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Mathematics. — *"On the necessary and sufficient conditions for the expansion of a function in a Binomial Series".* By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 3, 1919)

PINCHERLE has given a necessary and sufficient condition for the expansion of a function in a *binomial series* (*Binomialkoeffizientenreihe*) [1]). It runs thus:

The necessary and sufficient condition that an analytic function $\omega(x)$ *may be expanded in a series of the form*

$$\omega(x) = \sum_{0}^{\infty}{}_{n}\, c_n \binom{x-1}{n} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

is that $\omega(x)$ *be* *coefficient-function* (*fonction coefficiente*) *of another analytic funtion* $\varphi(t)$, *which is regular and zero at infinity and whose singularities lie all within the circle* (1, 1), *with centre* $t=1$ *and radius* $r=1$, *or on the circumference of it, provided that, in the latter case, the order of* $\varphi(t)$ *on the circumference, taken in the sense defined by* HADAMARD, *be finite or negative infinite* [2]).

By a coefficientfunction $\omega(x)$ of an analytic function $\varphi(t)$ of the kind mentioned PINCHERLE means a function which can be deduced from $\varphi(t)$ in a more or less simple manner, according to the order of $\varphi(t)$. The relation between the two functions is, however, always such that conversely $\varphi(t)$, called by PINCHERLE the *generating* function (fonction génératrice) of $\omega(x)$, follows from $\omega(x)$ by the equation

$$\varphi(t) = \sum_{0}^{\infty}{}_{n}\, \frac{\omega(n+1)}{t^{n+1}}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

This means: the *coefficients* of the series of negative integral powers of t, in which $\varphi(t)$ may be expanded in a neighbourhood of $t=\infty$, are equal to the values of $\omega(x)$ for positive integral values of x; the name *coefficientfunction* for $\omega(x)$ is due to this circumstance.

The question now arises, how it must be discriminated if a *given* function may oe expanded in a binomial series. *This question is not*

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1) S. PINCHERLE, "Sur les fonctions déterminantes", Annal. de l'Ecole Normale, 1905.

2) A circle with centre α and radius r will be denoted by (α, r).

answered by the theorem of PINCHERLE, at least not in a simple manner, as will appear from what follows. In order to investigate the question we should commence to deduce the series (2) from the given function $\omega(x)$. Next we should examine whether the function $\varphi(t)$ represented by it has the required properties: to be regular without the circle (1,1), and on the circumference of it of finite order. For this we should try to transform the above series into another according to negative integral powers of $t-1$

$$\varphi(t) = \sum_{0}^{\infty}{}_{n} \frac{c_n}{(t-1)^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The relation between the coefficients of the two series is given by the equations

$$c_n = \omega(n+1) - \binom{n}{1} \omega(n) + \ldots + (-1)^n \omega(1) = \Delta^n [\omega(1)] \text{ [1]} \quad . \quad (4)$$

and

$$\omega(n+1) = c_0 + \binom{n}{1} c_1 + \ldots + c_n \quad . \quad . \quad . \quad . \quad (4')$$

By means of (4) we must see if the series (3) converges without the circle (1,1), and further if the *characteristic* λ' of the coefficients c_n, defined by

$$\lambda' = \overline{\lim_{n=\infty}} \frac{\log |c_n|}{\log n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is not positive infinite; the latter being the condition that $\varphi(t)$ shall be of non-positive infinite order on the circle (1,1).

But the relation (4) is rather intricate and so it may be very difficult, if not impossible, to perform the just mentioned research. Suppose this, however, possible, and let λ' differ from $+\infty$. Then we have to examine whether the given function $\omega(x)$ is really the coefficientfunction of $\varphi(t)$. For there are a great many functions $\Omega(x)$ giving rise to the *same* generating function $\varphi(t)$, viz. all those contained in the equation

$$\Omega(x) = \omega(x) + F(x)$$

where $F(x)$ is a function that *vanishes* for positive integral values of x. It is therefore necessary to consult the definition of the coefficient-function given by PINCHERLE and to apply it to the obtained $\varphi(t)$ in order to see if the original function $\omega(x)$ is the result. But this is again not very easy. If the characteristic λ' defined by (5) is less than -1, then $\varphi(t)$ is finite and continuous along the circum-

[1]) This symbol denotes the n-th difference of $\omega(x)$ at $x=1$, the increase Δx of the argument x being equal to unity.

