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d $c = b$. Their angle $2V$ is small or very small, sometimes these ıphiboles are nearly uniaxial and are connected with the distinctly ıxial ones in zonal crystals.

These amphiboles sometimes are intergrown with biotite or aegi- ıe and also with a bluish green amphibole, in which the plane of tic axes is also normal to the plane of symmetry, if they have ; same crystallographic orientation as the brownish green amphi- les. In sections parallel to (100) of the latter ones, the prism axis parallel to the fast ray in the bluish green amphiboles, whilst in :tions parallel to (010) it is nearly parallel to the slow ray.

From the facts, which have been mentioned above, it is evident, ıt amphiboles, in which the plane of optic axes lies in the plane symmetry, very probably occur at the same locality.

ıtronomy. — *"On canonical elements."* By Prof. W. DE SITTER.

In the developments of the planetary theory each of the three omalies has been used as independent variable: the mean anomaly LAGRANGE, the excentric anomaly by HANSEN and the true anomaly GYLDÉN. All systems of canonical elements, however, which have en in use up to the present time, are only modifications of the stem of DELAUNAY, which is based on the use of the *mean* anomaly. Recently [1]) LEVI-CIVITA has proposed a new system of elements, in ıich the *excentric* anomaly appears instead of the mean anomaly, most simultaneously [2]) HILL has called attention to another system which the *true* anomaly appears as one of the variables. The ethod by which HILL arrived at his system is, however, very ferent from that by which the systems of DELAUNAY and LEVI-CIVITA e developed. The object of the present paper is to show how these :ee systems, as well as others, can be derived from the same ndamental principle.

Let x_i be the co-ordinates of a body P, and $y_i = m\dfrac{dx_i}{dt}$ the com- nents of its momentum ($i = 1, 2, 3$). The equations of motion ; then

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad \ldots\ldots\ldots \quad (1)$$

·) T. LEVI-CIVITA. Nuovo sistema canonico di elementi ellittici. Annali di Mate- tica, Ser. III, Tom. XX, p. 153 (Aprile 1913).

²) G. W. HILL. Motion of a system of material points under the action of avitation. Astronomical Journal, Vol. XXVII, Nr. 646—647, p. 171 (1913 April 28).

where

$$H = T - K,$$

T representing the kinetic energy and K the force-function. In the problem of planetary motion we have

$$K = \frac{k}{r} + S,$$

where S is the perturbative function. According to a theorem discovered by JACOBI, any new system of canonical variables p_ι, q_ι can be derived from an arbitrary function $\Phi(x_\iota, q_\iota)$ of x_ι and q_ι, by putting

$$\frac{\partial \Phi}{\partial x_\iota} = y_\iota \quad , \quad \frac{\partial \Phi}{\partial q_\iota} = p_\iota \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If then, by means of (2), we replace x_ι and y_ι in H by p_ι and q_ι, the equations for the new variables are

$$\frac{dp_\iota}{dt} = \frac{\partial H}{\partial q_\iota} \quad , \quad \frac{dq_\iota}{dt} = -\frac{\partial H}{\partial p_\iota} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

JACOBI's method of integration, which has led to the system of canonical elements introduced into astronomical practice by DELAUNAY, consists in so choosing Φ that the equations (3) are of a much simpler form than (1). For this purpose JACOBI chooses for Φ an integral of the partial differential equation, which bears his name, and which is constructed as follows. In the function $H(x_\iota, y_\iota)$ replace y_ι by $\frac{\partial \Phi}{\partial x_\iota}$, then JACOBI's equation is

$$H\left(x_\iota, \frac{\partial \Phi}{\partial x_\iota}\right) = h.$$

The constant h is the energy of the motion.

