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Mathematics. — *"Remark on Multiple Integrals."* By Prof. L. E. J. BROUWER.

(Communicated in the meeting of June 28, 1919).

The object of this communication is to make two remarks in conjunction with the first part of my paper: *"Polydimensional Vectordistributions"* [1]) presented at the meeting of May 26, 1906.

§ 1.

I. The proof of the generalisation of STOKES's theorem given l.c. pp. 66—70 provides this generalisation not only in the Euclidean but also in the following ametric form:

THEOREM. *In the n-dimensional space* $(x_1, \ldots x_n)$ *let the* $(p-1)$*-tuple integral*

$$\int \Sigma F_{\alpha_1 \ldots \alpha_{p-1}} (x_1, \ldots x_n)\, dx_{\alpha_1} \ldots dx_{\alpha_{p-1}} \quad . \quad . \quad . \quad . \quad (1)$$

be given, where the F's are continuous and finitely and continuously differentiable; consider also the p-tuple integral

$$\int \Sigma f_{\alpha_1 \ldots \alpha_p} (x_1, \ldots x_n)\, dx_{\alpha_1} \ldots dx_{\alpha_p}\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$f_{\alpha_1 \ldots \alpha_p} = \sum_{\nu=1}^{p} \frac{\partial F_{j}}{\partial x_{\alpha_\nu}},$$

(indicatrix j, α_ν *aeq. indicatrix* $\alpha_1 \ldots \alpha_p$*).*

Then, if the two-sided p-dimensional fragment [2]) *G is bounded by the two-sided* $(p-1)$*-dimensional closed space g, the indicatrices of G and g corresponding and both G and g possessing a continuously varying plane tangent space, the value of* (1) *over g is equal to the value of* (2) *over G.*

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[1]) See Vol. IX, pp. 66—78; we take the definitions modified in accordance with note [1]) on p. 116 l. c. I take this opportunity of pointing out that on p. 76 l. c. lines 13 and 14 "finite sourceless current system" should be read instead of "system of finite closed currents".

[2]) Math. Annalen 71, p. 306.

Of this theorem, which was enunciated by POINCARÉ [1]) in 1899 already, without proof however, and in a form expressing the rule of signs in a less simple manner, I shall here give the proof anew, editing it somewhat more precisely than in my quoted paper.

II. In the n-dimensional space $(x_1, \dots x_n)$, which we shall denote by S, let the p-tuple integral

$$\int \Sigma\, \varphi_{\alpha_1 \dots \alpha_p} (x_1, \dots x_n)\, dx_{\alpha_1} \dots dx_{\alpha_p} \quad \dots \dots \quad (3)$$

be given, where the φ's are continuous.

Let Q be a two-sided p-dimensional net fragment [2]) provided with an indicatrix and situated in S, σ a base simplex of Q with the vertex indicatrix $A_1 A_2 \dots A_p A_{p+1}$, A an arbitrary point of σ, ${}_A\varphi_{\alpha_1 \dots \alpha_p}$ the value of $\varphi_{\alpha_1 \dots \alpha_p}$ at A, ${}_\nu x_\mu$ the value of x_μ at A_ν. For every σ we determine the value of

$${}_\sigma\varphi \equiv \Sigma\, {}_A\varphi_{\alpha_1 \dots \alpha_p} \cdot {}_\sigma i_{\alpha_1 \dots \alpha_p},$$

where

$${}_\sigma i_{\alpha_1 \dots \alpha_p} \equiv \frac{1}{p\,!} \begin{vmatrix} {}_1x_{\alpha_1} & \dots & {}_{p+1}x_{\alpha_1} \\ \cdot & \cdot & \cdot \\ {}_1x_{\alpha_p} & \dots & {}_{p+1}x_{\alpha_p} \\ 1 & \dots & 1 \end{vmatrix},$$

and where, for different terms under the Σ sign A may be chosen differently; and we sum ${}_\sigma\varphi$ over the different base simplexes of Q. The upper and lower limit between which this latter sum varies on account of the free choice of the points A, we call *the upper and lower value of* (3) *over* Q.

If we now subject Q to a sequence of indefinitely condensing simplicial divisions which give rise to a sequence Q', Q'', ... of net fragments covering Q, then, as ν increases indefinitely, the upper and lower value of (3) over $Q^{(\nu)}$ converge to the same limit, which we call *the value of* (3) *over* Q.