2*

ference of the circle (1,1). The coefficient-function is then defined by the integral

$$\frac{1}{2\pi i}\int_{(1,1)} \varphi(t)\, t^{x-1}\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

taken round the just mentioned circle, and this integral is easily seen to be equal to the binomial series (1) in the half-plane of x on the right of the imaginary axis, provided definite agreements as to the value of t^{x-1} be made. It may, however, be difficult to investigate whether the integral (6) is equal to the given function $\omega(x)$. And in any case the investigation is intricate if $\lambda' > -1$, especially when the difference between λ' and -1 is rather great. For the coefficient-function of $\varphi(t)$ is then brought into relation with that of another generating function, with $\lambda' < -1$, by means of a polynomial consisting of a very large number of terms.

The question therefore naturally suggests itself, if it is possible, to find simpler tests which are *sufficient* for a function to be expanded in a binomial series. This is, indeed, the case, and we may, moreover, say that the obtained properties are about *necessary* [1]). For we can prove the following theorem:

If a function $\omega(x)$ *is regular in the finite part of the half-plane* $R(x) \geqq \gamma$ [2]) ($\gamma = real$), *and if, in that domain, it satisfies the inequality*

$$|\,\omega(x)\,| < M\,|\,(x+b)^l\, a^{x-\gamma}\,| \;, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where M *is a positive and* l *and* b *are real numbers, the latter such that* $b + \gamma > 0$, *further* a *a complex number on the circumference of the circle* (1,1), *variable with the argument* ψ *of* $x-\gamma$, *the argument* α *of* a *being equal to* $-\psi$ *so that*

$$a = 2\cos\psi\, e^{-i\psi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

then, in the domain ($\beta = real > \gamma$).

$$R(x) > l + \beta - \tfrac{1}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

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1) For the sake of comparison we observe that for the expansion of a function in a *series of factorials*

$$\sum_{0}^{\infty}{}_n \frac{n!\, a_n}{x(x+1)\ldots(x+n)}$$

there is a necessary and sufficient condition, stated by Nielsen and simplified by Pincherle, which has some similarity with the above condition for binomial series. But it is not possible to find *simple* tests for the expansion of a function in a series of factorials. The only simple *sufficient* condition which may be given in this case, is that a function can be expanded in a series of factorials, if it is *regular and of zero value* at infinity. But this condition is far from necessary.

2) $R(x)$ means the real part of x.

the function $\omega(x)$ *can be expanded as a binomial series*

$$\omega(x) = \sum_{0}^{\infty} {}_n \, c_n \binom{x-\beta}{n} \text{[1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

if $l + \beta - \frac{1}{2} \gtrless \gamma$, *and otherwise the expansion is possible in the domain* $R(x) \gtrless \gamma$.

The special value $-\psi$ of the argument α of a is such that the expression $a^{x-\gamma}$, $x-\gamma$ being given, has the greatest modulus compared to those for other values of a on the circumference of the circle (1,1). *If the inequality* (7) *holds for fixed a-value on the circle* (1,1), *then expansion of* $\omega(x)$ *in the series* (10) *is possible in the domain*

$$R(x) > l + \beta - 1 \; , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

if $l + \beta - 1 \gtrless \gamma$, *and otherwise in the domain* $R(x) \gtrless \gamma$.

The sufficient condition for the expansion of a function in a binomial-series contained in the above theorem seems, indeed, very simple. If a function $\omega(x)$ can be represented by the equality

$$\omega(x) = (x + b)^k \, c^x \, \mu(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where c is a fixed number *within* the circle (1,1) and $\mu(x)$ a function remaining within finite limits in $R(x) > \gamma$, then it satisfies the inequality (7) for a value of l differing arbitrarily little from $-\infty$, and therefore it can be expanded in a binomial-series in the domain $R(x) \gtrless \gamma$. For $c = 1$ formula (11) gives an expression which shows that all functions regular in the finite part of the half-plane $R(x) \gtrless \gamma$ and vanishing at infinity may be expanded as a binomial-series in that domain; further all functions becoming infinitely large at infinity of an order lower than a certain finite power of x; so all irrational and logarithmic expressions.

