1\ 3360_3\ 3680_6\ 1440_7\ 162_4)$	$\frac{3}{10}$	$ce_1\ e_3\ C_8$	P_{CO}	(4 ; 6)	—	$e_1\ e_3\ S(5)$	3″	8, 5, 3, 1	7	4, 2, 1
$ce_2\ e_3\ e_4\ C_{10}$	=	$e_1\ e_2\ C_{32}$	[32100] $\sqrt{2}$	$(480_1\ 1440_3\ 1520_4\ 640_4\ 82_3)$	$\frac{3}{10}$	$ce_2\ e_3\ C_8$	P_O	—	—	$e_1\ e_2\ S(5)$	3″	9, 6, 3, 1	6	3, 1
$ce_1\ e_2\ e_3\ e_4\ C_{10}$	=	$e_1\ e_2\ e_3\ C_{32}$	[43210] $\sqrt{2}$	$(1920_1\ 4800_4\ 4160_7\ 1440_7\ 162_4)$	$\frac{3}{10}$	$ce\ e_2\ e_3\ C_8$	P_{tO}	(4 ; 6)	—	$e_1\ e_2\ e_3\ S(5)$	4″	10, 6, 3, 1	10	6, 3, 1

$n = 3$

	g_3	g_2	g_1	g_0		p.	$(l)_1$	$(l)_2$	$(l)_3$			
1	$[1\,1\,1] = C$					r.	1	1	1	1		
e_1	$[1'1'1] = tC$			$[1\,0\,0]\sqrt{2} = O$	$ce_2\,e_3$		2	2	2	$\frac{3}{8}$	17	$4_3\,(o,\ 2tc_{38}),\ 1_4\,(4tc_{88})$
e_2	$[1'1\,1] = RCO$		$[1][1\,0]\sqrt{2} = P_4$	$[1\,1\,0]\sqrt{2} = CO$	$ce_1\,e_3$		2	2	3	$\frac{3}{8}$	20	$4_3\,(c,\ co_{34},\ rco_{34}),\ 2_3\,(c,\ 2rco_{44})$
$e_1\,e_2$	$[2'1'1] = tCO$		$[1][1\,0]\sqrt{2} = P_4$	$[2\,1\,0]\sqrt{2} = tO$	$ce_1\,e_2\,e_3$		3	3	3	$\frac{3}{8}$	21	$1_3\,(c,\ 2tco_{48}),\ 2_3\,(c,\ tco_{46},\ to_{46}),\ 1_3\,(2tco_{68},\ to_{66})$
e_3	$[1\,1\,1] = C$	$[1\,1][1] = P_4$	$[1][1\,1] = P_4$	$[1\,1\,1] = C$	e_3	r.	1	1	1	1		
$e_1\,e_3$	$[1'1'1] = tC$	$[1'1][1] = P_8$	$[1][1\,1] = P_4$	$[1'1\,1] = RCO$	$e_2\,e_3$		3	3	4	$\frac{3}{8}$	22	$1_4\,(c,\ 2p^8_{48},\ tc_{88}),\ 2_3\,(c,\ p^8_{44},\ rco_{34}),\ 2_3\,(p^8_{48},\ tc_{38},\ rco_{34})$
$e_1\,e_2\,e_3$	$[2'1'1] = tCO$	$[1'1][1] = P_8$	$[1][1'1] = P_8$	$[2'1'1] = tCO$	$e_1\,e_2\,e_3$	s. p.	2	3	2	$\frac{3}{8}$	24^{bis}	$2_3\,(p^8_{48},\ p^8_{44},\ tco_{48}),\ 2_3\,(p^8_{48},\ tco_{68},\ tco_{46})$
ce_1	$[1\,1\,0]\sqrt{2} = CO$			$[1\,0\,0]\sqrt{2} = O$	ce_2		1	2	2	$\frac{1}{2}$	18	$8_3\,(2co_{34},\ o)$
$ce_1\,e_2$	$[2\,1\,0]\sqrt{2} = tO$			$[2\,1\,0]\sqrt{2} = tO$	$ce_1\,e_2$	s. p.	1	2	1	$\frac{1}{2}$	14	$4_3\,(2to_{46},\ 2to_{66})$

$n = 4$

	g_4	g_3	g_2	g_1	g_0		p.	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	
1	$[1\,1\,1\,1] = 1$						r.	1	1	1	1	1
e_1	$[1'1'1'1] = e_1$				$[1\,0\,0\,0]\sqrt{2} = ce_3 = C_{16}$	$ce_3\,e_4$		2	2	2	2	$\frac{3}{10}$
e_2	$[1'1'1\,1] = e_2$			$[1][1\,0\,0]\sqrt{2} = P_O$	$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$	$ce_2\,e_4$		2	2	3	3	$\frac{3}{10}$
e_3	$[1'1\,1\,1] = e_3$		$[1\,1][1\,0]\sqrt{2} = (p_4; p_4) = C_8$	$[1][1\,1\,0]\sqrt{2} = P_{CO}$	$[1\,1\,1\,0]\sqrt{2} = ce_1$	$ce_1\,e_4$		2	2	4	4	$\frac{3}{10}$
$e_1\,e_2$	$[2'2'1'1] = e_1\,e_2$			$[1][1\,0\,0]\sqrt{2} = P_O$	$[2\,1\,0\,0]\sqrt{2} = ce_2\,e_3$	$ce_2\,e_3\,e_4$		2	4	4	3	$\frac{3}{10}$
$e_1\,e_3$	$[2'1'1'1] = e_1\,e_3$		$[1'1][1\,0]\sqrt{2} = (p_8; p_4)$	$[1][1\,1\,0]\sqrt{2} = P_{CO}$	$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$	$ce_1\,e_3\,e_4$	s. p.	2	3	5	4	$\frac{3}{10}$
$e_2\,e_3$	$[2'1'1\,1] = e_2\,e_3$		$[1\,1][1\,0]\sqrt{2} = (p_4; p_4) = C_8$	$[1][2\,1\,0]\sqrt{2} = P_{tO}$	$[2\,2\,1\,0]\sqrt{2} = ce_1\,e_2$	$ce_1\,e_2\,e_4$		2	3	5	4	$\frac{3}{10}$
$e_1\,e_2\,e_3$	$[3'2'1'1] = e_1\,e_2\,e_3$		$[1'1][1\,0]\sqrt{2} = (p_8; p_4)$	$[1][2\,1\,0]\sqrt{2} = P_{tO}$	$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$	$ce_1\,e_2\,e_3\,e_4$	s. p.	2	3	5	4	$\frac{3}{10}$
e_4	$[1\,1\,1\,1] = 1$	$[1\,1\,1][1] = P_C = C_8$	$[1\,1][1\,1] = (p_4; p_4) = C_8$	$[1][1\,1\,1] = P_C = C_8$	$[1\,1\,1\,1] = 1$	e_4	r.	1	1	1	1	1
$e_1\,e_4$	$[1'1'1'1] = e_1$	$[1'1'1][1] = P_{tC}$	$[1'1][1\,1] = (p_8; p_4)$	$[1][1\,1\,1] = P_C = C_8$	$[1'1\,1\,1] = e_3$	$e_3\,e_4$		2	3	5	5	$\frac{3}{10}$
$e_2\,e_4$	$[1'1'1\,1] = e_2$	$[1'1\,1][1] = P_{RCO}$	$[1\,1][1\,1] = (p_4; p_4) = C_8$	$[1][1'1\,1] = P_{RCO}$	$[1'1'1\,1] = e_2$	$e_2\,e_4$	s. p.	2	2	4	3	$\frac{3}{10}$
$e_1\,e_2\,e_4$	$[2'2'1'1] = e_1\,e_2$	$[2'1'1][1] = P_{tCO}$	$[1'1][1\,1] = (p_8; p_4)$	$[1][1'1\,1] = P_{RCO}$	$[2'1'1\,1] = e_2\,e_3$	$e_2\,e_3\,e_4$		2	4	7	5	$\frac{3}{10}$
$e_1\,e_3\,e_4$	$[2'1'1'1] = e_1\,e_3$	$[1'1'1][1] = P_{tC}$	$[1'1][1'1] = (p_8; p_8)$	$[1][1'1'1] = P_{tC}$	$[2'1'1'1] = e_1\,e_3$	$e_1\,e_3\,e_4$	s. p.	2	3	4	3	$\frac{3}{10}$
$e_1\,e_2\,e_3\,e_4$	$[3'2'1'1] = e_1\,e_2\,e_3$	$[2'1'1][1] = P_{tCO}$	$[1'1][1'1] = (p_8; p_8)$	$[1][2'1'1] = P_{tCO}$	$[3'2'1'1] = e_1\,e_2\,e_3$	$e_1\,e_2\,e_3\,e_4$	s. p.	2	3	5	3	$\frac{3}{10}$
ce_1	$[1\,1\,1\,0]\sqrt{2} = ce_1$				$[1\,0\,0\,0]\sqrt{2} = ce_3 = C_{16}$	ce_3		1	2	2	2	$\frac{2}{5}$
ce_2	$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$				$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$	ce_2	r.	1	1	1	1	1
$ce_1\,e_2$	$[2\,2\,1\,0]\sqrt{2} = ce_1\,e_2$				$[2\,1\,0\,0]\sqrt{2} = ce_2\,e_3$	$ce_2\,e_3$		2	3	3	2	$\frac{3}{10}$
$ce_1\,e_3$	$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$		$[1\,0]\sqrt{2}\,[1\,0]\sqrt{2} = (p_4; p_4) = C_8$		$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$	$ce_1\,e_3$	s. p.	1	2	2	2	$\frac{2}{5}$
$ce_1\,e_2\,e_3$	$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$		$[1\,0]\sqrt{2}\,[1\,0]\sqrt{2} = (p_4; p_4) = C_8$		$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$	$ce_1\,e_2\,e_3$	s. p.	2	2	2	2	$\frac{3}{10}$

