Mathematics. — *Ordered groups.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

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§ 1. *Definition.* A set G of elements $a, b, \ldots$ is said to form an ordered group if the conditions expressed in postulates I, II are satisfied:

I. G is a group;

II. G is a simply ordered set, that is, for each pair of distinct elements either $a < b$ or $a > b$, while α) $a < b$, $b < c$ includes $a < c$ and β) moreover $a < b$ includes: $ac < bc$ and $ca < cb$. So the ordering of a group implies the ordering of all its undergroups.

From a topological point of view the ordered groups are to be considered as Hausdorff-spaces; their topological structure is simple on the ground of the property β): the ordered groups are homogeneous in this sense that topological properties of one element are adherent to all elements. If e.g. an element of G is a point of accumulation, then all elements of G satisfy this condition and the group is called dense in itself.

If an ordered set O is dense in itself, then generally O need not be dense-ordered. (A set O is be called dense-ordered if for each pair of elements a and b of O there is at least one element c "between" a and b). For ordered groups however we prove:

1. 1. If G is an ordered group dense in itself, then G is densely ordered. If there was namely between two elements a and b no further element, then every element had an immediate successor and predecessor and thus G was not dense in itself. Conclusion: An ordered group G is dense in itself if and only if G is densely ordered. If on the contrary one element is isolated, then every element is necessarily isolated; consequently:

1. 2. An ordered group G is either a discretely ordered group, which means that every element has an immediate successor and predecessor, or a densely ordered group. The property of an ordered group as being discrete resp. dense is as a topological property invariant for a mapping preserving the order.

If two ordered groups of the same ordering-structure are simply-isomorphic, we call them ordered-isomorphic.

1. 3. An ordered group is not finite; neither a first nor a last element exists. For natural m: $a^m > a^{m-1}$ for $a \neq e$ and $a > e$, in other words no element except e is of finite order. It follows from $a > b$ that $a \cdot b^{-1} \cdot a > a$, thus G cannot contain a last element a, neither a first element.

§ 2. If for two elements a and b ($a \geqq b > e$) of an ordered group there exists a natural number n with $b^n > a$, then b is said to be archimedian with regard to a. For $b > e$ and $a \geqq b^{-1} > e$, then b will be called archimedian with regard to a, if b^{-1} is archimedian with regard to a; if $a < e$ as well as $b < e$ and $a^{-1} \geqq b^{-1}$ e, then b is said to be archimedian with regard to a, if b^{-1} is archimedian with regard to a^{-1}. If for each pair of elements a and b of an ordered group G a is archimedian with regard to b, then we say that G has an archimedian order and otherwise a non-archimedian order.

If for two elements a and b $(e < a < b)$ a is archimedian with regard to b, then a and b are said to have the same "rank". If $e < a < b$ and for all n: $a^n < b$ then a is said to be of lower rank then b, $a \prec b$. For $a < e$, $b > e$, we define $a \prec b$, if $a^{-1} \prec b$ and if $a < e$, $b < e$, we define $a \prec b$, if $a^{-1} \prec b^{-1}$.

Thus the elements of an archimedian ordered group — exept e — are of the same rank. The property of two elements as being of the same rank, is obviously reflexive, symmetrical and transitive. This enables us to divide the ordered group G into classes of elements: one class contains those and only those elements that are in respect to each other of the same rank. Thus every element of G is contained in only one class. Except for the unity-class E, which contains only the unity-element e, we can state:

2. 1. A class C is not finite, because C contains with an element a also the whole cyclic group $\{a\}$ except the element e.

If A and B are two classes, $a \subset A$, $b \subset B$ and $a \prec b$ then it is obvious that this relation exists also between $a' \subset A$ and $b' \subset B$. For this reason we define the class A of lower rank then the class B, $A \prec B$, if the elements of A are of lower rank than those of B. The relation $A \prec B$ is transitive, in other words:

2. 2. The classes of an ordered group define an ordered set, called the class-set Λ of the ordered group G. The type or order of Λ we call the class-type of G. Λ has the property, that the unity-class E consisting of e precedes all other classes. The class-set of a discretely ordered group moreover has the property that the successor of e belongs to the lowest class after E. If we speak in the following about Λ, then we mean the class-set without the unity-class E, thus:

2. 3. The class-set of a discretely ordered group has a first element. Therefore Λ itself need not be a discretely ordered set. Archimedian ordered groups are of the class-type 1.

We add to every element a of an ordered group a rank $\mathfrak{r}_a$ with the following conditions:

1. $\mathfrak{r}_a = \mathfrak{r}_b$ if and only if a and b belong to one class;
2. $\mathfrak{r}_a < \mathfrak{r}_b$ if and only if a belongs to a lower class than b.

