M. Hentschel and J.U. Nöckel

The Sequential-Reflection Model in Deformed Dielectric Cavities

Abstract

The stationary states of a microlaser are related to the decaying quasibound states of the corresponding passive cavity. These are interpreted classically as originating from sequential escape attempts of an ensemble of rays obeying a curvature-corrected Fresnel formula. Polarization-dependent predictions of this model, and its limitations for stable orbits in partially chaotic systems are discussed.

As a mechanism for achieving mode confinement, waveguiding by total internal reflection is ubiquitous in optics. However, in dielectric microresonators where three-dimensionally confined mode volumes are desired, there is always leakage because the ray picture, in which Fresnel's formulas describe the outcoupling, acquires corrections. Leaky modes corresponding to classically confined rays can be found, e.g., in optical fibers as "spiral" modes [1], or in latterally structured cylindrical VCSELs [2] as well as in microdisk lasers [3]. The classically forbidden loss in such modes is analogous to tunneling through an effective potential barrier [4].

The highest Q is achieved for modes which semiclassically correspond to rays almost at grazing incidence. Resonators with a circular cross section are a particularly simple realization of this requirement, because they exhibit *whispering-gallery* (WG) modes characterized by high intensity in an annular region near the surface. However, even Lord Rayleigh who first described the acoustic analog that gave the phenomenon its name, concluded [5] that it requires only an everywhere positive curvature, not necessarily rotational symmetry. A rigorous proof of this is difficult because in the short-wavelength limit, this "clinging" of waves to the walls has to carry over to the ray picture, in which a generic oval cavity exhibits a transition to chaos [6, 7]. Notwithstanding, this problem is fundamental to microresonator design [8], because the availability of high-Q modes is the foremost selection criterium in an otherwise unbounded space of potential resonator shapes [9]. Chaos can in fact make WG modes more useful, and moreover create other types of modes with desirable properties, such as the bowtie pattern whose confocal geometry points the way toward the strong-coupling regime in combination with focused emission [10, 11].

The robustness of whispering-gallery type intensity patterns in the modes of convex resonators extends even to nonlinear media [12]. However, in that case the distinction to the widely studied phenomenon of vortex formation [13] becomes washed out: a WG mode is also a vortex with a phase singularity at points of vanishing intensity; for a circular resonator where the field is proportional to a Bessel function $J_m(kr) \approx r^m$ near the center $r = 0$, the vorticity is simply the angular momentum quantum number m. Therefore, since our aim is to address the fundamental aspect of the shape dependence of high-Q modes in microresonators, we focus here on linear media where amplification is taken into account by a negative imaginary part of the refractive index $\tilde{n}$.

The model considered here can be derived from a homogenous cylinder by deforming its cross section and considering only propagation transverse to its axis. In this case, TE and TM polarization are decoupled and one has to consider only a scalar wave equation

$$\nabla^2\psi + \tilde{n}^2 k^2 \psi = 0, \tag{1}$$

assuming a steady state time dependence so that k is real. Here, $\tilde{n} \equiv n - i\,n'$ inside the resonator and $\tilde{n} = 1$ outside, giving rise to an exterior and interior field, ψ_{ext} and ψ_{int}, both of which are connected by the proper matching conditions at the dielectric interface, depending on polarization. For TM modes, ψ denotes the electric field, which is parallel to the cylinder axis. In this case one finds that ψ and its normal derivative are continuous at the interface, in analogy to quantum mechanics.

The system is open because it radiates energy into the environment via its modal losses. This openness increases as $n \to 1$, and the closed-resonator limit is approached for $n \to \infty$. This can be understood from Fresnel's formulas which imply total internal reflection for all angles of incidence χ satisfying $\sin\chi > 1/n \equiv \sin\chi_c$ (χ_c is the *critical angle*). Equation (1) can be recast as

$$\nabla^2\psi + n^2\tilde{k}^2 \psi = 0, \tag{2}$$

where n is the real part of $\tilde{n}$ as defined above, and $\tilde{k} \equiv k - i\,k\,n'/n$ is a complex wavenumber inside the cavity but reduces to $\tilde{k} = k$ outside. If instead of this we also had $\tilde{k} = k - i\,k\,n'/n$ outside, the solutions of Eq. (2) would be the *quasibound states* of the passive resonator, as they arise when one assumes a decaying time dependence $\propto \exp[-ick\,t - \gamma\,t]$, where $\gamma = ck\,n'/n$.