The way in which we have arrived at our theorem is substantially the same as that followed in the ordinary theory of functions of a complex variable, in order to obtain the expansion of a function in a *power-series*; it is founded upon the fundamental theorem of Cauchy. According to this we have

$$\omega(x) = \frac{1}{2\pi i} \int_W \frac{\omega(z)\, dz}{z - x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where the integral is taken round a closed curve W, *within* and *upon* which $\omega(x)$ is regular, and which contains the point $z = x$ in its interior. If we wish to deduce from this integral an expansion according to positive integral powers of $x-a$, a being a number

[1]) This series is taken instead of (1) for the sake of generality.

within W, then we start from the *known* expansion with *known remainder* of $1:(z-x)$ in such a series. In the same way we may reach our present purpose, if we use the *known* expansion with *known remainder* of the just mentioned elementary function in a binomial series, viz.

$$\frac{1}{z-x}=\sum_{0}^{n-1}{}_m \frac{(x-\beta)\ldots(x-\beta-m+1)}{(z-\beta)\ldots(z-\beta-m)}+\frac{(x-\beta)\ldots(x-\beta-n+1)}{(z-\beta).\ .(z-\beta-n+1)}\cdot\frac{1}{z-x}.$$

Substituting this expression for $1/z-x$ in the integral (12) and choosing the path of integration so as to include, besides the point $z=x$, the points $z=\beta,\ \beta+1,\ldots,\beta+n-1$, we find [1])

$$\omega(x)=\sum_{0}^{n-1}{}_m\binom{x-\beta}{m}\Delta^m\omega(\beta)+R_n\ .\quad .\quad .\quad .\quad (13)$$

where

$$R_n=\frac{1}{2\pi i}\int_W\omega(z)\frac{(x-\beta)\ldots(x-\beta-n+1)}{(z-\beta)\ldots(z-\beta-n+1)}\cdot\frac{dz}{z-x}\quad .\quad .\quad (13')$$

Formula (13) is the ordinary formula of interpolation of NEWTON with a *remainder* added to it and valid for all complex x-values lying within W.

If all points $z=\beta,\ \beta+1,\ldots,\beta+n-1$, are to lie within the integration-curve W, this curve will in general have to be modified with increasing n. It is required to choose W as fit as possible, that is to say: so that the remainder (13') tends to zero with indefinite increase of n, and that yet the aggregate of functions $\omega(x)$ for which this takes place, is as extensive as possible. If, now, the form (7) is taken as majorant-value of these functions, where the number a is, as yet, left undetermined and the number γ, in order to have a definite case, is chosen zero (so that $\beta>0$), it is found after a rather long but principally not difficult inquirement: 1. that the most favourable integration-curve is a circle with $z=n$ as centre and n as radius so that it passes through the origin; 2. that for a a complex number may be taken lying on the circumference of the circle (1,1), with the specifications concerning the domains of validity already mentioned in the Statement of the above theorem.

We may further observe that, in case the number c in formula

[1]) If a few points $\beta,\ \beta+1,\ldots$, are *excluded* from the closed curve W, we obtain an expression the further examination of which leads to the so-called *zero-expansions*, which are treated in an elementary way by PINCHERLE (Rendic. d. R. Accad. d Lincei, 1902, 2e Sem.)

(11) is real and not greater than 1, a *fixed* integration-path W for the remaining integral may be chosen, as soon as the number n attains a certain magnitude, and for this we may take the imaginary axis in this case. The proof that $\lim R_n = 0$ for $n = \infty$ is then very simple, so that the above mentioned particular cases in which a function can be expanded in a binomial series, may be derived in a *short* manner from CAUCHY's integral.