If we take $S=0$, and, instead of x_ι, y_ι introduce polar coordinates r, s, w, and the corresponding momenta $r' = m\frac{dr}{dt}$, $s' = mr^2\frac{ds}{dt}$, $w' = mr^2 \cos^2 s \frac{dw}{dt}$, the energy function becomes

$$H_0 = \frac{1}{2m}\left(r'^2 + \frac{s'^2}{r^2} + \frac{w'^2}{r^2 \cos^2 s}\right) - \frac{k}{r} = h \quad . \quad . \quad . \quad . \qquad (4)$$

Then JACOBI's equation admits the integral

$$\Phi_0 = \Theta w + \int_0^s \sqrt{G^2 - \frac{\Theta^2}{\cos^2 s}}\, ds + \int_{r_0}^r \sqrt{2hm + \frac{2km}{r} - \frac{G^2}{r^2}}\, dr,$$

where Θ and G are constants of integration. JACOBI now takes

$\Phi = \Phi_0$ and for the variables q_i he takes Θ, G and h. In order to get a more general point of departure I take for the function Φ which serves to define the new variables

$$\Phi = \Theta w + \int_0^s Q ds + \int_{r_0}^r R dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$\left.\begin{aligned} Q^2 &= G^2 - \frac{\Theta^2}{\cos^2 s}, \\ R^2 &= m\left(-\alpha^2 + \frac{2\beta^2}{r} - \frac{\gamma^2}{r^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We have thus

$$r' = \frac{\partial \Phi}{\partial r} = R \quad , \quad s' = \frac{\partial \Phi}{\partial s} = Q, \quad , \quad w' = \frac{\partial \Phi}{\partial w} = \Theta,$$

and therefore

$$H = -\frac{\alpha^2}{2} + \frac{\beta^2 - k}{r} + \left(\frac{G^2}{m} - \gamma^2\right)\frac{1}{2\,r^2} - S \quad . \quad . \quad . \quad . \quad (7)$$

I will now for two of the variables q_i take Θ and G, for the third I take either α, β or γ, or a function of one of these parameters.

We have thus in all cases

$$\vartheta = \frac{\partial \Phi}{\partial \Theta} = w - \int_0^s \frac{\partial Q}{\partial \Theta} ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If now we introduce the auxiliary angle ζ' by

$$\zeta' = w - \vartheta = \int_0^s \frac{\partial Q}{\partial \Theta} ds,$$

and then construct the right-angled spherical triangle of which the sides next to the right angle are ζ' and s, it is easily seen that in this triangle we shall have $\frac{d\zeta'}{ds} = \frac{\partial Q}{\partial \Theta}$ if we put

$$\Theta = G \cos i,$$

where i is the angle opposite the side s. Consequently i and ϑ are the inclination and node of the instantaneous orbital plane, i. e. the plane which contains the origin of co-ordinates and the velocity of the body P. Introducing now the argument of the latitude ζ, i. e. the angle between the line of nodes and the radius-vector, or the side opposite the right angle in the above mentioned triangle, we find from simple geometrical considerations $\zeta' \cos i + \int_0^s \frac{Q}{G} ds = \zeta$, and consequently

$$\Phi = \Theta \vartheta + G\zeta + \int_{r_0}^{r} R\, dr \; . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Next calling the values of r for which R vanishes $a(1-e)$ and $a(1+e)$ respectively, we find

$$a = \frac{\beta^2}{\alpha^2} \quad , \quad a^2(1-e^2) = \frac{\gamma^2}{\alpha^2}, \; . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$R^2 = \alpha^2 m \left[-1 + \frac{2a}{r} - \frac{a^2(1-e^2)}{r^2} \right].$$

I now introduce a new parameter δ by

$$\gamma = \frac{G}{\sqrt{m}} + \delta \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We have then

$$g = \frac{\partial \Phi}{\partial G} = \zeta + \frac{1}{\sqrt{m}} \int_{r_0}^{r} \frac{\partial R}{\partial \gamma} dr. \; . \quad . \quad . \quad . \quad . \quad (12)$$

Putting now

$$f = \zeta - g = -\frac{1}{\sqrt{m}} \int_{r_0}^{r} \frac{\partial R}{\partial \gamma} dr,$$

we find from (12) and (10)

$$\frac{dr}{df} = \frac{r^2 R}{\gamma \sqrt{m}} = \sqrt{-\frac{r^4}{a^2(1-e^2)} + \frac{r^3}{a(1-e^2)} - r^2} \; . \quad . \quad . \quad (13)$$