	g_5	g_4	g_3	g	g_1	g_0	$p.$	$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	
1	$[1\,1\,1\,1\,1]=$ M_5						$r.$					1
e_1	$[1'1'1'1'1]=$ e_1 ,,					$[10000]\sqrt{2}=$ $ce_4\,M_5$		1	2			$\frac{1}{4}$
e_2	$[1'1'1'1\,1]=$ e_2 ,,				$[1][1000]\sqrt{2}=P_{ce_3C_8}$	$[11000]\sqrt{2}=$ ce_3 ,,		1	2			$\frac{1}{4}$
e_3	$[1'1'1\,1\,1]=$ e_3 ,,			$[1\,1][100]\sqrt{2}=(p_4;O)$	$[1][1100]\sqrt{2}=P_{ce_2C_8}$	$[11100]\sqrt{2}=$ ce_2 ,,		1	2			$\frac{1}{4}$
e_4	$[1'1\,1\,1\,1]=$ e_4 ,,		$[1\,1\,1][10]\sqrt{2}=(C;p_4)=M_5$	$[1\,1][110]\sqrt{2}=(p_4;CO)$	$[1][1110]\sqrt{2}=P_{ce_1C_8}$	$[11110]\sqrt{2}=$ ce_1 ,,		1	2			$\frac{1}{4}$
$e_1\,e_2$	$[2'2'2'1'1]=$ $e_1\,e_2$,,				$[1][1000]\sqrt{2}=P_{ce_3C_8}$	$[21000]\sqrt{2}=$ $ce_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_3$	$[2'2'1'1'1]=$ $e_1\,e_3$,,			$[1'1][100]\sqrt{2}=(p_8;O)$	$[1][1100]\sqrt{2}=P_{ce_2C_8}$	$[21100]\sqrt{2}=$ $ce_2\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_4$	$[2'1'1'1'1]=$ $e_1\,e_4$,,		$[1'1'1][10]\sqrt{2}=(tC;p_4)$	$[1'1][110]\sqrt{2}=(p_8;CO)$	$[1][1110]\sqrt{2}=P_{ce_1C_8}$	$[21110]\sqrt{2}=$ $ce_1\,e_4$,,		1	2			$\frac{1}{4}$
$e_2\,e_3$	$[2'2'1'1\,1]=$ $e_2\,e_3$,,			$[1\,1][100]\sqrt{2}=(p_4;O)$	$[1][2100]\sqrt{2}=P_{ce_2e_3C_8}$	$[22100]\sqrt{2}=$ $ce_2\,e_3$,,		1	2			$\frac{1}{4}$
$e_2\,e_4$	$[2'1'1'1\,1]=$ $e_2\,e_4$,,		$[1'1\,1][10]\sqrt{2}=(RCO;p_4)$	$[1\,1][110]\sqrt{2}=(p_4;CO)$	$[1][2110]\sqrt{2}=P_{ce_1e_3C_8}$	$[22110]\sqrt{2}=$ $ce_1\,e_3$,,		1	2			$\frac{1}{4}$
$e_3\,e_4$	$[2'1'1\,1\,1]=$ $e_3\,e_4$,,		$[1\,1\,1][10]\sqrt{2}=(C;p_4)=M_5$	$[1\,1][210]\sqrt{2}=(p_4;tO)$	$[1][2210]\sqrt{2}=P_{ce_1e_2C_8}$	$[22210]\sqrt{2}=$ $ce_1\,e_2$,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_3$	$[3'3'2'1'1]=$ $e_1\,e_2\,e_3$,,			$[1'1][100]\sqrt{2}=(p_8;O)$	$[1][2100]\sqrt{2}=P_{ce_2e_3C_8}$	$[32100]\sqrt{2}=$ $ce_2\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_4$	$[3'2'2'1'1]=$ $e_1\,e_2\,e_4$,,		$[2'1'1][10]\sqrt{2}=(tCO;p_4)$	$[1'1][110]\sqrt{2}=(p_8;CO)$	$[1][2110]\sqrt{2}=P_{ce_1e_3C_8}$	$[32110]\sqrt{2}=$ $ce_1\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_3\,e_4$	$[3'2'1'1'1]=$ $e_1\,e_3\,e_4$,,		$[1'1'1][10]\sqrt{2}=(tC;p_4)$	$[1'1][210]\sqrt{2}=(p_8;tO)$	$[1][2210]\sqrt{2}=P_{ce_1e_2C_8}$	$[32210]\sqrt{2}=$ $ce_1\,e_2\,e_4$,,		1	2			$\frac{1}{4}$
$e_2\,e_3\,e_4$	$[3'2'1'1\,1]=$ $e_2\,e_3\,e_4$,,		$[1'1\,1][10]\sqrt{2}=(RCO;p_4)$	$[1\,1][210]\sqrt{2}=(p_4;tO)$	$[1][3210]\sqrt{2}=P_{ce_1e_2e_3C_8}$	$[33210]\sqrt{2}=$ $ce_1\,e_2\,e_3$,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_3\,e_4$	$[4'3'2'1'1]=$ $e_1\,e_2\,e_3\,e_4$,,		$[2'1'1][10]\sqrt{2}=(tCO;p_4)$	$[1'1][210]\sqrt{2}=(p_8;tO)$	$[1][3210]\sqrt{2}=P_{ce_1e_2e_3C_8}$	$[43210]\sqrt{2}=$ $ce_1\,e_2\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
e_5	$[1\,1\,1\,1\,1]=$ M_5	$[1\,1\,1\,1][1]=P_{C_8}=M_5$	$[1\,1\,1]\,[1\,1]=(C;p_4)=M_5$	$[1\,1]\,[1\,1\,1]=(p_4;C)=M_5$	$[1]\,[1\,1\,1\,1]=P_{C_8}=M_5$	$[1\,1\,1\,1\,1]=$ M_5	$r.$					1
$e_1\,e_5$	$[1'1'1'1'1]=$ e_1 ,,	$[1'1'1'1][1]=P_{e_1C_8}$	$[1'1'1]\,[1\,1]=(tC;p_4)$	$[1'1]\,[1\,1\,1]=(p_8;C)$	$[1]\,[1\,1\,1\,1]=P_{C_8}=M_5$	$[1'1\,1\,1\,1]=$ e_4 ,,		1	2			$\frac{1}{4}$
$e_2\,e_5$	$[1'1'1'1\,1]=$ e_2 ,,	$[1'1'1\,1][1]=P_{e_2C_8}$	$[1'1\,1]\,[1\,1]=(RCO;p_4)$	$[1\,1]\,[1\,1\,1]=(p_4;C)=M_5$	$[1]\,[1'1\,1\,1]=P_{c_3C_8}$	$[1'1'1\,1\,1]=$ e_3 ,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_5$	$[2'2'2'1'1]=$ $e_1\,e_2$,,	$[2'2'1'1][1]=P_{e_1e_2C_8}$	$[2'1'1]\,[1\,1]=(tCO;p_4)$	$[1'1]\,[1\,1\,1]=(p_8;C)$	$[1]\,[1'1\,1\,1]=P_{e_3C_8}$	$[2'1'1\,1\,1]=$ $e_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_3\,e_5$	$[2'2'1'1'1]=$ $e_1\,e_3$,,	$[2'1'1'1][1]=P_{e_1e_3C_8}$	$[1'1'1]\,[1\,1]=(tC;p_4)$	$[1'1]\,[1'1\,1]=(p_8;RCO)$	$[1]\,[1'1'1\,1]=P_{e_2C_8}$	$[2'1'1'1\,1]=$ $e_2\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_4\,e_5$	$[2'1'1'1'1]=$ $e_1\,e_4$,,	$[1'1'1'1][1]=P_{e_1C_8}$	$[1'1'1]\,[1'1]=(tC;p_8)$	$[1'1]\,[1'1'1]=(p_8;tC)$	$[1]\,[1'1'1'1]=P_{e_1C_8}$	$[2'1'1'1'1]=$ $e_1\,e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$
$e_2\,e_3\,e_5$	$[2'2'1'1\,1]=$ $e_2\,e_3$,,	$[2'1'1'1][1]=P_{e_2e_3C_8}$	$[1'1\,1]\,[1\,1]=(RCO;p_4)$	$[1\,1]\,[1'1\,1]=(p_4;RCO)$	$[1]\,[2'1'1\,1]=P_{e_2e_3C_8}$	$[2'2'1'1\,1]=$ $e_2\,e_3$,,	$s.\,p.$	1	2			$\frac{1}{4}$
$e_1\,e_2\,e_3\,e_5$	$[3'3'2'1'1]=$ $e_1\,e_2\,e_3$,,	$[3'2'1'1][1]=P_{e_1e_2e_3C_8}$	$[2'1'1]\,[1\,1]=(tCO;p_4)$	$[1'1]\,[1'1\,1]=(p_8;RCO)$	$[1]\,[2'1'1\,1]=P_{e_2e_3C_8}$	$[3'2'1'1\,1]=$ $e_2\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_4\,e_5$	$[3'2'2'1'1]=$ $e_1\,e_2\,e_4$,,	$[2'2'1'1][1]=P_{e_1e_2C_8}$	$[2'1'1]\,[1'1]=(tCO;p_8)$	$[1'1]\,[1'1'1]=(p_8;tC)$	$[1]\,[2'1'1'1]=P_{e_1e_3C_8}$	$[3'2'1'1'1]=$ $e_1\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$e_1\,e_2\,e_3\,e_4\,e_5$	$[4'3'2'1'1]=$ $e_1\,e_2\,e_3\,e_4$,,	$[3'2'1'1][1]=P_{e_1e_2e_3C_8}$	$[2'1'1]\,[1'1]=(tCO;p_8)$	$[1'1]\,[2'1'1]=(p_8;tCO)$	$[1]\,[3'2'1'1]=P_{e_1e_2e_3C_3}$	$[4'3'2'1'1]=$ $e_1\,e_2\,e_3\,e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$
ce_1	$[11110]\sqrt{2}=$ $ce_1\,M_5$					$[10000]\sqrt{2}=$ $ce_4\,M_5$		1	1	2		$\frac{1}{3}$
ce_2	$[11100]\sqrt{2}=$ ce_2 ,,					$[11000]\sqrt{2}=$ ce_3 ,,		1	1	1	2	$\frac{1}{2}$
$ce_1\,e_2$	$[22210]\sqrt{2}=$ $ce_1\,e_2$,,					$[21000]\sqrt{2}=$ $ce_3\,e_4$,,		1	2			$\frac{1}{4}$
$ce_1\,e_3$	$[22110]\sqrt{2}=$ $ce_1\,e_3$,,			$[10]\sqrt{2}[100]\sqrt{2}=(p_4;O)$		$[21100]\sqrt{2}=$ $ce_2\,e_4$,,		1	2			$\frac{1}{4}$
$ce_1\,e_4$	$[21110]\sqrt{2}=$ $ce_1\,e_4$,,		$[110]\sqrt{2}[10]\sqrt{2}=(CO;p_4)$	$[10]\sqrt{2}[110]\sqrt{2}=(p_4;CO)$		$[21110]\sqrt{2}=$ $ce_1\,e_4$,,	$s.\,p.$	1	1	2		$\frac{1}{3}$
$ce_2\,e_3$	$[22100]\sqrt{2}=$ $ce_2\,e_3$,,					$[22100]\sqrt{2}=$ $ce_2\,e_3$,,	$s.\,p.$	1	1	2		$\frac{1}{3}$
$ce_1\,e_2\,e_3$	$[33210]\sqrt{2}=$ $ce_1\,e_2\,e_3$,,			$[10]\sqrt{2}[100]\sqrt{2}=(p_4;O)$		$[32100]\sqrt{2}=$ $ce_2\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$ce_1\,e_2\,e_4$	$[32210]\sqrt{2}=$ $ce_1\,e_2\,e_4$,,		$[210]\sqrt{2}[10]\sqrt{2}=(tO;p_4)$	$[10]\sqrt{2}[110]\sqrt{2}=(p_4;CO)$		$[32110]\sqrt{2}=$ $ce_1\,e_3\,e_4$,,		1	2			$\frac{1}{4}$
$ce_1\,e_2\,e_3\,e_4$	$[43210]\sqrt{2}=$ $ce_1\,e_2\,e_3\,e_4$,,		$[210]\sqrt{2}[10]\sqrt{2}=(tO;p_4)$	$[10]\sqrt{2}[210]\sqrt{2}=(p_4;tO)$		$[43210]\sqrt{2}=$ $ce_1\,e_2\,e_3\,e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$

