Mathematics. — *A complex of Conics.* By Prof. JAN DE VRIES.

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The conics k^2 cutting the fixed conic α^2 in the plane α twice and resting on the lines b_1, b_2, b_3, form a *complex*, Γ.

1. *Pairs of lines in* Γ. (a.) Any transversal t_{123} of the lines b forms a k^2 with any line r cutting it in the plane α; to any r there correspond two t_{123}.

(b.) Any transversal t_{12} of b_1, b_2 and α^2 forms a k^2 with any line r that rests on t_{12}, b_3 and α^2. The lines t_{12} form a scroll of the fourth degree; the lines r belong to a congruence [2,2]; any r defines three t_{12}.

(c.) Each of the four transversals a of α^2, b_1, b_2, b_3, is completed to a k^2 by any ray r of the congruence [1,2] that has a and α^2 as directrices.

(d.) The line a_{12} lying in α and resting on b_1 and b_2 forms a k^2 with any line r, that cuts a_{12} and b_3.

(e.) Any line r_{12} resting on b_1 and b_2 is completed to a k^2 by the line r in α that cuts r_{12} and b_3.

Evidently each of (b), (d) and (e) represent three different systems.

2. *Conics through a given point.* The k^2 through a point C form a surface Ψ; in order to determine the degree we shall investigate the intersection with the plane α.

Any plane through C and a point A of α^2 contains a k^2; these k^2 form a dimonoid of the fourth degree which has five points A^* outside A in common with α^2, hence α^2 is a *fivefold* curve on Ψ. Through C there passes a transversal r of a_{kl} and b_m; accordingly Ψ contains the lines a_{12}, a_{23}, a_{31}. The transversal of b_k and b_l through C yields a line of α that cuts it and b_m.

The intersection of Ψ and α consists, therefore, of the five-fold α^2 and six straight lines; consequently the surface is of the degree 16.

Its intersection with the plane Cb_1 consists of the following figures. In the first place a k^2 cutting b_1 twice, which must, therefore, be counted double. Through C there pass two lines r cutting b_1 and α^2 each of which defines three lines t_{23} and is, accordingly, a triple line of Ψ. Further Cb_1 contrains three lines as component parts of pairs of lines of which one line lies in α (§ 1, d and e). And now it appears that b_1 must be a *triple* line.

This may be confirmed in the following way. The conics through C

and a point of b_1 which cut a^2 twice and rest on b_2, form a cubic dimonoid; this contains, therefore, three k^2 that also cut b_3.

Now the intersection with Cb_k shows that C is an *eleven-fold* point of Ψ^{16}.

Ψ contains the four transversals of a^2, b_1, b_2, b_3, and the corresponding lines r through C. Further the 18 lines t_{kl} corresponding to the 6 above mentioned triple lines r.

3. *Surface of the k^2 that cut a given line l twice.* The k^2 that cut l in the point L, form a Ψ^{16} with an 11-fold point L; hence there are 5 k^2 which cut l in another point L^*. As each plane through l contains one k^2, the said surface is of the degree *seven*.

Its intersection with a consists of a^2 and five lines three of which rest on l and a line b whereas the other two cut l and a transversal of l and the three lines b_k. The lines b are single on Φ^7.

There are ten k^2 that touch the line l.

4. *Surface of the k^2 that cut the lines b in projective point ranges.* The planes of these k^2 osculate a twisted cubic; a^2 is, therefore, a triple conic of this surface Ω. The intersection of b_k and a defines a pair of lines of which one lies in a and the other one rests on b_l and b_m. Consequently the degree of Ω is *nine*.

The scroll defined by two of the point ranges cuts a^2 four times; hence Ω^9 contains twelve pairs of lines of which one of the lines rests on two b_k, the other one on the third line b.

5. *Surface Λ of the k^2 that cut two more lines, b_4 and b_5.* The k^2 of Γ that cut b_4 in a point B_4, form a surface of the degree 16 (§ 2); accordingly there are 16 k^2 which also rest on b_5. Hence on the surface Λ the 5 lines b are 16-fold.

a contains two lines that cut a line b and one of the two transversals of the other four lines b. Further a contains the lines which each cut two lines b and rest on a transversal of the other three lines b and which are, accordingly, double lines. Consequently in all 10 lines and 10 double lines of Λ lie in a.

The k^2 through a point A of a^2 resting on the 5 lines b, form a surface Φ of the degree 18 ($P\nu^6 = 18$). Its intersection with the plane Ab_5 consists of a k^2 that cuts b_5 twice and is, accordingly, a double curve, the quadruple line b_5 ($P^2\nu^4 = 4$), two lines through A that cut a transversal of b_1, b_2, b_3 and b_4, and four double lines through A each of which cuts one of the lines b and two transversals of the other three. Accordingly a straight line through A in Ab_5, has 6 points outside A in common with Φ^{18}. As this is also the case in each of the other planes Ab_k, A is a 12-fold point of Φ; hence a^2 and Φ^{18} have 24 points outside A in common and a^2 is a 24-*fold* curve of Λ.