This is the differential equation of an ellipse of which a is the semi major axis and e the excentricity. If the constant of integration is so chosen that $r_0 = a(1-e)$, then f is the true anomaly. We have then

$$r = \frac{a(1-e^2)}{1+e\cos f} \quad , \quad R = \alpha \sqrt{m} \frac{e \sin f}{\sqrt{1-e^2}} \; . \quad . \quad . \quad (14)$$

We can now in this ellipse introduce by definition the excentric anomaly E and the mean anomaly M. We find

$$r = a(1 - e\cos \mathrm{E}) \quad , \quad R = \alpha \sqrt{m} \frac{ae \sin \mathrm{E}}{r},$$

$$d\mathrm{E} = \frac{a}{2\beta\sqrt{m}} \frac{\partial R}{\partial \beta} dr \,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

$$\mathrm{M} = \mathrm{E} - e \sin \mathrm{E} \qquad , \qquad d\mathrm{M} = -\frac{a^2}{\beta^2 \sqrt{m}} \frac{\partial R}{\partial \alpha} dr \quad . \quad . \quad . \qquad (16)$$

In all these formulas a and e are written as abbreviations for certain functions of α, β, γ defined by (10).

All this is independent of the choice of the third pair of canonical elements. We must now specialize the values of the parameters α, β, δ, which were so far left entirely indeterminate. Now we can distinguish three cases. In each case two of these parameters are constant, while the third is variable, and a function of it is taken as the element q_3.

Case I. $\qquad \beta = \beta_0 = const. \qquad , \qquad \delta = \delta_0 = const.$

The third linear element is a function of α and will be called L. Therefore the conjugated variable l is given by

$$l = \frac{\partial \Phi}{\partial L} = \frac{\partial \Phi}{\partial \alpha} \frac{d\alpha}{dL} = \frac{d\alpha}{dL} \int \frac{\partial R}{\partial \alpha} dr = -\frac{\beta_0^{\,2} \sqrt{m}}{a^2} \frac{d\alpha}{dL} \int d\mathrm{M}.$$

Thus, if we wish to get

$$l = \mathrm{M} = mean \;\; anomaly,$$

we must take

$$\frac{dL}{d\alpha} = -\frac{\beta_0^{\,2} \sqrt{m}}{a^2},$$

from which

$$L = \frac{\beta_0^{\,2} \sqrt{m}}{\alpha} = \beta_0 \sqrt{m} \,.\, \sqrt{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Since β_0 and m are constants, the semi major axis a is variable. We find at once from (10)

$$L \sqrt{1 - e^2} = G + \delta_0 \sqrt{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Case II. $\quad \alpha = \alpha_0 = const., \quad \delta = \delta_0 = const.$

The third linear variable U is a function of β. Therefore the conjugated variable is

$$u = \frac{\partial \Phi}{\partial U} = \frac{d\beta}{dU} \int \frac{\partial R}{\partial \beta} dr = \frac{2\beta\sqrt{m}}{\alpha_0} \frac{d\beta}{dU} \int d\mathrm{E}.$$

Thus in order to get

$$u = \mathrm{E} = excentric \;\; anomaly,$$

we must take

$$\frac{dU}{d\beta} = \frac{2\beta\sqrt{m}}{\alpha_0},$$

and consequently

$$U = \frac{\beta^2 \sqrt{m}}{\alpha_0} = \alpha_0 \sqrt{m} \, . \, a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Here again α_0 and m being constant, a is variable. We find further

$$U \sqrt{1-e^2} = G + \sigma_0 \sqrt{m} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Case III. $\alpha = \alpha_0$ *const.*, $\beta = \beta_0$ *const.*

The third linear element V is now a function of σ. Therefore

$$v = \frac{\partial \Phi}{\partial V} = \frac{d\sigma}{dV} \frac{\partial \Phi}{\partial \sigma} = \frac{d\sigma}{dV} \int \frac{\partial R}{\partial \gamma} dr = - \sqrt{m} \frac{d\sigma}{dV} \int df.$$