Thus the set of all ranks is a set with the same order-structure as Λ. It is easy to see that

2. 4. α) $\mathfrak{r}_a = \mathfrak{r}_{a^{-1}}$; β) from $a < c < b$ follows $\mathfrak{r}_a \leqq \mathfrak{r}_c = \mathfrak{r}_b$; γ) $\mathfrak{r}_{ab} = \max(\mathfrak{r}_a; \mathfrak{r}_b)$ and only if $\mathfrak{r}_a < \mathfrak{r}_b$, then $\mathfrak{r}_{ab} = \mathfrak{r}_b$; δ) we prove: $\mathfrak{r}_{pq} = \mathfrak{r}_{p^{-1}q}$ if $\mathfrak{r}_p \neq \mathfrak{r}_q$; if namely $\mathfrak{r}_p < \mathfrak{r}_q$, then $\mathfrak{r}_{pq} = \mathfrak{r}_q$; as $\mathfrak{r}_p = \mathfrak{r}_{p^{-1}}$, so also $\mathfrak{r}_{p^{-1}} < \mathfrak{r}_q$, therefore $\mathfrak{r}_{p^{-1}q} = \mathfrak{r}_q$ and $\mathfrak{r}_{p^{-1}q} = \mathfrak{r}_{pq}$. For $\mathfrak{r}_p > \mathfrak{r}_q$ there is a similar reasoning. For $\mathfrak{r}_p = \mathfrak{r}_q$ the statement is generally not valid.

2. 5. The classes of densely ordered groups are at once open and closed sets. If a and b are elements of a class A, then $\mathfrak{r}_a = \mathfrak{r}_b$. If c belongs to the interval $(a\,;b)$, then we conclude (on the ground of 2. 4. β) $\mathfrak{r}_a = \mathfrak{r}_b = \mathfrak{r}_c$, in other words the interval $(a\,;b)$ together with c also contains an interval, that c contains and that is contained in $(a\,;b)$, on the ground of the dense ordering. Thus every class A is open and therefore A, being the complement of the sum of all the other classes, is closed.

2. 6. An ordered group is a topological group.

a. The inverse element x^{-1} of an element x is a continuous function of x while the mapping $a(x) = x^{-1}$ presents an inverse ordering mapping, in other words, a is a topological mapping;

b. The product xy of two elements x and y is a continuous function of x and y. Therefore it is sufficient to prove, that if $x \to x_0$ and $y \to y_0$ that $xy \to x_0 y_0$. We have

$$(xy)\,(x_0 y_0)^{-1} = (xx_0)^{-1} \cdot [x_0\,(yy_0^{-1})\,x_0^{-1}].$$

The second term is the product of two elements approaching to e; and so the first term has the limit e and xy has the limit $x_0 y_0$.

2. 7. If an ordered group G has a class A of lowest rank then A-with the exception of e-forms a group; if moreover G is discretely ordered then this group of lowest rank is a cyclic group.

If a and b are elements of A, so $\mathfrak{r}_a = \mathfrak{r}_b$; it follows that $\mathfrak{r}_{ab} \leqq \mathfrak{r}_a$. If $\mathfrak{r}_{ab} < \mathfrak{r}_a$ then necessarily $ab = e$ and a and b are inverse elements. If a is not the inverse element of b, then $\mathfrak{r}_{ab} = \mathfrak{r}_a$ and ab also belongs to the class of lowest rank. If G is discrete then this group of lowest rank is a cyclic one: if namely a is the immediate successor of e then the lowest class A contains — with the exception of e — the cyclical group $\{a\}$.

If moreover A contained an element b not belonging to $\{a\}$, then b would, on the ground of the archimedian ordering of a class, be an element of an interval $(a^m\,; a^{m+1})$ with the consequence that $e < b\,.\,a^{-m} < a$, contrary to the definition of a. According to this theorem the discrete and archimedian ordered groups are cyclic groups. We shall return to this subject later on.

2. 8. A discretely ordered group G is archimedian ordered if and only if every Dedekind-cut defines a sault [1]).

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1) If we divide an ordered set N into two subsets N_1 and N_2 with the conditions:

I. Every element of N belongs to one, but only one of the sets N_1 and N_2;

II. Neither N_1 nor N_2 is empty;

III. All elements of N_1 precede those of N_2,

then we call this division of N a Dedekind-cut $N_1 \mid N_2$ in N. There are only the following cases that exclude each other:

1. N_1 has a last, N_2 a first element; then we call the cut $N_1 \mid N_2$ a sault in N.

2^a. N_1 has a last, N_2 no first element;

2^b. N_1 has no last, N_2 has a first element; we call the cut in 2^a and 2^b a continuous cut.