For a quasibound (or metastable) state, the field at distances larger than $\approx c/(2\gamma)$ from the cavity grows epxonentially due to retardation, but within this physical range ψ_{ext} vanishes as $\gamma \to 0$, so that one can write $\psi_{\text{ext}}(\mathbf{r}) \approx \gamma\zeta(\mathbf{r})$. If one expands the dependence of ψ_{int} and ζ on γ in a Taylor series, then to linear order the γ-dependence of ζ, but not that of ψ_{int}, can be dropped in the full solution. Therefore, the stationary state of the active medium and the metastable decaying state are identical to first order in γ within an area of order γ^{-2}.

This approximate equivalence establishes a connection to the study of S-matrix poles from which quasibound states arise, see [14], and to dissipation in quantum mechanics [15, 16]. The recent resurgence of interest in these problems is motivated to a significant extent by our lack of understanding of the quantum-to-classical transition, in particular in the presence of classical chaos. Precisely this constellation is also present in Eq. (2) when one considers its short-wavelength limit for the generic case of a deformed cavity.

In the context of laser resonators, there are three main differences to previous work on open systems in the context of quantum chaos, see also [17, 18]: firstly, we are interested in the properties of *individual* states of an open system, as opposed to a statistical ensemble. Secondly, an important quantity that can be studied for such individual states is their *emission directionality*, which in other open systems of chemical or nuclear physics is averaged out. Finally, the classical limits of quantum mechanics with smooth potentials and optics with discontinuous refractive indices are qualitatively different [19]: the first yields deterministic Hamiltonian mechanics; the second leads to the probabilistic Fresnel formulas which moreover depend on polarization.

In principle, Eq. (2) can be solved numerically to find the discrete complex $\tilde{k}$ and the corresponding modes. One approach is based on the Rayleigh hypothesis [20, 21] which in our implementation for quasibound states [7] assumes that the fields can be expanded in cylinder functions as

$$\begin{aligned} a\psi_{\text{int}}(r,\phi) &= \sum_m A_m\, J_m(kr)\, e^{im\phi}, \\ \psi_{\text{ext}}(r,\phi) &= \sum_m B_m\, H_m^{(1)}(kr)\, e^{im\phi}. \end{aligned} \tag{3}$$

a where a polar coordinate system with suitably chosen origin is used. These expansions always work inside some circle of convergence for ψ_{int} and outside some other circle for ψ_{ext}, and for a large range of resonator shapes both convergence domains contain the dielectric interface where the matching conditions are imposed to obtain equations for the unknown coefficients A_m and B_m.

Computational cost can be high here, especially at short wavelengths, and hence a semiclassical approximation can lead to simplifications while preserving physical insight. The ray picture is a cornerstone of classical optics, but its value in the study of open resonators only unfolds when the ray dynamics is studied in *phase space* [22, 7, 23, 24], because Fresnel's formulas determine escape probabilities according to the angle of incidence χ, not the position of impact. One can make use of the physical information contained in this picture in two ways: Either one starts from Eq. (2) and takes a short-wavelength limit [25]; or alternatively, one starts from the classical dynamics and makes *classical* approximations that allow one to impose simple quantization conditions and thus make the connection to the resonator modes [8]. The question whether these different routes meet "in the middle" is not straightforward because the problem of semiclassical quantization in a generic deformed resonator is not completely solved as yet, owing to the coexistence of both regular and chaotic motion in their classical phase space.