Consequently, if we wish to have

$$v = f = \textit{true anomaly}$$

we must take

$$\frac{dV}{d\sigma} = - \sqrt{m},$$

and therefore

$$V = V_0 - \sigma \sqrt{m}.$$

Now we can introduce a new variable ν by

$$\sigma \sqrt{m} = \frac{\beta_0^2}{\alpha_0} (\sqrt{m} - \nu).$$

Putting then

$$V_0 = \frac{\beta_0^2 \sqrt{m}}{\alpha_0} = \beta_0 \sqrt{m} \, . \, \sqrt{a} = \alpha_0 \sqrt{m} \, . \, a, \quad . \quad . \quad . \quad . \qquad (21)$$

we find

$$V = \frac{\beta_0^2 \nu}{\alpha_0} = \beta_0 \nu \sqrt{a} = \alpha_0 \nu a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

In this case, α_0 and β_0 being constant a is also constant, by (10), and ν is variable. We have now

$$V_0 \sqrt{1-e^2} = G + \sigma \sqrt{m} = G + V_0 - V \, . \quad . \quad . \quad . \qquad (23)$$

The energy H is in the three cases:

$$\left.\begin{aligned}
&I. \quad H = -\frac{\beta_0^4 m}{2L^2} + \frac{\beta_0^2 - k}{r} - \sigma_0 \left(\frac{2G}{\sqrt{m}} + \sigma_0\right) \frac{1}{2r^2} - S. \\
&II. \quad H = -\frac{\alpha_0^2}{2} + \left(\frac{\alpha_0}{\sqrt{m}} U - k\right) \frac{1}{r} - \sigma_0 \left(\frac{2G}{\sqrt{m}} + \sigma_0\right) \frac{1}{2r^2} - S. \\
&III. \quad H = -\frac{\alpha_0^2}{2} + \frac{\beta_0^2 - k}{r} + \frac{(V - V_0)(2G + V_0 - V)}{m} \cdot \frac{1}{2r^2} - S.
\end{aligned}\right\} \quad (24)$$

Here r must be understood to be written for brevity's sake instead of its expression in function of the elements.

In the cases I and II it is advantageous to take $\delta_0 = 0$.

In the cases II and III the value of α_0 is of course immaterial, the first term of H may as well be omitted. As to the value of β_0 in the cases I and III, it is customary in classical celestial mechanics (case I) to take $\beta_0 = \sqrt{k}$. This however is not at all necessary, and the term $\frac{\beta_0^2 - k}{r}$ can be taken advantage of by an appropriate choice of β_0 to cancel a term in S. This is also advocated by HILL in the paper already quoted. Though HILL does not say so (and doubtlessly does not intend to say), a casual reader may easily be led to assume that the possibility of this device is one of the advantages of the system of elements of case III. It is therefore well to point out that it does not depend on the choice of elements, and can as well be applied in case I.

By each of the three sets of elements

$$\begin{Bmatrix} L, & G, & \Theta \\ l, & g, & \vartheta \end{Bmatrix}, \quad \begin{Bmatrix} U, & G, & \Theta \\ u, & g, & \vartheta \end{Bmatrix}, \quad \begin{Bmatrix} V, & G, & \Theta \\ v, & g, & \vartheta \end{Bmatrix}$$

the motion of the body P is described as a Keplerian motion in an ellipse with varying parameters. In the cases I and II the variable instantaneous ellipse has a point of contact with the true orbit, and can therefore be called an osculating ellipse. But the definition of this osculating ellipse is different in each case. In fact at every point of the orbit there is an infinity of ellipses having that point and the tangent at that point in common with the orbit and all having one and the same given point as a focus. In case I we choose from this family of ellipses that ellipse that would be described by a body of mass m starting from the given point with the given velocity nnder the action of a central force $\frac{\beta_0^2}{r^2}$ emanating from the common focus. The constant β_0^2 here has a prescribed value, the same for all points of the orbit. The elements thus derived are those of DELAUNAY. They are called by LEVI-CIVITA *isodynamic* elements.