CROSS POLYTOPE NETS (IN S_4).

Table VII.

			g_4		g_3		g_2		g_1		g_0		
1	1	1	$[2, 0, 0, 0]$									ce_4	1
2	e_1	e_1	$[4, 2, 0, 0]$							ce_3	$[2, 2, 0, 0]$	$ce_3 e_4$	$\frac{3}{10}$
3	e_2	e_2	$[4, 2, 2, 0]$					P_C	$[2, 0][2, 0]$	ce_2	$[4, 2, 2, 0]$	$ce_2 e_4$	$\frac{2}{5}$
4	e_3	e_3	$[2+\sqrt2, \sqrt2, \sqrt2, \sqrt2]$			(3 ; 3)	$(\frac43-\frac13\sqrt2, -\frac23-\frac13\sqrt2, -\frac23-\frac13\sqrt2)[\sqrt2]$ $(\frac43+\frac23\sqrt2, -\frac23+\frac23\sqrt2, -\frac23+\frac23\sqrt2)\ 0$	P_{CO}	$\frac12[\sqrt2+1, \sqrt2-1][\sqrt2, \sqrt2]$ $[1, 1][2\sqrt2, 0]$	ce_1	$[3\sqrt2, \sqrt2, \sqrt2, \sqrt2]$ $[2\sqrt2, 2\sqrt2, 2\sqrt2, 0]$	$ce_1 e_4$	$\frac{3}{10}$
5	$e_1 e_2$	$e_1 e_2$	$[6, 4, 2, 0]$					P_C	$[2, 0][2, 0]$	$ce_2 e_3$	$[6, 4, 2, 0]$	$ce_2 e_3 e_4$	$\frac{3}{10}$
6	$e_1 e_3$	$e_1 e_3$	$[4+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$			(3 ; 6)	$(2-\frac13\sqrt2, -\frac13\sqrt2, -2-\frac13\sqrt2)[\sqrt2]$ $(2+\frac23\sqrt2, \frac23\sqrt2, -2+\frac23\sqrt2)\ 0$	P_{CO}	$\frac12[\sqrt2+1, \sqrt2-1][\sqrt2, \sqrt2]$ $[1, 1][2\sqrt2, 0]$	$ce_1 e_3$	$[2+3\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$ $[2+2\sqrt2, 2+2\sqrt2, 2\sqrt2, 0]$	$ce_1 e_3 e_4$	$\frac{3}{10}$
7	$e_2 e_3$	$e_2 e_3$	$[4+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$			(3 ; 3)	$(4+\sqrt2, 2+\sqrt2, 2+\sqrt2)[\sqrt2]$ $(4+2\sqrt2, 2+2\sqrt2, 2+2\sqrt2)\ 0$	P_{tC}	$\frac12[2+\sqrt2, \sqrt2][2+\sqrt2, \sqrt2]$ $[2, 0][2+2\sqrt2, 0]$	$ce_1 e_2$	$[4+3\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$ $[4+2\sqrt2, 2+2\sqrt2, 2+2\sqrt2, 0]$	$ce_1 e_2 e_4$	$\frac{3}{10}$
8	$e_1 e_2 e_3$	$e_1 e_2 e_3$	$[6+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$			(3 ; 6)	$(2-\frac13\sqrt2, -\frac13\sqrt2, -2-\frac13\sqrt2)[\sqrt2]$ $(2+\frac23\sqrt2, \frac23\sqrt2, -2+\frac23\sqrt2)\ 0$	P_{tC}	$\frac12[2+\sqrt2, \sqrt2][2+\sqrt2, \sqrt2]$ $[2, 0][2+2\sqrt2, 0]$	$ce_1 e_2 e_3$	$[6+3\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$ $[6+2\sqrt2, 4+2\sqrt2, 2+2\sqrt2, 0]$	$ce_1 e_2 e_3 e_4$	$\frac{3}{10}$
9	e_4	1	$[2, 0, 0, 0]$	P_T	$(\frac32-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac12-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac12+\frac12\sqrt2)$	(3 ; 3)	$(\frac43-\frac23\sqrt2, -\frac23-\frac23\sqrt2, -\frac23-\frac23\sqrt2)\ 0$ $(\frac43+\frac13\sqrt2, -\frac23+\frac13\sqrt2, -\frac23+\frac13\sqrt2)[\sqrt2]$	P_O	$\frac12[\sqrt2+1, \sqrt2-1]\ 0, 0$ $(1, -1)[\sqrt2, \sqrt2]$	1	$[2\sqrt2, 0, 0, 0]$ $[\sqrt2, \sqrt2, \sqrt2, \sqrt2]$	e_4	$\frac{3}{10}$
10	$e_1 e_4$	e_1	$[4, 2, 0, 0]$	P_{tT}	$(\frac52-\frac12\sqrt2, \frac12-\frac12\sqrt2, -\frac32-\frac12\sqrt2, -\frac32-\frac12\sqrt2)$ $(\frac52+\frac12\sqrt2, \frac12+\frac12\sqrt2, -\frac32+\frac12\sqrt2, -\frac32+\frac12\sqrt2)$	(3 ; 6)	$(2-\frac23\sqrt2, -\frac23\sqrt2, -2-\frac23\sqrt2)\ 0$ $(2+\frac13\sqrt2, \frac13\sqrt2, -2+\frac13\sqrt2)[\sqrt2]$	P_O	$\frac12[\sqrt2+1, \sqrt2-1]\ 0, 0$ $(1, -1)[\sqrt2, \sqrt2]$	e_3	$[2+2\sqrt2, 2, 0, 0]$ $[2+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$	$e_3 e_4$	$\frac{3}{10}$
11	$e_2 e_4$	e_2	$[4, 2, 2, 0]$	P_{CO}	$(2-\frac12\sqrt2, -\frac12\sqrt2, -\frac12\sqrt2, -2-\frac12\sqrt2)$ $(2+\frac12\sqrt2, \frac12\sqrt2, \frac12\sqrt2, -2+\frac12\sqrt2)$	(3 ; 3)	$(\frac43-\frac23\sqrt2, -\frac23-\frac23\sqrt2, -\frac23-\frac23\sqrt2)\ 0$ $(\frac43+\frac13\sqrt2, -\frac23+\frac13\sqrt2, -\frac23-\frac13\sqrt2)[\sqrt2]$	P_{RCO}	$\frac12[2+\sqrt2, \sqrt2][2, 0]$ $[2, 0][2+\sqrt2, \sqrt2]$	e_2	$[4+2\sqrt2, 2, 2, 0]$ $[4+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$	$e_2 e_4$	$\frac{3}{10}$
12	$e_3 e_4$	e_3	$[2+\sqrt2, \sqrt2, \sqrt2, \sqrt2]$	P_T	$(\frac32-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac12-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac12+\frac12\sqrt2)$	(6 ; 3)	$(\frac43-\sqrt2, -\frac23-\sqrt2, -\frac23-\sqrt2)[\sqrt2]$ $(\frac43+\sqrt2, -\frac23+\sqrt2, -\frac23+\sqrt2)[\sqrt2]$ $(\frac43, -\frac23, -\frac23)[2\sqrt2]$	P_{tO}	$\frac12[2\sqrt2+1, 2\sqrt2-1][\sqrt2, \sqrt2]$ $(1, -1)[3\sqrt2, \sqrt2]$ $\frac12[\sqrt2+1, \sqrt2-1][2\sqrt2, 2\sqrt2]$	e_1	$[5\sqrt2, \sqrt2, \sqrt2, \sqrt2]$ $[3\sqrt2, 3\sqrt2, 3\sqrt2, \sqrt2]$ $[4\sqrt2, 2\sqrt2, 2\sqrt2; 2\sqrt2]$	$e_1 e_4$	$\frac{3}{10}$
13	$e_1 e_2 e_4$	$e_1 e_2$	$[6, 4, 2, 0]$	P_{tO}	$(3-\frac12\sqrt2, 1-\frac12\sqrt2, -1-\frac12\sqrt2, -3-\frac12\sqrt2)$ $(3+\frac12\sqrt2, 1+\frac12\sqrt2, -1+\frac12\sqrt2, -3+\frac12\sqrt2)$	(3 ; 6)	$(2-\frac23\sqrt2, -\frac23\sqrt2, -2-\frac23\sqrt2)\ 0$ $(2+\frac13\sqrt2, \frac13\sqrt2, -2+\frac13\sqrt2)[\sqrt2]$	P_{RCO}	$\frac12[2+\sqrt2, \sqrt2][2, 0]$ $[2, 0][2+\sqrt2, \sqrt2]$	$e_2 e_3$	$[6+2\sqrt2, 4, 2, 0]$ $[6+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$	$e_2 e_3 e_4$	$\frac{3}{10}$
14	$e_1 e_3 e_4$	$e_1 e_3$	$[4+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$	P_{tT}	$(\frac52-\frac12\sqrt2, \frac12-\frac12\sqrt2, -\frac32-\frac12\sqrt2, -\frac32-\frac12\sqrt2)$ $(\frac52+\frac12\sqrt2, \frac12+\frac12\sqrt2, -\frac32+\frac12\sqrt2, -\frac32+\frac12\sqrt2)$	(6 ; 6)	$(2-\sqrt2, -\sqrt2, -2-\sqrt2)[\sqrt2]$ $(2+\sqrt2, \sqrt2, -2+\sqrt2)[\sqrt2]$ $(2, 0, -2)[2\sqrt2]$	P_{tO}	$\frac12[2\sqrt2+1, 2\sqrt2-1][\sqrt2, \sqrt2]$ $(1, -1)[3\sqrt2, \sqrt2]$ $\frac12[\sqrt2+1, \sqrt2-1][2\sqrt2, 2\sqrt2]$	$e_1 e_3$	$[2+5\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$ $[2+3\sqrt2, 2+3\sqrt2, 3\sqrt2, \sqrt2]$ $[2+4\sqrt2, 2+2\sqrt2, 2\sqrt2, 2\sqrt2]$	$e_1 e_3 e_4$	$\frac{3}{10}$