Among the advantages of the ray-based approach [8] are its flexibility and computational ease. However, in order for the prescription outlined in Ref. [8] to correctly describe the limiting case of a circular cylinder, one must include the tunneling which in the circle is the only loss mechanism. This can be done in the ray picture with a curvature- and wavelength dependent "rounding" of Fresnel's formulas which the simulation uses at each reflection along a ray path. The idea used in [8] was to interpret the resonance widths of a cirular cylinder in terms of a "sequential-tunneling" ansatz: if the intensity of a quasibound state decays as $\exp[-2\gamma t]$, this can be interpreted in the ray picture as the result of ν sequential escape attempts with reflection probability p_0, where ν is the number of reflections the ray undergoes during the time t. In a circle of radius R, a trajectory characterized by the angle of incidence

χ has $\nu = ct/(nL)$ reflections during t (c/n is the speed of light in the passive medium and $L = 2R\cos\chi$ is the geometric path length between reflections). Therefore, one expects a decay law $\propto p_0^\nu = \exp[ct\ln p_0/(nL)]$. Comparison with the wave result yields

$$p_0 = \exp(-2\,nL\gamma/c) \tag{4}$$

An analytic approximation for γ in the circle with TM polarization has been derived in [26],

$$\gamma \approx -\frac{c}{2nR}\ln\left[\frac{n-1}{n+1}\right]\times \frac{J_m(kR)Y_{m-1}(kR)-J_{m-1}(kR)Y_m(kR)}{J_m^2(kR)+Y_m^2(kR)}. \tag{5}$$

Using the semiclassical expression

$$m = nkR\sin\chi \tag{6}$$

for the angular momentum [7], one then obtains the reflecticity in terms of purely classical variables, $p_0(kR,\sin\chi)$. It reduces to Fresnel's formula in the limit of large radius of curvature R, and by construction reproduces the width of a mode in the circle if applied locally at each reflection in in our classical ray model. The latter does not hold uniformly for a similar correction derived in [27].

One drastic consequence of the different classical limits for smooth and discontinuous potentials is that in the latter, all resonances of the passive dielectric are narrower than a certain *maximum width* if we choose the polarization in which ψ is continuosly differentiable as in quantum mechanics. But in quantum mechanics (or for smooth index profiles) one generally finds resonances of arbitrarily large width at increasing k. The classical argument for this statement will be given further below.

From the classical limit, it follows that there exists an *upper bound* on resonance widths for dielectric cavities with stepped index profiles and "quantum-mechanical" continuity conditions on ψ, because the reflectivity p_0, (for polarization perpendicular to the plane of incidence), is bounded away from zero. This minimum $p_{0,min}$ will limit the width of resonances in a cavity of characteristic size l to $\gamma_{max} = -c\ln p_{0,min}/(2nl)$. Smooth index profiles can also appear discontinuous on the scale of the wavelength but are eventually resolved as $k\to\infty$, allowing arbitrarily small reflectivities at perpendicular incidence.

However, extending these arguments to *TE polarization* where the electric field is in the plane of incidence, we furthermore conclude that a similar upper bound on the widths does *not exist* even for sharp interfaces. The reason is that Fresnel's formula yields zero reflectivity at the *Brewster angle* χ_B at $\sin\chi_B = (1+n^2)^{-1}$. The normal derivative of ψ (which now represents the magnetic field) exhibits a jump proportional to n^2 at the dielectric interface – reminding us that this is a situation unique to optics. These general considerations have important implications for microresonator design especially at the large n typical for semicondutors, because in that case $\sin\chi_B \to 1/n$, i.e., the "hole" in the reflectivity for TE polarization approaches χ_c for total internal reflection from below. Taking tunneling due to finite curvature into account as in Eq. (4), the rounded Fresnel formula then exhibits reduced

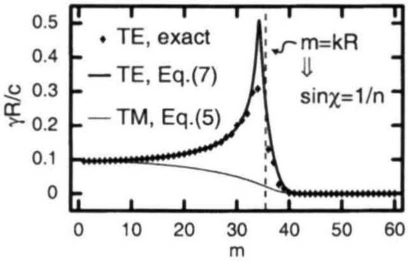

Figure 1: Exact resonance widths of a circle (radius R, refractive index $n = 3.29$) at $35 < kR < 35.5$ versus angular momentum m. Dashed line indicates χ_c in Fresnel's law, using Eq. (6). Also shown are the TM widths. The Brewster angle is at $m \approx 34$ and causes a peak in the TE widths.

