Mathematics. — "*A general definition of limit with application to limit-theorems*" [1]). By Prof. FRED. SCHUH. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of May 3, 1919).

1. Let us assume an aggregate V of real or complex numbers, in which the same number may occur repeatedly. This can take place, if a mode of arising of the numbers of V has been given and various modes of arising may lead to the same number. Those equal numbers however, are considered as different elements of V, so that we distinguish between a number having arisen in the former and the same number having arisen in the latter manner.

2. Next we assume a law given *by which every positive number* δ *is made to correspond to a part* V_δ *(consisting of at least one element) of* V (covering of the aggregate of the positive numbers by the aggregate of the parts of V), in this way, *that* $V_{\delta'}$ *is a part of* V_δ, *if* $\delta' < \delta$; here V_δ is called a part of V if each element of V_δ is an element of V, so that the part can be identical with the whole aggregate. If now L is a (real or complex) number with the property *that corresponding to every positive number* δ *there exists such a positive number* ε *that every element* E *of* V_δ *satisfies the inequality* $|E-L| < \varepsilon$, then L is called the LIMIT *of the aggregate* V *as regards the covering by which* V_δ *is made to correspond to* δ. It is clear that there can be at most but one number with the beforementioned property.

3. The covering observed in n°. 2 we call EQUIVALENT to a second covering by which a positive number δ is made to correspond to a part V'_δ of V, *if corresponding to every positive number* δ *a positive number* δ_1 *can be found so that* V'_{δ_1} *is a part of* V_δ *and* V_{δ_1} *is a part of* V'_δ. It is evident *that this equivalence is a transitive one*. It furthermore easily appears *that two equivalent coverings both give the same limit, or no limit exists in the two cases.* Hence equivalent coverings can be considered as the same limittransition, so that with a MANNER OF LIMITTRANSITION we mean a *set of mutually equivalent coverings of the kind mentioned in n°. 2.*

4. *The limit defined in n°. 2 exists, and exists only then, if corre-*

[1]) For further particulars c.f. Hand. van het Nat. en Geneeskundig Congres 1919.

sponding to every positive number ε a positive number δ can be found so that every pair of elements E and E' of V_δ satisfies the inequality $|E-E'|<\varepsilon$ (GENERAL PRINCIPLE OF CONVERGENCE). That this condition is *necessary* for the existence of a limit is obvious.

That this condition moreover is *sufficient*, appears in the case of real numbers by noticing that (the condition being fulfilled) a number satisfying the definition of a limit is furnished by the upper boundary of the numbers a, for which there exists a V_δ, all elements of which are $>a$. In the case of complex numbers the theorem is further proved by applying the theorem for real numbers to the real parts and the coefficients of i of the complex nnmbers.

5. Let V and W be two aggregates of numbers, the elements of which being placed into correspondence. We suppose the coverings of the positive numbers by the aggregates of the parts of V resp. W to be of such a nature, that for each positive number δ the parts V_δ and W_δ in the correspondence between V and W are corresponding ones.

We now form an aggregate U by adding the corresponding elements of V and W, at the same time transferring the covering to U. *If now with these coverings V shows a limit L_v and W a limit L_w, then also U has a limit, viz. L_v+L_w,* as may easily may be deduced from the definition of limit.

Other known limit-theorems also can be stated in this manner in a general way.

6. We now suppose, that the elements of the aggregate V are *real* numbers. About the existence however of a limit of V, as regards the chosen covering, nothing is assumed to be known. We can then consider the *lower boundary of the upper boundaries of the aggregates V_δ*, and call it the UPPER LIMIT *of the aggregate V as regards the considered covering.* The upper limit B is $+\infty$, if all aggregates V_δ are unbounded to the right, and $-\infty$, if the aggregate of the upper boundaries of the aggregates V_δ is unbounded to the left.

We likewise call the *upper boundary O of the lower boundaries of the aggregates V_δ* the LOWER LIMIT *of V as regards the considered covering*; this lower limit can also be $\pm\infty$. It is easily proved, *that B and O always satisfy the inequality $O \leqq B$.*

The aggregate V has a limit L then only, when O and B are equal and finite; we then have $L=O=B$. If O and B are both $+\infty$, we speak of an IMPROPER LIMIT $+\infty$, if O and B are both $-\infty$, of an IMPROPER LIMIT $-\infty$.