Mathematics. — "*On* MAC LAURIN's *Theorem in the Functional Calculus*". By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. L. E. J. BROUWER).

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In the third communication of my paper "Some observations on complete transmutation"[1]) I proved a restricted validity of MAC LAURIN's theorem in the Functional Calculus for a *normal* additive transmutation. A normal transmutation was defined by me as follows:

1. There is a functional field $F(T)$, the functions u of which *belong to*[2]) the very same circle (σ), and for these functions the transmutation T produces functions belonging to the very same circle (α), concentric with (σ).

2. All rational integral functions are included in the functional field.

3. The transmutation T is continuous in the pair of associated fields $F(T)$ and (α)[3]).

From 2 it can be derived that to any such transmutation T another transmutation P *formally* corresponds, which is given by

$$Pu = a_0 u + \frac{a_1 u'}{1} + \frac{a_2 u''}{2\,!} + \ldots + \frac{a_m u^{(m)}}{m\,!} + \ldots, \quad . \quad . \quad . \quad (1)$$

where the quantities $u^{(m)}$ are the derivatives of the subject of operation u and the quantities a_m functions of the numerical variable x, which by means of the formula

$$a_m = \xi_m - m_1 x \xi_{m-1} + \ldots + (-1)^m x^m \xi_0 \text{ [4])} \quad . \quad . \quad . \quad . \quad (2)$$

can be derived from the functions

$$\xi_k = T(x^k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[1]) These Proc. Vol. XIX N°. 6 and 8 and Vol. XX N°. 3—7, to be quoted as l.c.

[2]) A function *belongs* to a circle, if it is regular *within and on* it. The symbol (σ) means a circle with radius σ.

[3]) See for the definition of continuity l.c. Vol. XIX N°. 8, where also definitions are given of the expressions *functional field* (*F. F.*). *numerical field of operation* (*N. F. O.*) (here the circle (α)) and *numerical field of functions* (*N. F. F.*) (here the circle (σ)).

[4]) By m_k the binomialcoefficient of order k of the number m is meant.

which are the transmutated of the successive positive integral powers of x: the latter functions exist according to 2 and belong to the circle (α). The above-mentioned theorem of MAC LAURIN consists in stating the *equality* of the transmutations T and P, in a certain numerical field (α'), which is part of (α) or identical with (α), *if the further condition is added, that the latter transmutation is complete in* (α')[1]).

In this statement there is something unsatisfying, if we compare it with TAYLOR's theorem for the *Theory of Functions*. The latter asserts: "if a function, in a certain circle, has some specified properties (viz. a *definite* differential coefficient) it can be expanded in TAYLOR's series in that circle". It is therefore not necessary *to impose further conditions on that series*. Accordingly it would be desirable that also in the *Functional Calculus* the theorem might be stated in such a way that it were not necessary to impose any further condition on the series P corresponding to the given transmutation T, but that *such conditions were implied in the properties of T*. At the time it was our opinion that this was not the case. But now we are in a position to prove the following proposition:

The series corresponding to a normal additive transmutation represents a complete transmutation.

For simplicity we consider a circular domain (σ) round the origin as a centre and in this the infinite sequence of functions

$$1, x, x^2, \ldots x^m, \ldots \qquad (4)$$

to which, by definition, the infinite sequence of transmuted

$$\xi_0, \xi_1, \xi_2, \ldots \xi_m, \ldots \qquad (5)$$

corresponds, the latter functions being regular in a circular domain (α). If we denote by ε an arbitrarily small positive quantity, then the sequence of functions

$$1, \frac{x}{\sigma+\varepsilon}, \frac{x^2}{(\sigma+\varepsilon)^2}, \ldots \frac{x^m}{(\sigma+\varepsilon)^m}, \ldots \qquad (6)$$

derived from (4) converges uniformly to zero in the domain (α). According to a simple property of continuity (l.c. II, n°. 11) the sequence of the transmuted of the latter functions, which, by the additive property of the transmutation, is represented by

1) A transmutation P, represented by a series of the form (1), is called by me *complete in a domain* (α), if there is a certain circle (ρ), concentric with (α), such that *all* functions belonging to (ρ) possess a transmuted, regular in (α). The minimum circle (β), which may be taken for (ρ), I called the domain *corresponding* to (α) (l.c. Vol. XIX; N°. 6).

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$$\xi_0, \frac{\xi_1}{\sigma + \varepsilon}, \frac{\xi_2}{(\sigma + \varepsilon)^2}, \ldots \frac{\xi_m}{(\sigma + \varepsilon)^m}, \ldots \quad (6')$$

will converge uniformly to zero in the domain (α); because a normal transmutation is continuous in a pair of conjugate fields. For sufficiently large m-values we have therefore in all points of (α)

$$|\xi_m| < (\sigma + \varepsilon)^m \quad (7)$$

From the equation (2) it is now easily derived that an analogous inequality holds for the functions a_m occurring in the series (1), that is to say, that these functions, too, are less in value than the m^{th} power of a certain number independent of m. For if (7) is valid for $m > m_0$, we have by (2), $|x|$ being at most equal to α,

$$|a_m| < \sum_{0}^{m_0} m_k \alpha^{m-k} |\xi_k| + \sum_{m_0+1}^{m} m_k \alpha^{m-k} (\sigma + \varepsilon)^k$$

$$< \longleftrightarrow + \sum_{0}^{m} \longleftrightarrow$$

$$< \longleftrightarrow + (\sigma + \alpha + \varepsilon)^m$$

The first part of the second member of this inequality consists of a fixed number of terms, each of which is, for sufficiently large m-values, less than $(\alpha + \varepsilon)^m$, so that the same holds for their sum. The second part is greater than the latter quantity, hence we have for sufficiently large m-values at all points of the domain (α)

$$|a_m| < (\sigma + \alpha + \varepsilon)^m \quad (8)$$

and therefore also

$$\overline{\lim_{m=\infty}} |a_m|^{\frac{1}{m}} \overline{<} \sigma + \alpha \quad (8')$$

Thus the upper limit in the left-hand member of this inequality is *finite*, and this is (Vol. XIX, N°. 6) the very condition under which the transmuting series (1) is *complete* in the domain (α); moreover we infer that the corresponding domain (β) has a radius β, no greater than $\sigma + 2\alpha$. For all functions u belonging to the circle $(\sigma + 2\alpha)$ the series P therefore produces a transmuted function Pu in the domain (α), and this transmuted is equal to Tu, according to the functional theorem of MAC LAURIN we gave in the form (Vol. XX, N°. 3):

If the series P, answering to a normal additive transmutation T, is complete in the circular domain (α) forming the N. F. O. of T, we have in this domain $Tu = Pu$ for such functions of the functional field $F(T)$ of T as belong to the circle (β) corresponding to (α).

But now that we have found that the series answering to a normal additive transmutation has *necessarily* the property of representing a *complete* transmutation, our theorem can be expressed in the following form, in which it is more really a "theorem of MAC LAURIN" for the functional calculus, the unsatisfying point mentioned above having disappeared:

A *normal additive transmutation can always, in its N. F. O.* (α), *be expanded in the functional series of* MAC LAURIN (1), *either for all functions of its F. F., or for a certain part of it, consisting of the functions belonging to a circle* $(\beta) > (\sigma)$, (σ) *being the N. F. F.; and the radius* β *is never greater than* $\sigma + 2a$.