Let F be a two-sided p-dimensional fragment provided with an

[1]) Les méthodes nouvelles de la mécanique céleste III, p. 10. The significance of the rule of signs here formulated, is apparent only after comparing former publications of the same author from the Acta Mathematica and the Journal de l'École Polytechnique in which the equivalence of the identically vanishing of (2) and the vanishing of (1) over every g was pronounced.

[2]) Math. Annalen 71, p. 316.

indicatrix and situated in S, f a sequence of indefinitely condensing simplicial approximations P', P'', ... of F corresponding to a category ψ of simplicial divisions. If the values of (3) over P', P'', ... converge to a limit which is independent of the choice of f so far as it is left free by ψ, then we call this limit *the value of* (3) *over* F *for* ψ.

III. We shall now occupy ourselves with the value of (1) over the boundary β of a p-dimensional simplex σ provided with an indicatrix and situated in S. In doing so we take it that the indicatrices of β and σ *correspond*, that is to say, the indicatrix of an arbitrary $(p-1)$-dimensional side of σ is obtained by placing the vertex of σ which does not belong to this side last in the indicatrix of σ and subsequently omitting it. We begin by confining ourselves to the contribution of the single term

$$\int F_{\alpha_1 \ldots \alpha_{p-1}} dx_{\alpha_1} \ldots dx_{\alpha_{p-1}}$$

to the value of (1) over β. By a suitable simplicial division ζ of the space $(x_{\alpha_1}, \ldots x_{\alpha_{p-1}})$ we determine a simplicial division of β, whose base simplexes correspond in pairs to those of $(x_{\alpha_1}, \ldots x_{\alpha_{p-1}})$. The family of those $(n-p+1)$-dimensional spaces within which $x_{\alpha_1}, \ldots x_{\alpha_{p-1}}$ are constant, cuts the plane p-dimensional space in which σ is contained, in a family of straight lines which connect pairs of corresponding base simplexes of β into p-dimensional truncated simplicial prisms. If e_1 and e_2 are a pair of corresponding base simplexes of β, d the concomitant truncated simplicial prism, l a line segment having components $r_{\alpha_p}, \ldots r_{\alpha_n}$ which leads from a point of e_1 to the corresponding point of e_2, then the contribution of the term $\int F_{\alpha_1 \ldots \alpha_{p-1}} dx_{\alpha_1} \ldots dx_{\alpha_{p-1}}$ to the value of (1) over e_1 and e_2 becomes

$$-\, {}_{e_1}i_{\alpha_1 \ldots \alpha_{p-1}} \cdot \sum_{\nu=p}^{n} r_{\alpha_\nu} \cdot {}_A\left\{\frac{\partial F_{\alpha_1 \ldots \alpha_{p-1}}}{\partial x_{\alpha_\nu}}\right\} + \varepsilon,$$

where A denotes a point of σ which may be different for the different terms under the Σ sign, and ε becomes indefinitely small with respect to ${}_{e_1}i_{\alpha_1 \ldots \alpha_{p-1}}$ for indefinite condensation of ζ.

Now let $B_1 B_2 \ldots B_p$ be a vertex indicatrix of e_1 and ${}_\nu x_\mu$ the value of x_μ at B_ν, then the value of $r_{\alpha_\nu} \cdot {}_{e_1}i_{\alpha_1 \ldots \alpha_{p-1}}$ can be expressed as

$$\frac{1}{(p-1)!}\begin{vmatrix} {}_1x_{\alpha_1}-{}_px_{\alpha_1} & \cdots\cdots & {}_{p-1}x_{\alpha_1}-{}_px_{\alpha_1} & 0 \\ \cdot & \cdots\cdots & \cdot & \cdot \\ {}_1x_{\alpha_{p-1}}-{}_px_{\alpha_{p-1}} & \cdots & {}_{p-1}x_{\alpha_{p-1}}-{}_px_{\alpha_{p-1}} & 0 \\ {}_1x_{\alpha_\nu}-{}_px_{\alpha_\nu} & \cdots\cdots & {}_{p-1}x_{\alpha_\nu}-{}_px_{\alpha_\nu} & r_{\alpha_\nu} \end{vmatrix},$$

thus also as

$$-\, d i_{\alpha_1 \ldots \alpha_{p-1}\alpha_\nu} + \varepsilon$$

so that the contribution of the term $\int F_{\alpha_1 \ldots \alpha_{p-1}}\, dx_{\alpha_1} \ldots dx_{\alpha_{p-1}}$ to the value of (1) over e_1 and e_2 can be expressed as