In the second case we choose that ellipse in which the energy of a Keplerian motion of a body of mass m starting with the given velocity from the given point has a prescribed fixed value $h_0 = -\frac{1}{2}\alpha_0^2$. The elements which we then get are those of LEVI-CIVITA, and are by him called *isoenergetic* elements.

In the third case the ellipse has a prescribed semi major axis $a = \frac{\beta_0^2}{\alpha_0^2}$. There is no osculation, the tangent of the ellipse in the common point being different from the tangent of the orbit.[1]) If a name analogous to those coined by Levi-Civita for the other two systems were required, we might call these elements *isoprotometric* elements, since the quantity a, which here remains constant, is called the protometer by Gyldén, who was the first to use a system of elements belonging to this class.

If at a given point of the true orbit, i.e. for given values of r, s, w, $\frac{dr}{dt}$, $\frac{ds}{dt}$, $\frac{dw}{dt}$, we wish to determine the instantaneous elements in the three cases, the method of procedure is as follows. First we determine geometrically the inclination i, and node ϑ of the plane containing the origin of coordinates and the velocity of the body P. With the aid of these we find ζ and $\frac{d\zeta}{dt}$. Then

$$G = mr^2 \frac{d\zeta}{dt} \qquad\qquad \Theta = G \cos i.$$

For the determination of the third linear element we require the living force, or kinetic energy:

$$2T = m\left(\frac{dr}{dt}\right)^2 + mr^2\left(\frac{d\zeta}{dt}\right)^2.$$

We have then in the three cases (taking $\sigma_0 = 0$ for the cases I and II):

$$\left.\begin{array}{ll} I. & 2T = \dfrac{2\beta_0^2}{r} - \dfrac{m\beta_0^4}{L^2} \\[2ex] II. & 2T = \dfrac{2\alpha_0 U}{r\sqrt{m}} - \alpha_0^2 \\[2ex] III. & 2T = \beta_0^2\left(\dfrac{2}{r} - \dfrac{1}{a}\right) + \dfrac{(V-V_0)(2G+V_0-V)}{mr^2} \end{array}\right\} \quad (25)$$

From these formulas we find L, U, V. Next a and e are determined by (17), (18), (19), (20), (21), (23) and then the ordinary elliptic formulae give v and the true, excentric or mean anomaly. Finally we have

$$g = \zeta - v.$$

The differential equations for the elements are given below for the three cases. In the cases I and II I take $\sigma_0 = 0$, or $\gamma = \frac{G}{\sqrt{m}}$ and in the cases I and III I put

[1]) Hill l.c. p. 176, states that the ellipse has a point of contact with the orbit. This, however, is an oversight.

$$S' = S + \frac{k-\beta_0^2}{r}.$$

I.
$$\left.\begin{aligned} \frac{dl}{dt} &= \frac{\beta_0^4 m}{L^3} - \frac{\partial S'}{\partial L} & \frac{dL}{dt} &= \frac{\partial S'}{\partial l} \\ \frac{dg}{dt} &= -\frac{\partial S'}{\partial G} & \frac{dG}{dt} &= \frac{\partial S'}{\partial g} \\ \frac{d\vartheta}{dt} &= -\frac{\partial S'}{\partial \Theta} & \frac{d\Theta}{dt} &= \frac{\partial S'}{\partial \vartheta} \end{aligned}\right\} \quad \ldots \quad (26)$$

II. Put
$$U = \frac{k\sqrt{m}}{a_0} + \Delta U$$

$$r = \frac{U}{a_0\sqrt{m}}(1 - e\cos u) \qquad \sqrt{1-e^2} = \frac{G}{U}$$

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{k}{Ur} + \frac{\Delta U(1-e^2)}{mr^2}\frac{\cos u}{e} - \frac{\partial S}{\partial U} & \frac{dU}{dt} &= -\frac{\Delta U}{mr^2} e\, U \sin u + \frac{\partial S}{\partial u} \\ \frac{dg}{dt} &= -\frac{\Delta U\sqrt{1-e^2}}{mr^2}\frac{\cos u}{e} - \frac{\partial S}{\partial G} & \frac{dG}{dt} &= \frac{\partial S}{\partial g} \\ \frac{d\vartheta}{dt} &= -\frac{\partial S}{\partial \Theta} & \frac{d\Theta}{dt} &= \frac{\partial S}{\partial \vartheta} \end{aligned}\right\} \quad (27)$$

If at $t=0$ we start with $\Delta U = 0$, and if $S=0$, then the motion is Keplerian: $U, G, \Theta, g, \vartheta$ are constants. In the general case, when S differs from zero, ΔU is of the order of S, i. e. of the order of the perturbing masses.