15	$e_2 e_3 e_4$	$e_2 e_3$	$[4+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$	P_{CO}	$(2-\frac12\sqrt2, -\frac12\sqrt2, -\frac12\sqrt2, -2-\frac12\sqrt2)$ $(2+\frac12\sqrt2, \frac12\sqrt2, \frac12\sqrt2, -2+\frac12\sqrt2)$	(6 ; 3)	$(\frac43-\sqrt2, -\frac23-\sqrt2, -\frac23-\sqrt2)[\sqrt2]$ $(\frac43+\sqrt2, -\frac23+\sqrt2, -\frac23+\sqrt2)[\sqrt2]$ $(\frac43, -\frac23, -\frac23)\ 0$	P_{tCO}	$\frac12[2+2\sqrt2, 2\sqrt2][2+\sqrt2, \sqrt2]$ $[2, 0][2+3\sqrt2, \sqrt2]$ $\frac12[2+\sqrt2, \sqrt2][2+2\sqrt2, 2\sqrt2]$	$e_1 e_2$	$[4+5\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$ $[4+3\sqrt2, 2+3\sqrt2, 2+3\sqrt2, \sqrt2]$ $[4+4\sqrt2, 2+2\sqrt2, 2+2\sqrt2, 2\sqrt2]$	$e_1 e_2 e_4$	$\frac{3}{10}$
16	$e_1 e_2 e_3 e_4$	$e_1 e_2 e_3$	$[6+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$	P_{tO}	$(3-\frac12\sqrt2, 1-\frac12\sqrt2, -1-\frac12\sqrt2, -3-\frac12\sqrt2)$ $(3+\frac12\sqrt2, 1+\frac12\sqrt2, -1+\frac12\sqrt2, -3+\frac12\sqrt2)$	(6 ; 6)	$(2-\sqrt2, -\sqrt2, -2-\sqrt2)[\sqrt2]$ $(2+\sqrt2, \sqrt2, -2+\sqrt2)[\sqrt2]$ $(2, 0, -2)[2\sqrt2]$	P_{tCO}	$\frac12[2+2\sqrt2, 2\sqrt2][2+\sqrt2, \sqrt2]$ $[2, 0][2+3\sqrt2, \sqrt2]$ $\frac12[2+\sqrt2, \sqrt2][2+2\sqrt2, 2\sqrt2]$	$e_1 e_2 e_3$	$[6+5\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$ $[6+3\sqrt2, 4+3\sqrt2, 2+3\sqrt2, \sqrt2]$ $[6+4\sqrt2, 4+2\sqrt2, 2+2\sqrt2, 2\sqrt2]$	$e_1 e_2 e_3 e_4$	$\frac{3}{10}$
17	ce_1	ce_1	$[2, 2, 0, 0]$							ce_3	$[2, 2, 0, 0]$	ce_3	1
18	ce_2	ce_2	$[2, 2, 2, 0]$							ce_2	$[4, 2, 2, 0]$	ce_2	$\frac{2}{5}$
19	ce_3	ce_3	$[2, 2, 2, 2]$							ce_1	$[3\sqrt2, \sqrt2, \sqrt2, \sqrt2]$ $[2\sqrt2, 2\sqrt2, 2\sqrt2, 0]$	ce_1	$\frac{2}{5}$
20	$ce_1 e_2$	$ce_1 e_2$	$[4, 4, 2, 0]$							$ce_2 e_3$	$[6, 4, 2, 0]$	$ce_2 e_3$	$\frac{3}{10}$
21	$ce_1 e_3$	$ce_1 e_3$	$[2+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$			(3 ; 3)	$(\frac23-\frac13\sqrt2, \frac23-\frac13\sqrt2, -\frac43-\frac13\sqrt2)[\sqrt2]$ $(\frac23+\frac23\sqrt2, \frac23+\frac23\sqrt2, -\frac43+\frac23\sqrt2)\ 0$			$ce_1 e_3$	$[2+3\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$ $[2+2\sqrt2, 2+2\sqrt2, 2\sqrt2, 0]$	$ce_1 e_3$	$\frac{3}{10}$
22	$ce_2 e_3$	$ce_2 e_3$	$[2+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$							$ce_1 e_2$	$[4+3\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$ $[4+2\sqrt2, 2+2\sqrt2, 2+2\sqrt2, 0]$	$ce_1 e_2$	$\frac{3}{10}$
23	$ce_1 e_2 e_3$	$ce_1 e_2 e_3$	$[4+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$			(3 ; 3)	$(\frac23-\frac13\sqrt2, \frac23-\frac13\sqrt2, -\frac43-\frac13\sqrt2)[\sqrt2]$ $(\frac23+\frac23\sqrt2, \frac23+\frac23\sqrt2, -\frac43+\frac23\sqrt2)\ 0$			$ce_1 e_2 e_3$	$[6+3\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$ $[6+2\sqrt2, 4+2\sqrt2, 2+2\sqrt2, 0]$	$ce_1 e_2 e_3$	$\frac{3}{10}$
24	ce_4									1	$[2\sqrt2, 0, 0, 0]$ $[\sqrt2, \sqrt2, \sqrt2, \sqrt2]$	1	1
25	$ce_1 e_4$	ce_1	$[2, 2, 0, 0]$	P_O	$(1-\frac12\sqrt2, 1-\frac12\sqrt2, -1-\frac12\sqrt2, -1-\frac12\sqrt2)$ $(1+\frac12\sqrt2, 1+\frac12\sqrt2, -1+\frac12\sqrt2, -1+\frac12\sqrt2)$	(3 ; 3)	$(\frac23+\frac13\sqrt2, \frac23+\frac13\sqrt2, -\frac43+\frac13\sqrt2)[\sqrt2]$ $(\frac23-\frac23\sqrt2, \frac23-\frac23\sqrt2, -\frac43-\frac23\sqrt2)\ 0$			e_3	$[2+2\sqrt2, 2, 0, 0]$ $[2+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$	e_3	$\frac{3}{10}$
26	$ce_2 e_4$	ce_2	$[2, 2, 2, 0]$	P_T	$(\frac12-\frac12\sqrt2, \frac12-\frac12\sqrt2, \frac12-\frac12\sqrt2, -\frac32-\frac12\sqrt2)$ $(\frac12+\frac12\sqrt2, \frac12+\frac12\sqrt2, \frac12+\frac12\sqrt2, -\frac32+\frac12\sqrt2)$					e_2	$[4+2\sqrt2, 2, 2, 0]$ $[4+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$	e_2	$\frac{3}{10}$
27	$ce_3 e_4$	ce_3	$[2, 2, 2, 2]$							e_1	$[5\sqrt2, \sqrt2, \sqrt2, \sqrt2]$ $[3\sqrt2, 3\sqrt2, 3\sqrt2, \sqrt2]$ $[4\sqrt2, 2\sqrt2, 2\sqrt2, 2\sqrt2]$	e_1	$\frac{2}{5}$
28	$ce_1 e_2 e_4$	$ce_1 e_2$	$[4, 4, 2, 0]$	P_{tT}	$(\frac32-\frac12\sqrt2, \frac32-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac52-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2, \frac32+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac52+\frac12\sqrt2)$	(3 ; 3)	$(\frac23+\frac13\sqrt2, \frac23+\frac13\sqrt2, -\frac43+\frac13\sqrt2)[\sqrt2]$ $(\frac23-\frac23\sqrt2, \frac23-\frac23\sqrt2, -\frac43-\frac23\sqrt2)\ 0$			$e_2 e_3$	$[6+2\sqrt2, 4, 2, 0]$ $[6+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$	$e_2 e_3$	$\frac{3}{10}$
29	$ce_1 e_3 e_4$	$ce_1 e_3$	$[2+\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$	P_O	$(1-\frac12\sqrt2, 1-\frac12\sqrt2, -1-\frac12\sqrt2, -1-\frac12\sqrt2)$ $(1+\frac12\sqrt2, 1+\frac12\sqrt2, -1+\frac12\sqrt2, -1+\frac12\sqrt2)$	(6 ; 3)	$(\frac23-\sqrt2, \frac23-\sqrt2, -\frac43-\sqrt2)[\sqrt2]$ $(\frac23+\sqrt2, \frac23+\sqrt2, -\frac43+\sqrt2)[\sqrt2]$ $(\frac23, \frac23, -\frac43)[2\sqrt2]$			$e_1 e_3$	$[2+5\sqrt2, 2+\sqrt2, \sqrt2, \sqrt2]$ $[2+3\sqrt2, 2+3\sqrt2, 3\sqrt2, \sqrt2]$ $[2+4\sqrt2, 2+2\sqrt2, 2\sqrt2, 2\sqrt2]$	$e_1 e_3$	$\frac{3}{10}$
30	$ce_2 e_3 e_4$	$ce_2 e_3$	$[2+\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$	P_T	$(\frac12-\frac12\sqrt2, \frac12-\frac12\sqrt2, \frac12-\frac12\sqrt2, -\frac32-\frac12\sqrt2)$ $(\frac12+\frac12\sqrt2, \frac12+\frac12\sqrt2, \frac12+\frac12\sqrt2, -\frac32+\frac12\sqrt2)$					$e_1 e_2$	$[4+5\sqrt2, 2+\sqrt2, 2+\sqrt2, \sqrt2]$ $[4+3\sqrt2, 2+3\sqrt2, 2+3\sqrt2, \sqrt2]$ $[4+4\sqrt2, 2+2\sqrt2, 2+2\sqrt2, 2\sqrt2]$	$e_1 e_2$	$\frac{3}{10}$
31	$ce_1 e_2 e_3 e_4$	$ce_1 c_2 c_3$	$[4+\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$	P_{tT}	$(\frac32-\frac12\sqrt2, \frac32-\frac12\sqrt2, -\frac12-\frac12\sqrt2, -\frac52-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2, \frac32+\frac12\sqrt2, -\frac12+\frac12\sqrt2, -\frac52+\frac12\sqrt2)$	(6 ; 3)	$(\frac23-\sqrt2, \frac23-\sqrt2, -\frac43-\sqrt2)[\sqrt2]$ $(\frac23+\sqrt2, \frac23+\sqrt2, -\frac43+\sqrt2)[\sqrt2]$ $(\frac23, \frac23, -\frac43)[2\sqrt2]$			$e_1 e_2 e_3$	$[6+5\sqrt2, 4+\sqrt2, 2+\sqrt2, \sqrt2]$ $[6+3\sqrt2, 4+3\sqrt2, 2+3\sqrt2, \sqrt2]$ $[6+4\sqrt2, 4+2\sqrt2, 2+2\sqrt2, 2\sqrt2]$	$e_1 e_2 e_3$	$\frac{3}{10}$