reflectivity even for incidence somewhat above χ_c. One can approximately obtain the TE widths of the circle from $\gamma = -c\,\mathrm{Im}[\sigma]/(nR)$,

$$\sigma \approx \arctan\left[\left(n\frac{H^{(1)}_{m-1}(kR)}{H^{(1)}_{m}(kR)} - \frac{m}{kR}(n - 1/n)\right)^{-1}\right] \tag{7}$$

This is illustrated in Fig. 1, where refractive index and kR are chosen close to those of Ref. [10]. The reason is that the *quantum-cascade* material used there emits preferentially TM polarization, whereas the pioneering MQW microdisk lasers with sub-micron thickness permit guiding in the vertical direction only for TE modes [cf. McCall [3]; there, TE/TM must be interchanged to get from the slab-waveguide to our cylinder convention]. It is thus important to ascertain whether the identical oval lateral design of the quantum-cascade lasers in Ref. [10] would also permit a microdisk laser to operate in TE polarization.

The lasing mode in Fig. 3D of [10] was identified as a bowtie-shaped pattern corresponding to a periodic ray path with angle of incidence given by $\sin\chi \approx 1/n$, i.e., directly at the critical angle. That this mode provides high Q can be seen by comparing to γ in Fig. 1: assuming that the width γ_B of a bowtie mode results from the sequential application of p_0 as determined for the circle, the argument leading to Eq. (4) implies that $\gamma_B \approx \gamma\, L/l$ where $L/l \approx 1.13$ is the ratio of the classical path lengths between reflections in the WG orbit and bowtie, respectively. One sees that the TM line intersects the critical angle (corresponding to $m = kR$) at a much smaller width than the TE curve, and this Q-spoiling due to Brewster transmission is borne out by the actual TE resonances as well. This leads to the prediction that conventional microdisk lasers with a shape designed to yield a bowtie pattern just at χ_c as in Fig. 3D of [10] will *not lase.*

These ray arguments are known to yield large deviations from the true resonance widths when the modes under consideration are quantized on stable phase-space domains in a partially chaotic system, cf. Ref. [8] where this was attributed to chaos-assisted tunneling. The latter yields enhanced outcoupling and hence the true widths are *underestimated* by the sequential ray picture. Therefore, the above Q-spoiling for TE modes is not counteracted by a correction of this nature. The prediction of an upper bound for TM widths is also not affected by chaos-assisted tunneling because it cannot be faster than the fasted classical pro-

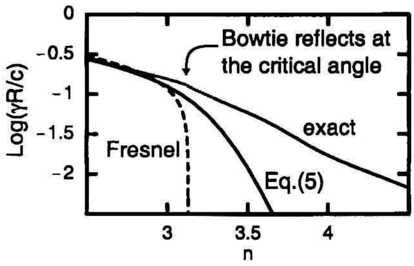

Figure 2: Width of a TM bowtie mode vs. refractive index, from numerical and ray calculations.

cess, which in turn is limited by p_0 at $\sin\chi = 0$. Beyond this, however, quantitative widths for stable-orbit modes in mixed phase spaces are not provided by the ray model.

The disagreement is illustrated in Fig. 2 for a bowtie mode similar to the ones studied in [25], as a function of n, but at a deformation of $\epsilon = 0.16$ [defined as in [10]] and $nkR \approx 119.8$. Since nk is the wavenumber inside the resonator, it should remain approximately independent of n as long the outcouplig can be taken into account in the form of a boundary phase shift intermediate between Dirichlet and Neumann. Indeed, for the state shown in Fig. 2, the change in nkR in the plotted range of n is only ≈ 0.2. The length scale R here is the radius of curvature at the points of reflection. At small n where χ_c is larger than the angle of incidence of the bowtie, escape is classically allowed in Fresnel's formula and hence curvature corrections are unimportant. At $n > 3$, the tunneling correction in Eq. (5) does improve on the classical Fresnel prediction ($\gamma = 0$) but clearly still underestimates the true width. As tunneling in general is definable only with respect to a classical expectation, we could again label the discrepancy as chaos-assisted tunneling. However, a semiclassical theory starting from Eq. (2) which reproduces the exact behavior in Fig. 2 very well [25] can shed more light on the physics of the phenomenon.