$$\varepsilon + \sum_{\nu=p}^{n} \left\{\frac{\partial F_{\alpha_1 \ldots \alpha_{p-1}}}{\partial x_{\alpha_\nu}}\right\}_A d i_{\alpha_1 \ldots \alpha_{p-1}\alpha_\nu},$$

and the value of (1) over β is obtained in the form:

$$\sum_{\alpha_1 \ldots \alpha_p} \sum_{\nu=1}^{p} \left\{\frac{\partial F_{j_\nu}}{\partial x_{\alpha_\nu}}\right\}_A {}_\sigma i_{\alpha_1 \ldots \alpha_p}$$

(*indicatrix* $j_\nu\, \alpha_\nu$ *aeq. indicatrix* $\alpha_1 \ldots \alpha_p$),

where A represents a point of σ which may be different for different terms under the $\Sigma\Sigma$ sign.

Hence it follows immediately, that, if Q is a two-sided p-dimensional net fragment situated in S, and R denotes the boundary of Q, while the indicatrices of Q and R correspond, then *the value of* (1) *over* R *is equal to the value of* (2) *over* Q.

IV. To complete the proof of the theorem formulated in I, we consider a category ψ of simplicial divisions of g such that the aggregate of the base sides possesses for ψ uniformly continuously varying plane tangent spaces, and the ratio of the volume of a base simplex to the $(p-1)^{\text{th}}$ power of its greatest coördinate fluctuation does not fall below a certain minimum for ψ. Let $g', g'', \ldots$ be a sequence of indefinitely condensing simplicial approximations of g corresponding to ψ. If, on $g^{(\mu)}$ we construct an approximating simplicial representation $g^{(\nu\mu)}$ of $g^{(\nu)}$, then, by choosing both μ and ν above a suitable limit, we can, in virtue of III, see to it that the values of (1) over $g^{(\nu\mu)}$ and $g^{(\nu)}$ differ from each other by as little as we please, whilst $g^{(\mu)}$ is covered by $g^{(\nu\mu)}$ with degree one, so that the values of (1) over $g^{(\mu)}$ and $g^{(\nu\mu)}$ are equal. Thus there exists a *value of* (1) *over* g *for* ψ which, naturally, does not change if, instead of ψ, some other category of the same kind is chosen.

Now let χ be a category of simplicial divisions of G analogous to ψ. The resulting simplicial divisions of g belong to a category ψ of the kind just described. Let $G', G'', \ldots$ be a sequence of indefinitely condensing simplicial approximations of G corresponding to χ, then, at the same time, there is hereby determined a sequence $g', g'', \ldots$ of indefinitely condensing simplicial approximations of g corresponding to ψ. Since, in virtue of III, the value of (1) over $g^{(\nu)}$ is equal to the value of (2) over $G^{(\nu)}$, *there exists, just as there does a value of* (1) *over* g *for* ψ, *a value of* (2) *over* G *for* χ, *both values being equal, and not changing if some other category of the same kind is chosen in the place of either* ψ *or* χ.

§ 2.

In introducing l.c. p. 70 the notion of a second derivative, we have omitted to give the definition of the underlying concept of *normality* of an S_p provided with an indicatrix and an S_{n-p} provided with an indicatrix which are perpendicular to each other in an S_n provided with an indicatrix. This definition we shall here give.

Let T be the point of intersection of S_p and S_{n-p}, $\alpha_1 \ldots \alpha_p T$ the indicatrix of S_p and $\beta_1 \ldots \beta_{n-p} T$ the indicatrix of S_{n-p}; we call S_p *normal to* S_{n-p} and S_{n-p} *postnormal to* S_p, if $\alpha_1 \ldots \alpha_p T \beta_{n-p} \ldots \beta_1$ is an indicatrix of S_n.

Thus, for some values of n the concepts *normal* and *postnormal* are equivalent, for other values not equivalent.

Furthermore we call a p-dimensional vector system V *normal* to an $(n-p)$-dimensional vector system W at the same point, and W *postnormal* to V, if, with respect to a rectangular system of coordinates the components of V are respectively normal to and of equal scalar values as the components of W.

In this terminology, the second derivative of the vector distribution X is the normal distribution of the first derivative of the postnormal distribution of ${}^{p}X$.