HMPD. IN S_3, S_4, S_5. Table VIII.

	Symbol of coordinates	Characteristic numbers.	Faces. p_3		p_4		p_6		Limiting polyhedra. T		O		P_3		tT		CO		P_6		tO	
HM_3																						
1 ,,	$\frac{1}{2}[1\ 1\ 1] = T$	(4, 6, 4)	4	3																		
e_2 ,,	$\frac{1}{2}[3\ 1\ 1] = tT$	(12, 18, 8)	4	1			4	2														
HM_4																						
1 ,,	$\frac{1}{2}[1\ 1\ 1\ 1] = C_{16}$	(8, 24, 32, 16)	32	12					16	8												
e_2 ,,	$\frac{1}{2}[3\ 3\ 1\ 1] = e_1$,,	(48, 120, 96, 24)	64	4			32	4			8	1			16	4						
e_3 ,,	$\frac{1}{2}[3\ 1\ 1\ 1] = ce_2$,,	(32, 96, 88, 24)	64	6	24	3			16	2							8	3				
$e_2\,e_3$,,	$\frac{1}{2}[5\ 3\ 1\ 1] = ce_1\,e_2$,,	(96, 192, 120, 24)	32	1	24	1	64	4							16	2					8	2
HM_5																						
1 ,,	$\frac{1}{2}[1\ 1\ 1\ 1\ 1]$	(16, 80, 160, 120, 26)	160	30					120	30												
e_2 ,,	$\frac{1}{2}[3\ 3\ 3\ 1\ 1]$	(160, 560, 640, 280, 42)	480	9			160	6	80	2	80	3			120	9						
e_3 ,,	$\frac{1}{2}[3\ 3\ 1\ 1\ 1]$	(160, 720, 880, 360, 42)	640	12	240	6			120	3	80	3	80	3			80	6				
e_4 ,,	$\frac{1}{2}[3\ 1\ 1\ 1\ 1]$	(80, 400, 720, 480, 82)	480	18	240	12			240	12			240	18								
$e_2\,e_3$,,	$\frac{1}{2}[5\ 5\ 3\ 1\ 1]$	(480, 1200, 1040, 360, 42)	320	2	240	2	480	6					80	1	200	5					80	4
$e_2\,e_4$,,	$\frac{1}{2}[5\ 3\ 3\ 1\ 1]$	(480, 1680, 1840, 720, 82)	800	5	720	6	320	4			80	1	240	3	160	4	80	2	160	4		
$e_3\,e_4$,,	$\frac{1}{2}[5\ 3\ 1\ 1\ 1]$	(320, 1120, 1280, 560, 82)	640	6	480	6	160	3	160	2			160	3	80	3	80	3	80	3		
$e_2\,e_3\,e_4$,,	$\frac{1}{2}[7\ 5\ 3\ 1\ 1]$	(960, 2400, 2080, 720, 82)	320	1	960	4	800	5					160	1	160	2			240	3	160	4