As further regards the question, how far the inequality (7) is *necessary* for the expansion of a function in a binomial series, the way in which the *sufficient* condition has been obtained gives us the conviction that the aggregate of functions determined by the latter condition is as large as possible. In order to come to certainty concerning this it is necessary to investigate how a function represented by a binomial series behaves in the domain of convergence of that series. This investigation may be effected by means of the statement, contained in the theorem of PINCHERLE, that a binomial series necessarily represents a *coefficientfunction*, at least in the domain of *absolute* convergence of that series, for to this only the proposition of PINCHERLE applies.

For simplicity we assume for the binomial series the original form (1), which is the one considered by PINCHERLE. If the characteristic λ' of the coefficients c_n is less than -1, then, as already mentioned, the binomial series can be represented by the integral (6) in the half-plane $R(x) > 0$ It can now be proved that this integral satisfies, in the domain mentioned, the condition (7), with $\gamma = 0$, the exponent l being subject to the inequality

$$l < -\tfrac{1}{2} + \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where σ is an arbitrarily small positive number. This condition can further be specified by *extending* a certain property of the operation I by means of which, according to the view of PINCHERLE, the generating function $\varphi(t)$ passes into the coefficient-function $\omega(x)$; we mean the property expressed by the equation

$$I\,[\varphi^{(r)}(t)] = \frac{(-1)^r\,\Gamma(x)}{\Gamma(x-r)}\,I\,[t^{-r}\,\varphi(t)]\,,$$

This equation is given by PINCHERLE (l.c. p. 30) for the case r is a positive integer. If r is replaced by $-\alpha$ and $\varphi(t)$ by $\varphi(t):(t-1)^\alpha$, the formula passes into

$$I\left[(-1)^\alpha\,D^{-\alpha}\,\frac{\varphi(t)}{(t-1)^\alpha}\right] = \frac{\Gamma(x)}{\Gamma(x+\alpha)}\,I\left[\frac{t^\alpha\,\varphi(t)}{(t-1)^\alpha}\right]. \quad . \quad . \quad (15)$$

The last equation appears indeed to be true for *arbitrary* positive values of α [1]), if for the derivative of any negative order the definition of RIEMANN is adopted, which in the present case, a neighbourhood of infinity being regarded, can be expressed by the identity

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{(u-t)^{\alpha-1}}{(u-1)^\alpha} \varphi(u)\, du \quad . \quad . \quad . \quad (16)$$

Since as domain of t and u the part of the plane *outside* a certain circle with centre (1) is considered, it will be convenient to assume for path of integration between $u = t$ and $u = \infty$ the half-line which has the same direction as the vector from $u = 1$ to $u = t$. The quantities $u - t$ and $u - 1$ then have the same arguments and $(u - t)^\alpha : (u - 1)^\alpha$ is real. With these agreements we have the expansion

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \sum_0^\infty{}_n \frac{\Gamma(n+1)\, c_n}{\Gamma(n+1+\alpha)\,(t-1)^{n+1}} \quad . \quad . \quad (17)$$

so that the derivative of negative order $-\alpha$ of the expression $\varphi(t) : (t - 1)^\alpha$ is, as $\varphi(t)$ itself, regular and zero at infinity. The characteristic of the derivative is, however, α less and this makes it possible, by means of (15), to express the coefficientfunction $\omega(x)$ of a generating function $\varphi(t)$ with characteristic $\lambda' < -1$ in terms of another generating function $\varphi_1(t)$ whose characteristic is *any small amount* less than -1. The function $\varphi_1(t)$ is constructed in such a way that the given function $\varphi(t)$ is the derivative of a certain negative order $-\alpha$ of $\varphi_1(t) : (t - 1)^\alpha$ and the number α is selected from an aggregate of positive values, whose upper limit is equal to the difference between λ' and -1. In other words, if $\varphi(t)$ is given by (3), we take

$$\varphi_1(t) = \sum_0^\infty{}_n \frac{c'_n}{(t-1)^{n+1}}$$

where the meaning of c'_n is given by

$$c_n = \frac{\Gamma(n+1)\, c'_n}{\Gamma(n+1+\alpha)}$$

with

$$\alpha = -1 - \lambda' - \delta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