References

[1] A.W. Poon, R.K. Chang, J.A. Lock. Opt. Lett. 23(1998) 1105

[2] J.C. Ahn, K.S. Kwak, B.H. Park, H.Y. Kang, J.Y. Kim, O'Dae Kwon. Phys. Rev. Lett. 82(1999) 536

[3] S.L. McCall, A.F. Levi, R.E. Slusher, S.J. Pearton, R.A. Logan. Appl. Phys. Lett. 60(1992) 289

[4] B.R. Johnson. J. Opt. Soc. Am. 10(1993) 343

[5] J.W. Strutt, Baron Rayleigh. *The Theory of Sound*, volume II. Dover, New York (1945)

[6] V.F. Lazutkin. *KAM Theory and Semiclassical Approximations to Eigenfunctions*. Springer, New York (1993)

[7] J.U. Nöckel, A.D. Stone. In R.K. Chang, A.J. Campillo, editors, *Optical Processes in Microcavities*. World Scientific, Singapore (1996).

[8] J.U. Nöckel, A.D. Stone. Nature 385(1997) 45

[9] G. Angelow, F. Laeri, T. Tschudi. Opt. Lett. 211324

[10] C. Gmachl, F. Capasso, E.E. Narimanov, J.U. Nockel, A.D. Stone, J. Faist, D.L. Sivco, A.Y. Cho. Science 280(1998) 1556

[11] S.E. Morin, C.C. Yu, T.W. Mossberg. Phys. Rev. Lett. 73(1994) 1489
[12] T. Harayama, P. Davis, K.S. Ikeda. Phys. Rev. Lett. 82(1999) 3803
[13] C.O. Weiss, M. Vaupel, K. Staliunas, G. Slekys, V.B. Taranenko. Appl. Phys. B 68(1999) 151
[14] M. Patra, H. Schomerus, C.W.J. Beenakker. cond-mat/9905019 (1999)
[15] I. Prigogine. Phys. Rep. 219(1992) 93
[16] D. Braun, P.L. Braun and F. Haake. Physica D 131(1999) 265
[17] G.Casati, G. Maspero, D.L. Shepelyanski. Phys. Rev. E 56(1997) R6233
[18] G.Casati, G. Maspero, D.L. Shepelyanski. Physica D 131(1999) 311
[19] A. Kohler, R. Blümel. Ann. Phys. 267(1998) 249
[20] P.M. van den Berg, J.T. Fokkema. IEEE Trans. Anten. Propag. AP-27(1979) 577
[21] J.P. Barton. Appl. Opt. 36(1997) 1312
[22] J.U. Nöckel, A.D. Stone, R.K. Chang. Optics Letters 19(1994) 1693
[23] J.U. Nöckel, A.D. Stone, G. Chen, H. Grossman, R.K. Chang. Opt. Lett. 21(1996) 1609
[24] A. Mekis, J.U. Nöckel, G. Chen, A.D. Stone, R.K. Chang. Phys. Rev. Lett. 75(1995) 2682
[25] E.E. Narimanov, G. Hackenbroich, P. Jaquod, A. Douglas Stone. cond-mat/9907109 (1999)
[26] J.U. Nöckel. Dissertation, Yale University (1997)
[27] A.W. Snyder, J.D. Love. IEEE Trans. MTT-23(1975) 134

Author's address

Max-Planck-Institut für Physik komplexer Systeme,
Nöthnitzer Str. 38, 01187 Dresden,
Germany.