$n = 3$ (limiting polyhedra); $n = 4$ (limiting polytopes); $n = 5$

Limiting polytopes (HM_5):

HM_5	10		40		16		16	
1 ,,	C_{16}	5					$S(5)$	5
e_2 ,,	e_1 ,,	3			$c\,e_1\,S(5)$	1	e_1 ,,	2
e_3 ,,	$c\,e_2$,,	2			$c\,e_1$,,	1	e_2 ,,	3
e_4 ,,	C_{16}	1	P_T	4	$S(5)$	1	e_3 ,,	4
$e_2\,e_3$,,	$c\,e_1\,e_2$,,	2			$c\,e_1\,e_2$,,	1	$e_1\,e_2$,,	2
$e_2\,e_4$,,	e_1 ,,	1	P_{tT}	2	e_2 ,,	1	$e_1\,e_3$,,	2
$e_3\,e_4$,,	$c\,e_2$,,	1	P_T	1	e_1 ,,	1	$e_2\,e_3$,,	3
$e_2\,e_3\,e_4$,,	$c\,e_1\,e_2$,,	1	P_{tT}	1	$e_1\,e_2$,,	1	$e_1\,e_2\,e_3$,,	2

Table IX.

Faces.

HM_6	Symbol of coordinates.	Characteristic numbers.	p_3		p_4		p_6	
1 ,,	$\frac{1}{2}$[1 1 1 1 1 1]	(32, 240, 640, 640, 252, 44)	640	60				
e_2 ,,	$\frac{1}{2}$[3 3 3 3 1 1]	(480, 2160, 3200, 2080, 636, 76)	2560	16			640	8
e_3 ,,	$\frac{1}{2}$[3 3 3 1 1 1]	(640, 3840, 5920, 3520, 876, 76)	4480	21	1440	9		
e_4 ,,	$\frac{1}{2}$[3 3 1 1 1 1]	(480, 3360, 7360, 6240, 1996, 236)	4480	28	2880	24		
e_5 ,,	$\frac{1}{2}$[3 1 1 1 1 1]	(192, 1440, 4000, 4800, 2344, 296)	2560	40	1440	30		
$e_2\,e_3$,,	$\frac{1}{2}$[5 5 5 3 1 1]	(1920, 5760, 6560, 3520, 876, 76)	2560	4	1440	3	2560	8
$e_2\,e_4$,,	$\frac{1}{2}$[5 5 3 3 1 1]	(2880, 12960, 18240, 10560, 2636, 236)	7680	8	8640	12	1920	4
$e_2\,e_5$,,	$\frac{1}{2}$[5 3 3 3 1 1]	(1920, 9600, 16800, 12480, 3656, 296)	7680	12	7200	15	1920	6
$e_3\,e_4$,,	$\frac{1}{2}$[5 5 3 1 1 1]	(1920, 7680, 10720, 6720, 1996, 236)	4480	7	4320	9	1920	6
$e_3\,e_5$,,	$\frac{1}{2}$[5 3 3 1 1 1]	(1920, 10560, 16960, 11040, 3016, 296)	8320	13	8640	18		
$e_4\,e_5$,,	$\frac{1}{2}$[5 3 1 1 1 1]	(960, 5280, 10720, 9120, 3016, 296)	5760	18	4320	18	640	4
$e_2\,e_3\,e_4$,,	$\frac{1}{2}$[7 7 5 3 1 1]	(5760, 17280, 19680, 10560, 2636, 236)	3840	2	10080	7	5760	6
$e_2\,e_3\,e_5$,,	$\frac{1}{2}$[7 5 5 3 1 1]	(5760, 20160, 25920, 14880, 3656, 296)	5760	3	14400	10	5760	6
$e_2\,e_4\,e_5$,,	$\frac{1}{2}$[7 5 3 3 1 1]	(5760, 23040, 31200, 17760, 4136, 296)	9600	5	15840	11	5760	6
$e_3\,e_4\,e_5$,,	$\frac{1}{2}$[7 5 3 1 1 1]	(3840, 15360, 21760, 13440, 3496, 296)	7680	6	11520	12	2560	4
$e_2\,e_3\,e_4\,e_5$,,	$\frac{1}{2}$[9 7 5 3 1 1]	(11520, 34560, 38400, 19200, 4136, 296)	3840	1	23040	8	11520	6

Limiting polyhedra.

HM_6	T		O		P_3		C		tT		CO		P_6		tO	
1	640	80														
e_2	960	8	480	6					640	16						
e_3	1120	7	960	9	960	9					480	9				
e_4	1920	16	480	6	3840	48										
e_5	1920	40			2880	90										
$e_2\,e_3$	480	1			960	3			1600	10					480	6
$e_2\,e_4$			960	2	5760	12			960	4	960	4	1920	8		
$e_2\,e_5$	1440	3	960	3	6720	21			1440	9			1920	12		
$e_3\,e_4$	960	2			2880	9			1440	9	480	3	960	6		
$e_3\,e_5$	1440	3	960	3	5760	18	1440	6			1440	9				
$e_4\,e_5$	2880	12			4800	30			480	6			960	12		
$e_2\,e_3\,e_4$					4800	5			1440	3			2880	6	1440	6
$e_2\,e_3\,e_5$					5760	6	1440	2	2400	5	480	1	3840	8	960	4
$e_2\,e_4\,e_5$			960	1	6720	7			2400	5	960	2	6720	14		
$e_3\,e_4\,e_5$	1920	2			5760	9	1440	3	960	3	960	3	1920	6	480	3
$e_2\,e_3\,e_4\,e_5$					3840	2	1440	1	1920	2			9600	10	2400	5

Limiting polytopes $(P)_4$.

HM_6	60	192	192	192	240	240	240	640	640
1	1 C_{16}	1 $S(5)$							
e_2	e_1 ,,	1 ,,	$e_1\,S(5)$	$c\,e_1\,S(5)$					
e_3	$c\,e_2$,,	e_2 ,,	$c\,e_1$,,	$c\,e_1$,,	P_T				
e_4	1 ,,	1 ,,	e_3 ,,	$c\,e_1$,,	2 P_T	P_O		(3 ; 3)	
e_5	2 ,,	1 ,,	1 ,,		5 P_T			(3 ; 3)	
$e_2\,e_3$	$c\,e_1\,e_2$,,	e_1 ,,	$e_1\,e_2$,,	$c\,e_1\,e_2$,,	P_T				
$e_2\,e_4$	e_1 ,,	e_2 ,,	e_2 ,,	$e_1\,e_3$,,	P_O	2 P_{tT}		(3 ; 3)	(3 ; 6)
$e_2\,e_5$	2 e_1 ,,	e_1 ,,	e_3 ,,	$c\,e_1$,,	P_T	2 P_O	4 P_{tT}	(3 ; 3)	(3 ; 6)
$e_3\,e_4$	$c\,e_2$,,	e_1 ,,	$e_1\,e_3$,,	$c\,e_1\,e_2$,,	2 P_T	P_{tT}		(3 ; 3)	
$e_3\,e_5$	2 $c\,e_2$,,	e_2 ,,	e_2 ,,	$c\,e_1$,,	4 P_T	3 P_{CO}		(3 ; 3)	
$e_4\,e_5$	2 ,,	1 ,,	e_1 ,,	e_3 ,,	6 P_T	P_{tT}		(6 ; 3)	
$e_2\,e_3\,e_4$	$c\,e_1\,e_2$,,	$e_1\,e_2$,,	$e_1\,e_2$,,	$e_1\,e_2\,e_3$,,	3 P_{tT}			(3 ; 3)	(3 ; 6)
$e_2\,e_3\,e_5$	2 $c\,e_1\,e_2$,,	$e_1\,e_2$,,	$e_1\,e_3$,,	$c\,e_1\,e_2$,,	4 P_{tT}	P_{CO}	2 P_{tO}	(3 ; 3)	(3 ; 6)
$e_2\,e_4\,e_5$	2 e_1 ,,	e_2 ,,	$e_1\,e_3$,,	$e_1\,e_3$,,	2 P_O	7 P_{tT}		(6 ; 3)	(6 ; 6)
$e_3\,e_4\,e_5$	2 $c\,e_2$,,	e_1 ,,	$e_1\,e_2$,,	$e_1\,e_3$,,	6 P_{tT}	2 P_{CO}	P_{tO}	(6 ; 3)	
$e_2\,e_3\,e_4\,e_5$	2 $c\,e_1\,e_2$,,	$e_1\,e_2$,,	$e_1\,e_2\,e_3$,,	$e_1\,e_2\,e_3$,,	6 P_{tT}	3 P_{tO}		(6 ; 3)	(6 ; 6)

Limiting polytopes $(P)_5$.