δ_1 being any small positive number. Then, according to (17)

$$\varphi(t) = (-1)^\alpha D^{\alpha} \frac{\varphi_1(t)}{(t-1)^\alpha}$$

[1]) I have communicated the proof of this truth in the Proceedings of the meeting of September 27, 1919.

and hence by (15)

$$\omega(x) = I[\varphi(t)] = \frac{\Gamma(x)}{\Gamma(x+\alpha)} I\left[\frac{t^\alpha \varphi_1(t)}{(t-1)^\alpha}\right] \quad . \quad . \quad . \quad . \quad (19)$$

Not only $\varphi_1(t)$, but also the function

$$\psi(t) = \frac{t^\alpha \varphi_1(t)}{(t-1)^\alpha}$$

has the property that the operation I applied to it gives a coefficient-function satisfying the condition (7), the inequality (14) for l being left unaltered. We only have in this case $\gamma = -\alpha$, instead of $\gamma = 0$, and the domain of validity is determined by $R(x) > -\alpha$, or, according to (18), by

$$R(x) > \lambda + \delta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$\lambda = \lambda' + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

That is: the domain of validity of (7) is the domain of *absolute* convergence of the series (1) (for δ_1 is arbitrarily small).

For the whole right-hand member of (19), that is for $\omega(x)$ we therefore have the inequality

$$|\,\omega(x)\,| < M\,|\,(x+b)^l\, a^{x-(\lambda+\delta_1)}\,| \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where l, now, satisfies the condition

$$l < \lambda - \tfrac{1}{2} + \delta_1 + \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If, at last, the characteristic λ' of $\varphi(t)$ is greater than -1 or equal to -1, then, after PINCHERLE, the coefficientfunction can be expressed in terms of that of another generating function $\varphi_1(t)$, with a characteristic less than -1. First, let

$$-1 \overline{<} \lambda' < 0,$$

then PINCHERLE considers the additional function

$$\varphi_1(t) = -D^{-1}\left(\frac{\varphi(t)}{t-1}\right),$$

having a characteristic $\lambda' - 1$, which, therefore, is less than -1, so that the corresponding coefficient-function $\omega_1(x)$ satisfies, in the domain $R(x) > \lambda - 1 + \delta_1$, the inequality

$$|\,\omega_1(x)\,| < M\,|\,(x+b)^l\, a^{x-(\lambda-1+\delta_1)}\,|$$

with

$$l < \lambda - \tfrac{3}{2} + \delta_1 + \delta$$

The coefficientfunction $\omega(x)$ of $\varphi(t)$ is connected with the latter by the formula[1])

$$\omega(x) = \Delta\,[(x-1)\,\omega_1(x-1)]$$

[1]) PINCHERLE, l. c., p. 64.

from which it follows that $\omega(x)$, precisely in the domain (20), satisfies the inequality (22) with, for l, the inequality (23). In this manner we may prove the same inequality for the intervals $(0,1)$, $(1,2)\ldots$ of λ' in succession.

If $\omega(x)$ satisfies the inequality (22) for a certain value of l, then, evidently, for all greater values. Thus there is a lower limit l_0 for all such values, but this may possibly not be substituted for l in (22). Instead of this we may however write

$$\omega(x) \sim (x+b)^{l_0} a^{x-(\lambda+\delta_1)} \quad \ldots\ldots \quad (24)$$

with the meaning that (22) holds for any $l > l_0$; we may call (24) an equation of *equivalence* and say that $\omega(x)$ is *equivalent* to the right-hand member of this equation. The exponent l_0 satisfies the condition

$$l_0 \overline{<} \lambda - \tfrac{1}{2} \quad \ldots\ldots \quad (25)$$

since δ and δ_1 were arbitrarily small. The proposition relating to the *necessary* condition for a function to be expanded in a binomial-series may thus be expressed in the following manner:

A binomial series of the form (1) *represents in any half-plane* $R(x) \overline{>} \lambda + \delta$, *differing arbitrarily little from its domain* $R(x) \overline{>} \lambda$, *of* *absolute convergence, a function* $\omega(x)$, *which satisfies the equation of equivalence* (24); *the exponent* l_0 *satisfies the inequality* (25).