III. Put
$$V = V_0 + \Delta V$$

$$r = \frac{a(1-e^2)}{1+e\cos v} \qquad \sqrt{1-e^2} = \frac{G-\Delta V}{V_0}$$

$$\left.\begin{aligned} \frac{dv}{dt} &= \frac{G-\Delta V}{mr^2} - \frac{\Delta V(2G-\Delta V)}{mr^3}\cdot\frac{\sqrt{1-e^2}}{eV_0}\frac{\partial r}{\partial e} - \frac{\partial S'}{\partial V} \\ \frac{dV}{dt} &= -\frac{\Delta V(2G-\Delta V)}{mr}\cdot\frac{e\sin v}{a(1-e^2)} + \frac{\partial S'}{\partial v} \\ \frac{dg}{dt} &= \frac{\Delta V}{mr^2} + \frac{\Delta V(2G-\Delta V)}{mr^3}\cdot\frac{\sqrt{1-e^2}}{eV_0}\cdot\frac{\partial r}{\partial e} - \frac{\partial S'}{\partial G} \qquad \frac{dG}{dt} = \frac{\partial S'}{\partial g} \\ \frac{d\vartheta}{dt} &= -\frac{\partial S'}{\partial \Theta} \qquad \frac{d\Theta}{dt} = \frac{\partial S'}{\partial \vartheta} \end{aligned}\right\} \quad (28)$$

ΔV again is of the order of the perturbing masses. For $S'=0$ the motion is Keplerian and $V, G, \Theta, g, \vartheta$ are constants.

In all cases the choice of the original variables x_i, y_i, is of course entirely free. It only affects the form of the perturbative function S, which plays no part in the definition of the elements. We can either use ordinary relative co-ordinates (S being in that case different for each planet), or we can introduce canonical relative co-ordinates, either by the method of JACOBI-RADAU ("élimination des noeuds") or by POINCARÉ's "transformation α" (Acta Mathematica, Vol. XXI, page 86). [In these last two cases the body P of course is not the true planet, but a fictitious planet, different according to the choice of co-ordinates]. LEVI-CIVITA uses POINCARÉ's co-ordinates, but this is not material: the isoenergetic elements may as well be used with any other system of relative co-ordinates.

Also it is hardly necessary to point out that in all three cases we can introduce new elements by canonical transformations and thus derive from the isoenergetic or the isoprotometric elements the same modifications which have been derived from DELAUNAY's elements. Thus e.g. we have the three corresponding transformations:

$$I. \quad \begin{array}{lll} \Lambda = L & \Pi = L - G & \Psi = G - \Theta \\ \lambda = l + g + \vartheta & \pi = -g - \vartheta & \psi = -\vartheta \end{array}$$

(where we have $\Pi = L(1 - \sqrt{1-e^2})$, $\Psi = 2G \sin^2 \frac{1}{2} i$)

$$II \quad \begin{array}{lll} H = U & \Pi = U - G & \Psi = G - \Theta \\ \eta = u + g + \vartheta & \pi = -g - \vartheta & \psi = -\vartheta \end{array}$$

$$(\Pi = U(1 - \sqrt{1-e^2}) \ , \quad \Psi = 2G \sin^2 \tfrac{1}{2} i)$$

$$III. \quad \begin{array}{lll} W = V & \Pi = V - G & \Psi = G - \Theta \\ w = v + g + \vartheta & \pi = -g - \vartheta & \psi = -\vartheta \end{array}$$

$$(\Pi = V_0(1 - \sqrt{1-e^2}) \ , \quad \Psi = 2G \sin^2 \tfrac{1}{2} i),$$

from which again we can derive the elements of POINCARÉ-HARZER:

$$\begin{array}{ll} h = \sqrt{2\Pi} \cos \pi & p = \sqrt{2\Psi} \cos \psi \\ k = \sqrt{2\Pi} \sin \pi & q = \sqrt{2\Psi} \sin \psi. \end{array}$$