HM_6	12	60	160	32	32
1	$\frac{1}{2}$[1 1 1 1 1]				$S(6)$
e_2	$\frac{1}{2}$[3 3 3 1 1]			$c\,e_1\,S(6)$	e_1 ,,
e_3	$\frac{1}{2}$[3 3 1 1 1]			$c\,e_2$,,	e_2 ,,
e_4	$\frac{1}{2}$[3 1 1 1 1]		(3 3 1) $\frac{1}{2}$[1 1 1]	$c\,e_1$,,	e_3 ,,
e_5	$\frac{1}{2}$[1 1 1 1 1]	(31) $\frac{1}{2}$[1 1 1 1]	(3 3 1) $\frac{1}{2}$[1 1 1]	$S(6)$	e_4 ,,
$e_2\,e_3$	$\frac{1}{2}$[5 5 3 1 1]			$c\,e_1\,e_2$,,	$e_1\,e_2$,,
$e_2\,e_4$	$\frac{1}{2}$[5 3 3 1 1]		(5 5 3) $\frac{1}{2}$[3 1 1]	$c\,e_1\,e_3$,,	$e_1\,e_3$,,
$e_2\,e_5$	$\frac{1}{2}$[3 3 3 1 1]	(53) $\frac{1}{2}$[3 3 1 1]	(5 3 3) $\frac{1}{2}$[3 1 1]	e_3 ,,	$e_1\,e_4$,,
$e_3\,e_4$	$\frac{1}{2}$[5 3 1 1 1]		(5 5 3) $\frac{1}{2}$[1 1 1]	$c\,e_1\,e_2$,,	$e_2\,e_3$,,
$e_3\,e_5$	$\frac{1}{2}$[3 3 1 1 1]	(53) $\frac{1}{2}$[3 1 1 1]	(5 3 3) $\frac{1}{2}$[1 1 1]	e_2 ,,	$e_2\,e_4$,,
$e_4\,e_5$	$\frac{1}{2}$[3 1 1 1 1]	(53) $\frac{1}{2}$[1 1 1 1]	(5 3 1) $\frac{1}{2}$[1 1 1]	e_1 ,,	$e_3\,e_4$,,
$e_2\,e_3\,e_4$	$\frac{1}{2}$[7 5 3 1 1]		(7 7 5) $\frac{1}{2}$[3 1 1]	$c\,e_1\,e_2\,e_3$,,	$e_1\,e_2\,e_3$,,
$e_2\,e_3\,e_5$	$\frac{1}{2}$[5 5 3 1 1]	(75) $\frac{1}{2}$[5 3 1 1]	(7 5 5) $\frac{1}{2}$[3 1 1]	$e_2\,e_3$,,	$e_1\,e_2\,e_4$,,
$e_2\,e_4\,e_5$	$\frac{1}{2}$[5 3 3 1 1]	(75) $\frac{1}{2}$[3 3 1 1]	(7 5 3) $\frac{1}{2}$[3 1 1]	$e_1\,e_3$,,	$e_1\,e_3\,e_4$,,
$e_3\,e_4\,e_5$	$\frac{1}{2}$[5 3 1 1 1]	(75) $\frac{1}{2}$[3 1 1 1]	(7 5 3) $\frac{1}{2}$[1 1 1]	$e_1\,e_2$,,	$e_2\,e_3\,e_4$,,
$e_2\,e_3\,e_4\,e_5$	$\frac{1}{2}$[7 5 3 1 1]	(97) $\frac{1}{2}$[5 3 1 1]	(9 7 5) $\frac{1}{2}$[3 1 1]	$e_1\,e_2\,e_3$,,	$e_1\,e_2\,e_3\,e_4$,,

																						CONSTITUENTS.					CONSTITUENTS in an other notation.				
Nets	A_4	A_t	A_3	A_0	B_4	B_3	B_2	B_1	B_0	C_4	C_3	C_2	C_0	D_4	D_3	D_t	D_0	E_a	E_b	E_c	E_d	A	B	C	base of D	E	C	B	A	D	E
NH_5	C_{16}	$S(5)$	—	—	$S(5)$	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	HM_5	Cr_5	Cr_5	—	—	—	[10000] 2	$\frac{1}{2}$[11111]	—	—
e_1 „	e_1 „	e_1 „	—	ce_1 $S(5)$	e_1 „	—	—	—	C_{16}	ce_1 $S(5)$	—	—	C_{16}	—	—	—	—	—	—	—	—	e_2 „	e_1 „	ce_1 „	—	—	[11000] 2	[21000] „	„ [33311]	—	—
e_2 „	ce_2 „	e_2 „	—	ce_1 „	e_2 „	—	—	P_O	ce_1 „	ce_1 „	—	—	ce_1 „	—	—	—	—	—	—	—	—	e_3 „	e_2 „	ce_2 „	—	—	[11100] „	[21100] „	„ [33111]	—	—
e_3 „	C_{16}	e_3 „	P_T	$S(5)$	e_3 „	—	(4 ; 3)	P_{CO}	ce_2 „	$S(5)$	—	—	ce_2 „	—	—	—	—	P_T	(4 ; 3)	—	—	e_4 „	e_3 „	ce_3 „	—	$(T ; p_4)$	[11110] „	[21110] „	„ [31111]	—	$\frac{1}{2}$[111][10] 2
e_4 „	C_{16}	$S(5)$	—	—	$S(5)$	P_T	(4 ; 3)	P_C	ce_3 „	—	—	—	ce_3 „	C_{16}	P_T	P_T	—	—	(4 ; 3)	—	P_T	e_5 „	e_4 „	ce_4 „	$\frac{1}{2}$[1111]	$(T ; p_4)$	[11111] $\sqrt{2}$	[1'1111] $\sqrt{2}$	„ [11111]	$\frac{1}{2}$[1111][1] $\sqrt{2}$	„ [111][11] $\sqrt{2}$
$e_1 e_2$ „	$ce_1 e_2$ „	$e_1 e_2$ „	—	$ce_1 e_2$ $S(5)$	$e_1 e_2$ „	—	—	P_O	e_1 „	$ce_1 e_2$ $S(5)$	—	—	e_1 „	—	—	—	—	—	—	—	—	$e_2 e_3$ „	$e_1 e_2$ „	$ce_1 e_2$ „	—	—	[22100] 2	[32100] 2	„ [55311]	—	—
$e_1 e_3$ „	e_1 „	$e_1 e_3$ „	P_{tT}	e_2 „	$e_1 e_3$ „	—	(4 ; 6)	P_{CO}	e_2 „	e_2 „	—	(4 ; 3)	e_2 „	—	—	—	—	P_{tT}	(4 ; 6)	(4 ; 3)	—	$e_2 e_4$ „	$e_1 e_3$ „	$ce_1 e_3$ „	—	$(tT ; p_4)$	[22110] „	[32110] „	„ [53311]	—	$\frac{1}{2}$[311][10] 2
$e_1 e_4$ „	e_1 „	e_1 „	—	ce_1 „	e_1 „	P_{tT}	(4 ; 6)	P_C	e_3 „	ce_1 „	P_O	(4 ; 3)	e_3 „	e_1 C_{16}	P_{tT}	P_{tT}	P_O	—	(4 ; 6)	(4 ; 3)	P_{tT}	$e_2 e_5$ „	$e_1 e_4$ „	$ce_1 e_4$ „	$\frac{1}{2}$[3311]	$(tT ; p_4)$	[1'1'111] $\sqrt{2}$	[2'1'111] $\sqrt{2}$	„ [33311]	$\frac{1}{2}$[3311][1] $\sqrt{2}$	„ [311][11] $\sqrt{2}$
$e_2 e_3$ „	ce_2 „	$e_2 e_3$ „	P_T	e_1 „	$e_2 e_3$ „	—	(4 ; 3)	P_{tO}	$ce_1 e_2$ „	e_1 „	—	—	$ce_1 e_2$ „	—	—	—	—	P_T	(4 ; 3)	—	—	$e_3 e_4$ „	$e_2 e_3$ „	$ce_2 e_3$ „	—	$(T ; p_4)$	[22210] 2	[32210] 2	„ [53111]	—	„ [111][10] 2
$e_2 e_4$ „	ce_2 „	e_2 „	—	ce_1 „	e_2 „	P_{CO}	(4 ; 3)	P_{RCO}	$ce_1 e_3$ „	ce_1 „	P_T	—	$ce_1 e_3$ „	ce_2 C_{16}	P_T	P_{CO}	P_T	—	(4 ; 3)	—	P_T	$e_3 e_5$ „	$e_2 e_4$ „	$ce_2 e_4$ „	$\frac{1}{2}$[3111]	$(T ; p_4)$	[1'1'1'11] $\sqrt{2}$	[2'1'1'11] $\sqrt{2}$	„ [33111]	$\frac{1}{2}$[3111][1] $\sqrt{2}$	„ [111][11] $\sqrt{2}$
$e_3 e_4$ „	C_{16}	e_3 „	P_T	$S(5)$	e_3 „	P_T	(8 ; 3)	P_{tC}	$ce_2 e_3$ „	$S(5)$	—	—	$ce_2 e_3$ „	C_{16}	P_T	P_T	—	P_T	(8 ; 3)	—	P_T	$e_4 e_5$ „	$e_3 e_4$ „	$ce_3 e_4$ „	„ [1111]	$(T ; p_8)$	[1'1'1'1'1] „	[2'1'1'1'1] „	„ [31111]	„ [1111][1] „	„ [111][1'1] „
$e_1 e_2 e_3$ „	$ce_1 e_2$ „	$e_1 e_2 e_3$ „	P_{tT}	$e_1 e_2$ „	$e_1 e_2 e_3$ „	—	(4 ; 6)	P_{tO}	$e_1 e_2$ „	$e_1 e_2$ „	—	(4 ; 3)	$e_1 e_2$ „	—	—	—	—	P_{tT}	(4 ; 6)	(4 ; 3)	—	$e_2 e_3 e_4$ „	$e_1 e_2 e_3$ „	$ce_1 e_2 e_3$ „	—	$(tT ; p_4)$	[33210] 2	[43210] 2	„ [75311]	—	„ [311][10] 2
$e_1 e_2 e_4$ „	$ce_1 e_2$ „	$e_1 e_2$ „	—	$ce_1 e_2$ „	$e_1 e_2$ „	P_{tO}	(4 ; 6)	P_{RCO}	$e_1 e_3$ „	$ce_1 e_2$ „	P_{tT}	(4 ; 3)	$e_1 e_3$ „	$ce_1 e_2$ C_{16}	P_{tT}	P_{tO}	P_{tT}	—	(4 ; 6)	(4 ; 3)	P_{tT}	$e_2 e_3 e_5$ „	$e_1 e_2 e_4$ „	$ce_1 e_2 e_4$ „	$\frac{1}{2}$[5311]	$(tT ; p_4)$	[2'2'1'11] $\sqrt{2}$	[3'2'1'11] $\sqrt{2}$	„ [55311]	$\frac{1}{2}$[5311][1] $\sqrt{2}$	„ [311][11] $\sqrt{2}$
$e_1 e_3 e_4$ „	e_1 „	$e_1 e_3$ „	P_{tT}	e_2 „	$e_1 e_3$ „	P_{tT}	(8 ; 6)	P_{tC}	$e_2 e_3$ „	e_2 „	P_O	(8 ; 3)	$e_2 e_3$ „	e_1 „	P_{tT}	P_{tT}	P_O	P_{tT}	(8 ; 6)	(8 ; 3)	P_{tT}	$e_2 e_4 e_5$ „	$e_1 e_3 e_4$ „	$ce_1 e_3 e_4$ „	„ [3311]	$(tT ; p_8)$	[2'2'1'1'1] „	[3'2'1'1'1] „	„ [53311]	„ [3311][1] „	„ [311][1'1] „
$e_2 e_3 e_4$ „	ce_2 „	$e_2 e_3$ „	P_T	e_1 „	$e_2 e_3$ „	P_{CO}	(8 ; 3)	P_{tCO}	$ce_1 e_2 e_3$ „	e_1 „	P_T	—	$ce_1 e_2 e_3$ „	ce_2 „	P_T	P_{CO}	P_T	P_T	(8 ; 3)	—	P_T	$e_3 e_4 e_5$ „	$e_2 e_3 e_4$ „	$ce_2 e_3 e_4$ „	„ [3111]	$(T ; p_8)$	[2'2'2'1'1] „	[3'2'2'1'1] „	„ [53111]	„ [3111][1] „	„ [111][1'1] „
$e_1 e_2 e_3 e_4$ „	$ce_1 e_2$ „	$e_1 e_2 e_3$ „	P_{tT}	$e_1 e_2$ „	$e_1 e_2 e_3$ „	P_{tO}	(8 ; 6)	P_{tCO}	$e_1 e_2 e_3$ „	$e_1 e_2$ „	P_{tT}	(8 ; 3)	$e_1 e_2 e_3$ „	$ce_1 e$ „	P_{tT}	P_{tO}	P_{tT}	P_{tT}	(8 ; 6)	(8 ; 3)	P_{tT}	$e_2 e_3 e_4 e_5$ „	$e_1 e_2 e_3 e_4$ „	$ce_1 e_2 e_3 e_4$ „	„ [5311]	$(tT ; p_8)$	[3'3'2'1'1] „	[4'3'2'1'1] „	„ [75311]	„ [5311][1] „	„ [311][1'1] „
	10	16	40	16	32	80	80	40	10	32	80	80	10	2	8	8	8	4	4	4	4										