3. N_1 has no last, N_2 has no first element; the cut is called a lacuna in the set N. The ordered set N is called continuously ordered, if no cut in N defines a sault or a lacuna.

Necessary. Suppose there be a cut that did not define a sault, then it followed that there existed two elements p and q $(e \leqq p < q)$, so that the interval $(p\,; q)$ contained an infinite number of elements. Let a be the immediate successor of e then the series of elements that followed p would be: p, $p \,.\, a$, $p \,.\, a^2$, ..., $p \,.\, a^n$, ... and for no finite value of n we should have $p \,.\, a^n \geqq q$, in other words the group would be non-archimedian ordered.

Sufficient. Suppose that G was non-archimedian ordered then there would exist at least one pair of elements a and b $(e < a < b)$, so that for no finite n: $a^n > b$. If we now divide G into two subsets A and B (with $A + B = G$): A contains all elements α with $\alpha \leqq a^n$ for all n; B contains the remaining elements. B is not empty while $b \subset B$. This cut, however, defines no sault, as A e.g. certainly has no last element. A discretely and archimedean ordered group has the properties: a) no first element, no last element; b) every cut defines a sault. Therefore

2. 9. A discretely and archimedean ordered group G is of the type $\omega^* + \omega$ of the set of all ordinary integers.

Suppose e to be the unity element of G, a the immediate successor of e then G is the cyclic group $\{a\}$ (see above) with the groupoperation $a^m \,.\, a^n = a^{m+n}$, $a^0 = e$, $a^m > a^n$ if $m > n$ with m and n integer. If we add to e the number o, to a^m the number m, then we have:

2. 10. Discretely and archimedian ordered groups G are free cyclic groups; they are ordered-isomorphic with the additive group of all ordinary integers and therefore commutative.

2. 11. A non-archimedean ordered group is not continuously ordered.

If the ordering was continuous then of every cut $A|B$ either A had a last, B no first element, or just the reverse. We shall try to prove, however, that in every non-archimedean ordered group there is to define a non-continuous cut. Suppose $e < a < b$ and a non-archimedean with regard to b. Let the class A contain all elements $\leqq a^n$ for a positive integer n, B the remaining elements; B is not empty, as $b \subset B$. A has no last element, nor has B a first element, for B also contains the elements $ba^{-1} > ba^{-2} > \ldots$, elements that precede b. Therefore a non-archimedean ordered group is never a continuous ordered set or directly: continuously ordered groups are necessarily archimedean ordered groups. Non-archimedean ordered groups therefore are in a topological sense unconnected. If we complete a dense-ordered group then the ordering becomes continuous. By completing such a group the original ordering is maintained. Therefore, non-archimedean ordered groups are not be completed in the usual way, but archimedean densely ordered groups can be completed, as van Dantzig proves us in this way that the transformed series of series converging to e are also e-convergent.

2. 12. A densely ordered group G has the property that to every element $a > e$ there exists an element $b > e$, so that $b^2 < a$.

We divide G into two classes N_1 and N_2 ; N_1 contains the elements b_1

with $b_1^2 \leqq a$ and N_2 the elements b_2 with $b_2^2 > a$. This division defines a Dedekind-cut, for $e \subset N_1$, $a \subset N_2$. Is the cut continuous then there exists an element $b > e$ so that $b^2 = a$ and for the elements $e < b_1 < b : b_1^2 < a$. If the cut defines a lacuna then N_1 has no last element, N_2 no first element, in other words e is not the last element with $e^2 < a$; there are other elements $b_1 > e$ with $b_1^2 < a$. Without a proof we pronounce the following theorem:

2. 13. In a continuous ordered group there exists to every element $a > e$ one element $b > e$ with $b^n = a$.

2. 14. In an archimedean densely ordered group G we can construct a series of elements converging towards e.

Suppose $a_0 > e$ to be a given element of G. According to 2. 12 there exists an element $a_1 > e$ with $a_1^2 < a_0$; further an element $a_2 > e$ with $a_2^2 < a_1$, ..., generally an element $a_n > e$ with $a_n^2 < a_{n-1}$. We prove that the series (a_i) $(i = 1, 2, 3, \ldots)$ penetrates into every interval $(e ; b)$ with $b > e$.

Suppose therefore $e < b_0 < a_0$ and $b^n > a_0$; from the construction of a_i follows: $a_1^2 < a_0$, $a_2^4 < a_0$, $a_3^8 < a_0$, ..., and thus $a_n^n < a_0$ and $a_n^n < b_n$ or $a_n < b$, with which we proved the existence of a series of elements converging towards e.