Finally α contains the k^2 which cuts the 5 lines b and which apparently must be counted six times. The entire intersection of Λ and α is, therefore, of the order $10 + 20 + 48 + 12$; hence Λ is a surface of the *degree* 90.

On Λ there lie $10 \times 4 \times 3$ pairs of lines (r, r'), of which r rests on α^2 and $3\, b_k$, r' on α^2, r, and the other $2\, b$.

The surface Φ^7 of the k^2 of Γ that have b_4 as chord, has 7 points in common with b_5. Hence Λ has besides $5 \times 7 = 35$ double curves k^2 the planes of which pass through one of the lines b.

6. Consequently there are 90 conics that cut a given conic twice and that rest on six given lines.

If we use the symbol k to indicate that a conic rests on a given conic, we may express the result found above by $k^2\nu^6 = 90$. This number (and other ones containing k) can be found in the following way by applying the principle of the conservation of the number.

In order to determine $P^2k^2\nu^2$ we place one of the lines, b_1, in the plane of the given conic α^2. The plane through the points P_1, P_2 and one of the points of intersection of b_1 and α^2 contains one k^2 which satisfies the conditions. The same also holds good for the configuration of $P_1\, P_2$ and the line of α resting on it and on b_2. Hence $P^2k^2\nu^2 = 3$ (cf. § 2).

In order to determine $Pk^2\nu^4$ we choose three lines b_1, b_2, b_3 in a plane φ [1]). In this case there satisfy in the first place the 3×3 figures k^2 through P and one of the points $b_1\, b_2$, $b_1\, b_3$, $b_2\, b_3$ which rest twice on α^2 and also on b_4. Further 7 pairs of lines (r, r') of which r lies in φ and is a chord of α^2 or rests on α^2 and b_4 or cuts α^2 and a transversal r' of α^2 and b_4. Consequently $Pk^2\nu^4 = 16$ (cf. § 2).

In order to find $k^2\nu^6$ we again choose b_1, b_2, b_3 in φ. In this case there satisfy in the first place the 3×16 k^2 through one of the points b_1b_2, b_1b_3 or b_2b_3, which rest on the other four b_k and cut α^2 twice. Further the k^2 in φ that cuts α^2 twice and rests on b_4, b_5, b_6; evidently this must be counted eight times.

There are three chords of α^2 each of which is completed to a k^2 by a line of φ which cuts one of the lines b_4, b_5, b_6 and three chords of α^2 to which there belongs a transversal in φ of two of these lines b.

Each of the six lines r of φ that rest on α^2 and one of the lines b_4, b_5, b_6 is completed to pairs of lines by three transversals of r, α^2 and the other two of these lines b. The chord of α^2 in φ belongs to two pairs of lines. Finally each of the four transversals of α^2, b_4, b_5, b_6 forms a pair of lines with the line that joins its intersection with φ to one of the intersection with α^2.

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[1]) In this way I have again determined some time ago the known numbers $P\nu^6 = 18$ and $\nu^8 = 92$. (These Proceedings, **4**, 181).

Consequently we find in all $48 + 8 + 3 + 3 + 18 + 2 + 8 = 90$ figures; hence $k^2\nu^6 = 90$.

7. A plane ϱ through b_1 has also a curve of the order 74 in common with the surface Λ^{90}. This cuts b_1 in the first place in the 7×2 points of intersection with the k^2, that have b_1 as chord. In each of the remaining 60 points of intersection the plane ϱ is touched by a k^2 of Γ. Hence the locus of the points of contact of conics k^2 with ϱ is a curve of the order 60 and the tangent k^2 form a surface of the 120nd degree. Hence $k^2\nu^5\varrho = 120$.

Applying a similar reasoning to the surface Ψ^{16} (§ 2) we find the number $Pk^2\nu^3\varrho = 22$.

Also these results are easily verified by the method of § 6.

In the first place we find $P^2k^2\nu\varrho = 4$ by remarking that any plane through P_1P_2 contains two conics which cut α^2 twice, touch the plane ϱ and pass through P_1 and P_2.

In order to arrive at $Pk^2\nu^3\varrho$ we again choose the three lines b in a plane φ. In this case there satisfy 3×4 k^2 through P and a point b_kb_l. Further the chord of α^2 in φ belongs to a pair of lines that must be counted twice. Finally there are four pairs of lines (r, r') to be counted twice, with a double point on $\varphi\varrho$ of which r passes through P and r' lies in φ. Hence $Pk^2\nu^3\varrho = 22$.

The number $k^2\nu^5\varrho$ is found in the following way. φ contains two conics each of which must be counted eight times. Through each point b_kb_l in φ there pass 22 k^2. The chord of α^2 in φ belongs to a pair of lines with double point on $\varphi\varrho$. There are eight pairs (r, r') of which r lies in φ and rests on α^2 and b_4 or b_5 and the double point lies on $\varphi\varrho$, Further there are eight pairs of which r' rests on α^2, b_4, b_5 and $\varphi\varrho$ and r lies in φ. Finally the two pairs (r, r') satisfy of which r' is a chord of α^2 and cuts the lines b_4 (b_5) and $\varphi\varrho$. Accordingly $16 + 66 + 2 + 16 + 16 + 4 = 120$.