CONSTITUENTS.

Nets		C	B	A	$A^{(5)}$	$A^{(4)}$	$A^{(3)}$
	NH_6	—	[100000] 2	½[111111]	—	—	—
e_1	,,	[110000] 2	[210000] ,,	,, [333311]	—	—	—
e_2	,,	[111000] ,,	[211000] ,,	,, [333111]	—	—	—
e_3	,,	[111100] ,,	[211100] ,,	,, [331111]	—	—	½[111][100] 2
e_4	,,	[111110] ,,	[211110] ,,	,, [311111]	—	½[1111][10] 2	—
e_5	,,	[111111] √2	[1'11111] √2	,, [111111]	½[11111][1] √2	—	—
$e_1 e_2$	,,	[221000] 2	[321000] 2	,, [555311]	—	—	—
$e_1 e_3$	,,	[221100] ,,	[321100] ,,	,, [553311]	—	—	½[311][100] 2
$e_1 e_4$	,,	[221110] ,,	[321110] ,,	,, [533311]	—	½[3311][10] 2	—
$e_1 e_5$	,,	[1'1'1111] √2	[2'1'1111] √2	,, [333311]	½[33311][1] √2	—	—
$e_2 e_3$	,,	[222100] 2	[322100] 2	,, [553111]	—	—	½[111][100] 2
$e_2 e_4$	,,	[222110] ,,	[322110] ,,	,, [533111]	—	½[3311][10] 2	—
$e_2 e_5$	,,	[1'1'1'111] √2	[2'1'1'111] √2	,, [333111]	½[33111][1] √2	—	—
$e_3 e_4$	,,	[222210] 2	[322210] 2	,, [531111]	—	½[1111][10] 2	½[111][210] 2
$e_3 e_5$	,,	[1'1'1'1'11] √2	[2'1'1'1'11] √2	,, [331111]	½[31111][1] √2	—	,, [111][1'11] √2
$e_4 e_5$	,,	[1'1'1'1'1'1] ,,	[2'1'1'1'1'1] ,,	,, [311111]	,, [11111][1] ,,	½[1111][1'1] √2	—
$e_1 e_2 e_3$	,,	[332100] 2	[432100] 2	,, [775311]	—	—	½[311][100] 2
$e_1 e_2 e_4$	,,	[332110] ,,	[432110] ,,	,, [755311]	—	½[5311][10] 2	—
$e_1 e_2 e_5$	,,	[2'2'1'111] √2	[3'2'1'111] √2	,, [555311]	½[55311][1] √2	—	—
$e_1 e_3 e_4$	,,	[332210] 2	[432210] 2	,, [753311]	—	½[3311][10] 2	½[311][210] 2
$e_1 e_3 e_5$	,,	[2'2'1'1'11] √2	[3'2'1'1'11] √2	,, [553311]	½[53311][1] √2	—	,, [311][1'11] √2
$e_1 e_4 e_5$	,,	[2'2'1'1'1'1] ,,	[3'2'1'1'1'1] 2	,, [533311]	,, [33311][1] ,,	½[3311][1'1] √2	—
$e_2 e_3 e_4$	,,	[333210] 2	[433210] 2	,, [753111]	—	½[3111][10] 2	½[111][210] 2
$e_2 e_3 e_5$	,,	[2'2'2'1'11] √2	[3'2'2'1'11] √2	,, [553111]	½[53111][1] √2	—	,, [111][1'11] √2
$e_2 e_4 e_5$	,,	[2'2'2'1'1'1] ,,	[3'2'2'1'1'1] ,,	,, [533111]	,, [33111][1] ,,	½[3111][1'1] √2	—
$e_3 e_4 e_5$	,,	[2'2'2'2'1'1] ,,	[3'2'2'2'1'1] ,,	,, [531111]	,, [31111][1] ,,	,, [1111][1'1] ,,	½[111][2'1'1] √2
$e_1 e_2 e_3 e_4$	,,	[443210] 2	[543210] 2	,, [975311]	—	,, [5311][10] 2	,, [311][210] 2
$e_1 e_2 e_3 e_5$	,,	[3'3'2'1'11] √2	[4'3'2'1'11] √2	,, [775311]	½[75311][1] √2	—	,, [311][1'11] √2
$e_1 e_2 e_4 e_5$	,,	[3'3'2'1'1'1] ,,	[4'3'2'1'1'1] ,,	,, [755311]	,, [55311][1] ,,	½[5311][1'1] √2	—
$e_1 e_3 e_4 e_5$	,,	[3'3'2'2'1'1] ,,	[4'3'2'2'1'1] ,,	,, [753311]	,, [53311][1] ,,	,, [3311][1'1] ,,	½[311][2'1'1] √2
$e_2 e_3 e_4 e_5$	,,	[3'3'3'2'1'1] ,,	[4'3'3'2'1'1] ,,	,, [753111]	,, [53111][1] ,,	,, [3111][1'1] ,,	,, [111][2'1'1] ,,
$e_1 e_2 e_3 e_4 e_5$	,,	[4'4'3'2'1'1] ,,	[5'4'3'2'1'1] ,,	,, [975311]	,, [75311][1] ,,	,, [5311][1'1] ,,	,, [311][2'1'1] ,,