2. 15. In an archimedean densely ordered group G there exists a denumerable densely ordered subgroup R dense in G.

We choose in G a series (a_i) converging towards e, constructed as in 2. 14. We prove that the intervals formed by the powers of a_i form a denumerable system of neighbourhoods. We simply prove that in every interval $(b ; c)$ is found an interval $(a_n^p ; a_n^{p+1})$. We choose namely an element a_n out of (a_i) so that $a_n^2 < cb^{-1}$. While G is archimedean ordered, there exists an integer number p, so that $a_n^{p-1} \leqq b$ and $a_n^p > b$. But then $a_n^{p+1} = a_n^{p-1} . a_n^2 \leqq a_n^2 . b < c . b^{-1} . b = c$, in other words that the interval $(a_n^p ; a_n^{p+1})$ is contained in the interval $(b ; c)$. So we have proved: All archimedean densely ordered groups G have a denumerable dense subset. Then the continuously archimedean ordered groups G satisfy the conditions: 1) continuous ordering; 2) no first, no last element; 3) G contains a denumerable, in G dense subset R. This means:

2. 16. A continuously archimedean ordered group G is of the same order-structure as the set of all real numbers, while G contains a denumerable subset R dense in G, so that R has the orderstructure of the rational numbers. We prove finally that every continuously archimedean ordered group is ordered-isomorphic with the additive group of all real numbers and to prove this we show first:

2. 17. Every continuously archimedean ordered group G contains a denumerable subgroup R, dense in G and ordered-isomorphic with an additive group R' of rational numbers. We construct the subgroup R with

the aid of a series (a_i) converging in G towards e and so that

$$a_1^{10} = a_0,\ a_2^{10} = a_1, \ldots, a_{n+1}^{10} = a_n, \ldots$$

which is possible according to (2. 13). The group R now exists of all products of a finite number of a_i ; R is denumerable and is dense in G. We add

$$a_0 \leftrightarrow 0,\ a_1 \leftrightarrow 0,1, \ldots, a_p \leftrightarrow 10^{-p},\ a_p \, . \, a_q \leftrightarrow 10^{-p} + 10^{-q}.$$

So for R we get a group ordered-isomorphic with an additive group of rational numbers, namely the additive group of all finite decimal fractions.

With this result we prove finally:

2. 18. A continuously archimedean ordered group G is ordered-isomorphic with the additive group G' of all real numbers.

We use the theorem that G contains a denumerable subgroup R, dense in G and ordered isomorphic with an additive group R' of rational numbers. Further we use 2. 16, namely that G and G' have the same orderstructure; if a and b are elements of G $(a > e, b > e)$, a' and b' their mapping-elements in G' then it is sufficient to prove: $ab \leftrightarrow a'b'$. Therefore we choose an element $x > e$ of R, so that $e < x < a$ and $e < x < b$. Then we have in G': $e' < x' < a'$, $e' < x' < b'$. According to the archimedean ordering of G there exist integer numbers n_1 and n_2 with $x^{n_1} \leqq a < x^{n_1+1}$ and $x^{n_2} \leqq b < x^{n_2+1}$. R and R' are ordered isomorphic, thus $x^k \leftrightarrow x'^k$. While G and G' are of the same order-structure, $x'^{n_1} \leqq a'$ and $x'^{n_1+1} > a$. For x and x' the corresponding n_1 and n_2 are equal. From $x^{n_1} \leqq a < x^{n_1+1}$ it follows $e \leqq ax^{-n} < x$ and $e \leqq bx^{-n_2} < x$. If now $x \rightarrow e$, which is possible in R, then $ax^{-n_1} \rightarrow e$, therefore

$$a = \lim_{\substack{x \rightarrow e \\ x > e}} x^{n_1} \text{ and } b = \lim_{\substack{x \rightarrow e \\ x > e}} x^{n_2}, \text{ but also } a' = \lim_{\substack{x' \rightarrow e' \\ x' > e'}} x'^{n_1} \text{ and } b' = \lim_{\substack{x' \rightarrow e' \\ x' > e'}} x'^{n_2}.$$

According to the continuity of the group

$$ab = \lim_{\substack{x \rightarrow e \\ x > e}} x^{n_1+n_2} \text{ and } a'\,b' = \lim_{\substack{x' \rightarrow e' \\ x' > e'}} x'^{n_1+n_2}$$

and therefore $ab \leftrightarrow a'b'$.

Moreover we see $ab = ba$. Every continuously archimedean ordered group therefore is ordered-isomorphic with the additive group of all real numbers.

2. 19. Every archimedean ordered group is ordered isomorphic with an additive group of real numbers.