If, now, this proposition is compared with that relating to the *sufficient* condition, then, to begin with, we find a complete accord between the majorant values (7) and (22). *These majorant-values are, therefore, both necessary and sufficient.* Further, as regards the domains of validity, the inequality (9) here becomes $R(x) > l + \frac{1}{2}$, since we had $\beta = 1$, or we may also write

$$R(x) > l_0 + \tfrac{1}{2},$$

if l_0 is again the lower limit of the l-values which may be taken for the given function. From (25) the same inequality follows with regard to the domain of *absolute* convergence. Since the domain of possibly *conditional* convergence extends at most over a strip of unity-breadth on the left of the domain of absolute convergence, the investigation performed by us leaves room for the possibility that a binomial series sometimes represents a function satisfying the condition (24) also in a strip determined by

$$l_0 - \tfrac{1}{2} < R(x) < l_0 + \tfrac{1}{2}$$

or in a certain part of it. In order to come to certainty concerning this point, we should have to examine how a function represented by a binomial series behaves in the domain of conditional convergence of that series. To such an investigation we have as yet not arrived; but we may already perceive that the result could not fill up the gap which, as regards the domains of validity, exists as yet between the necessary and the sufficient condition. First: if a function $\omega(x)$ satisfies the equation of equivalence (24) for a *fixed* value of a on the circle (1,1) and for a certain minimum-exponent l_0, then, on account of what has been remarked on the expression $a^{x-\gamma}$, immediately after formula (10), that function satisfies the same inequality, when the number a *varies*, in the specified mode, together with the argument ψ of x. The index l_0 cannot, however, be diminished, because it must at all events be taken for $\psi = -\alpha$, if α is the argument of the original *fixed* number a. The statement belonging to the inequality (9′) informs us, however, that in this case expansion of $\omega(x)$ in a binomial series is possible for $R(x) > l_0$. The function

$$2^x = \sum_0^{\infty} {}_n \binom{x-1}{n},$$

for which we have $a = 2$, $l_0 = 0$, affords an illustration of this fact, for the expansion is really valid for $R(x) > 0$, and it is *conditionally* convergent for $R(x) \lessgtr 1$. Therefore we can never find $R(x) > l_0 + \frac{1}{2}$ as a *necessary* condition whereas our theorem concerning the *sufficient* condition only says that expansion is *possible* in the domain defined by the last inequality.

Secondly the last condition only holds in case $\omega(x)$ has no singularities in the finite part of the domain $R(x) > l_0 + \frac{1}{2}$; for otherwise for the latter domain the one must be substituted where $\omega(x)$ is regular and that was defined by the inequality $R(x) > \gamma$.

Thus the proposition regarding the *necessary* condition states that for points in the domain of absolute convergence of the given binomial series we have $R(x) > l_0 + \frac{1}{2}$, but conversely it is not true that in the domain determined by this inequality there is certainly absolute convergence. A simple example is furnished by the function

$$\omega(x) = \frac{1}{x}$$

For this function $l_0 = -\infty$, and yet the function can only in the domain of regularity $R(x) > 0$ be expanded in a binomial series of the form (1).

From these remarks it will be clear that in order to fill up the gap existing as yet between the necessary and sufficient conditions we must give more specified propositions for both conditions. In other words we should have to succeed in dividing the functional aggregate of all functions satisfying the equation (24) by *special* characterising properties; and so also the aggregate of binomial series, in such a way that between the two kinds of sub-aggregates there existed a *complete* correspondence, such that functions of some sub-aggregate K could only be expanded in binomial series belonging to the sub-aggregate K and in no others. But the problem to find suchlike characterising properties will perhaps be very difficult, since it is required for it to derive the character of a function from that of the coefficients of the series representing it; a problem which already causes the greatest difficulties when it regards the more known power-series.