If in case III we make the transformation

$$\left.\begin{array}{ll} F = V - G & Z = G \\ f = v & \zeta = v + g \end{array}\right\}$$

we find the elements used by HILL. We have indeed $F = m \, . \, \eta$, $Z = m \, . \, \text{υ}$, $\zeta = u$ (where η, υ and u are the symbols used by HILL), and the letter f is used by HILL with the same meaning as in the present paper.

These elements can also be derived directly from the function Φ. The condition (11) must then be omitted: R must be assumed not to contain G.

If then we write Z for G, we find at once

$$\frac{\partial \Phi}{\partial Z} = \zeta.$$

The element F now is a function of γ, and consequently

$$f = \frac{\partial \Phi}{\partial F} = \frac{d\gamma}{dF}\int \frac{\partial R}{\partial \gamma}\, dr = -\sqrt{m}\frac{d\gamma}{dF}\int df.$$

Therefore

$$\frac{dF}{d\gamma} = -\sqrt{m},$$

from which

$$F = const. - \gamma\sqrt{m} = V_0 - \gamma\sqrt{m}.$$

Now, by (10) we have $\gamma = \frac{\beta_0^2}{\alpha_0}\sqrt{1-e^2}$, therefore, with the value (21) of V_0, we find

$$F = \frac{\beta_0^2\sqrt{m}}{\alpha_0}(1-\sqrt{1-e^2}).$$

To the elements I corresponds the classical development of the perturbative function according to the sines and cosines of multiples of the mean anomalies. The development of S according to excentric anomalies, which is required for the elements II, has been given by NEWCOMB in Vol III of the Astron. Papers of the Am. Eph. For the development in function of true anomalies, which is needed when using the elements III, the foundations have been laid down by HILL in the paper already quoted.

Case IV. $\alpha = \alpha_0 = const.$, $\beta = \beta_0 = const.$, $\delta = \delta_0 = 0$.

The third linear element is a function of $\varkappa$. It will be called M. We have

$$\mu = \frac{\partial \Phi}{\partial M} = \frac{d\varkappa}{dM}\int \frac{\partial R}{\partial K}\, dr = \frac{\beta_0^2}{\alpha_0}\frac{d\varkappa}{dM}\int d\mu.$$

Consequently we must take

$$\frac{dM}{d\varkappa} = \frac{\beta_0^2}{\alpha_0}$$

from which

$$M = \frac{\beta_0^2\varkappa}{\alpha_0} = \beta_0\varkappa\sqrt{a} = \alpha_0\varkappa a. \quad \ldots \ldots \quad (32)$$

The semi major axis a is constant, as it was in case III, and $\varkappa$ is variable. The meaning of $\varkappa$ is however different from that of ν in formula (22). From (10) we find

$$M\sqrt{1-e^2} = G \quad \ldots \ldots \quad (33)$$

Here again the motion is described as a Keplerian motion in an ellipse with varying elements. The ellipse has a point of contact with the true orbit, and therefore belongs to the family of ellipses mentioned above. The body P in its orbit, and the fictitious planet in its ellipse, however, have not the same velocity, but the same momentum. Since they have different masses, they have also different velocities, agreeing only in direction.

The energy is now

$$IV.\quad H = -\frac{M^2}{2ma^2} + \left(\frac{M^2}{ma} - k\right)\frac{1}{r} - S \quad . \quad . \quad . \quad . \quad (34)$$

and the living force

$$2mT = \frac{M^2}{a}\left(\frac{2}{r} - \frac{1}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

If we put $M = M_0 + \Delta M$,

$$M_0 = \frac{\beta_0 \sqrt{k}}{a_0}\sqrt{m},$$

then the differential equations become

$$\left.\begin{aligned}
&\frac{d\mu}{dt} = \frac{M}{am}\left(\frac{2}{r} - \frac{1}{a}\right) - \frac{\Delta M(2M_0 + \Delta M)}{amr^2}\frac{\partial r}{\partial M} - \frac{\partial S}{\partial M} \\
&\qquad\qquad\qquad \frac{dM}{dt} = \frac{\Delta M(2M_0 + \Delta M)}{amr^2}\frac{\partial r}{\partial \mu} + \frac{\partial S}{\partial \mu} \\
&\frac{dg}{dt} = -\frac{\Delta M(2M_0 + \Delta M}{amr^2}\frac{\partial r}{\partial G} - \frac{\partial S}{\partial G} \qquad \frac{dG}{dt} = \frac{\partial S}{\partial g} \\
&\frac{d\vartheta}{dt} = -\frac{\partial S}{\partial \Theta} \qquad\qquad\qquad \frac{d\Theta}{dt} = \frac{\partial S}{\partial \vartheta}
\end{aligned}\right\} \quad (36)$$

In the same way as the systems I, II, and III, we can of course derive other systems of elements. A system in which, as in III, the semi major axis is constant, but with osculation, is obtained as follows. We take the same function Φ, given by (5) or (9), but now we put

$$R^2 = \varkappa^2\left(-\alpha^2 + \frac{2\beta^2}{r} - \frac{\gamma^2}{r^2}\right).$$

The function R thus now contains four parameters. The elements I, II, III are derived as above by assigning to the fourth parameter a constant value $\varkappa = r_0 = \sqrt{m}$.

The equation (11) now becomes

$$\gamma = \frac{G}{\varkappa} + \delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

We have now, remembering that finally we will put $\delta = 0$.

$$\frac{\partial R}{\partial \varkappa} = \frac{R}{\varkappa} - \frac{G}{\varkappa^2}\frac{\partial R}{\partial \gamma} = \frac{1}{\varkappa}\left(R - \gamma\frac{\partial R}{\partial \gamma}\right) = \frac{1}{\varkappa}\left(\alpha\frac{\partial R}{\partial \alpha} + \beta\frac{\partial R}{\partial \beta}\right).$$

By the aid of (10) and (14) to (16) we find easily

$$\frac{\partial R}{\partial \varkappa}\, dr = \frac{\beta^2}{\alpha}(2d\mathrm{E} - d\mathrm{M}) = \frac{\beta^2}{\alpha}\, d\mu \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Here an angle μ has been introduced, of which the geometrical meaning is easily seen. If we take polar co-ordinates ϱ and φ with the second (empty) focus as origin, then μ bears the same relation to φ as the mean bears to the true anomaly. Therefore, since

$$r^2\, df = a^2 \sqrt{1-e^2}\, d\mathrm{M},$$

the equation connecting φ and μ is similarly

$$\varrho^2\, d\varphi = a^2 \sqrt{1-e^2}\, d\mu.$$

We have the formulas

$$\mu = \mathrm{E} + e \sin \mathrm{E}$$

$$\varrho \cos \varphi = a(\cos \mathrm{E} + e) \qquad \varrho = a(1 + e \cos \mathrm{E}) \quad . \quad (31)$$

$$\varrho \sin \varphi = a\sqrt{1-e^2} \sin \mathrm{E} \qquad \varrho = \frac{a(1-e^2)}{1-e\cos\varphi}.$$

The angle μ is easily seen to be proportional to the "action", if for the mass we take $\varkappa^2$. In that case the components of the momentum become $y_i = \varkappa^2 \frac{dx_i}{dt}$, and

$$\int 2T d\mathrm{M} = a^2\mu$$

I now take the fourth parameter $\varkappa$ as variable. We then have Here $r = 2a - \varrho$ must be expressed as a function of the elements by (31).

ΔM is of the order of the perturbing masses. If $S = 0$ the motion is Keplerian: $M, G, \Theta, g, \vartheta$ are constants.

For use with the elements IV, for which I will not try to coin a name. a development of the perturbative function S according to the trigonometric functions of multiples of μ would be required. This can be derived from the well known development in function of the mean anomaly by substituting ϱ for r, φ for v, $-e$ for e and μ for l.

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