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Mathematics. — "*Null systems determined by linear systems of plane algebraic curves*". By Prof. JAN DE VRIES.

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1. A triply infinite system (complex) $S^{(3)}$ of plane algebraic curves c^n contains a twofold infinity of nodal curves; for an arbitrarily chosen point D is node of a *nodal curve* δ^n belonging to $S^{(3)}$.

I shall now consider the *null system* in which the tangents d, d' of δ^n are associated as *null rays* with D as *null point.*

The nodal curves of a net belonging to $S^{(3)}$ have their nodes on the Jacobian, which is a curve of order $3(n-1)$. It has in common with the Jacobian of a second net the $3(n-1)^2$ nodes, which occur in the pencil common to the two nets. The remaining intersections of the two loci are *critical points,* i.e. nodes for the curves of a pencil. *The null system, therefore, has* $6(n-1)^2$ *singular null points.*

2. Let a be an arbitrary straight line, P an arbitrary point. The δ^n, which has its node D on a, intersects the ray PD, moreover, in $(n-2)$ points E. If E is to get into P, δ^n must belong to the net that possesses a base-point in P; D lies then on the Jacobian of that net. The locus (E) of the points E passes, therefore, $3(n-1)$ times through P, and is consequently a curve of order $(4n-5)$. Each intersection of (E) with a is node of a δ^n, of which one of the tangents d passes through P.

There is therefore a curve $(D)_P$ of order $(4n-5)$ which contains the nodes of the nodal curves δ^n, which send one of their tangents d through a given point P. It will be called the *null-curve* of P. For a singular point S it has in S a triple point. As P evidently is node of (D), there lie on a ray d passing through P $(4n-7)$ points D, for which d is one of the tangents of the corresponding curve δ^n. From which ensues: *an arbitrary straight line* d has $(4n-7)$ null-points D.

3. The null-curves $(D)_P$ and $(D)_Q$ have the $6(n-1)^2$ singular points in common; for, a critical point bears ∞^1 pairs d, d'.

The two curves pass further through the $(4n-7)$ null points of the straight line PQ. Each of the remaining intersections is a point D, for which d passes through P, d' through Q. In other words, if d revolves round P, d' will envelop a curve of class

$(10n^2 - 32n + 26)$. To the straight lines d', which pass through P, belong the tangents of the δ^n, which has its node in P. Each of the remaining $(10n^2 - 32n + 24)$ straight lines d' evidently coincides with a ray d, and therefore contains a null-point D, for which the two null-rays have coincided. If such a straight line is called a *double null-ray*, it ensues from the above that *the double null-rays envelop a curve of class* $2(n-2)(5n-6)$ [1].

4. The null-rays d, which have a null-ray D on the straight line p, envelop a curve (p) of class $(4n-5)$, which has p as $(4n-7)$-fold tangent. It, therefore, intersects p in $(4n-5)(4n-6) - (4n-7)(4n-6)$ points, which bear each two coinciding null-rays.

The locus of the points C, *which bear a double null-ray, is, therefore, of order* $4(2n-3)$.

The curve (C) is evidently the locus of the cusps of the complex. As the order of (C) may also be determined in another way, it appears at the same time that the curve (p) has no other multiple tangents.

5. The case $n = 2$ deserves a separate treatment. In the first place each line d has now only *one* null-point; this is the node of the conic, which is indicated by three points of d.

The locus (C) is now of the fourth order, and consists of four straight lines c_k. For, if the two straight lines of a nodal c^2 coincide, c_k is a double line. The complex contains, therefore, *four double lines*, and they are at the same time *singular null-rays*.

The vertices S_{kl} of the complete quadrilateral formed by them are the singular points of the null-system.

The curves $(p)_3$ and $(q)_3$, cf. § 4, have, besides the null-rays of the point pq, *seven* tangents in common, which have each a null-point on p and a null-point on q, and are consequently *singular null-rays*. To them belong the four straight lines c_k. Each of the remaining three singular null-rays s must belong to ∞^1 nodal conics. S_{12} bears as singular point, ∞^1 pairs of lines, which form an involution of rays; so $S_{12} S_{34}$ belongs to two, and then to ∞^1, pairs of lines and consequently must be singular. The diagonals of the quadrilateral, which is formed by the four straight lines c, are consequently the three singular null-rays required.

[1]) In other words, the *cuspidal tangents* of the cuspidal curves of a complex envelop a curve of class $2(n-2)(5n-6)$. In my paper on the characteristic numbers of a complex (These Proceedings, Vol. XVII, page 1055, § 13) the influence of the critical points in the determination of the class has been overlooked.

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6. If the complex $\{c^2\}$ has a base-point B, it is at the same time singular null-point, for two points on a ray passing through B, determine a nodal δ^2, with node in B. The double rays of the involution formed by the curves δ^2 with node B are double lines of $\{c^2\}$, consequently singular null-rays. Other double null-rays do not exist, for if a straight line d of δ^2 does not pass through B, d' does. As B is node of the Jacobian of each net belonging to $\{c^2\}$, this point replaces four critical points. Two more singular points, therefore, lie outside B; they are connected by a singular null-ray.

7. In a fourfold linear system $S^{(4)}$, each point D is node to a pencil (δ^n). Two of those curves have a *cusp* in $C \equiv D$.

I now consider the *null-system* in which to the *null-point* C are associated the cuspidal tangents c, c' of the two cuspidal curves γ^n, which have their cusps in C.

The straight line d is touched in each of its points D by a nodal δ^n, which has its node in D. With the straight line PD δ^n has moreover $(n—2)$ points E in common. In order to find the locus of the points E, I shall inquire how often E gets into P. In this case δ^n belongs to the complex that has a base point in P; in it occur $(4n—7)$ δ^n, which touch at d (§ 2). Consequently (E) is a curve of order $(5n—9)$.

If E lies on d, $PE \equiv d'$ touches in that point at a δ^n, which has its node on D. Every straight line d therefore is nodal tangent of $(5n—9)$ curves δ^n, of which the second tangent d' passes through P. If d is now made to revolve round a point Q, the point D describes a curve (D) of which every point is node of a δ^n, which sends its tangents d and d' through Q and P. In Q a δ^n is touched by QP, so Q and consequently P is a point of (D), so that this curve is of order $(5n—8)$.

If C is one of the $(5n—10)$ points, which (D') has in common with the straight line PQ, besides P and Q, the tangents d, d' fall both along PQ, so that C is a cusp of a cuspidal curve γ^n, which has $c \equiv PQ$ as cuspidal tangent.

In the above null-system a straight line therefore has $5(n—2)$ *null-points.*

If c revolves round a point M, the null-points C describe a curve of order $(5n—8)$, with node M (the *null curve* of M).

8. The system $S^{(4)}$ contains a number of curves with a triple point. If $S^{(4)}$ is represented by the equation

$$\alpha A + \beta B + \gamma C + \delta D + \varepsilon E = 0,$$

the co-ordinates of a triple point have to satisfy the six equations:

$$\alpha A_{kl} + \beta B_{kl} + \gamma C_{kl} + \delta D_{kl} + \varepsilon E_{kl} = 0,$$

in which A_{kl} etc. represent derivatives according to x_k and x_l.

The number of points has to be found, for which

$$\begin{Vmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{23} & A_{31} \\ B_{11} & B_{22} & B_{33} & B_{12} & B_{23} & B_{31} \\ C_{11} & C_{22} & C_{33} & C_{12} & C_{23} & C_{31} \\ D_{11} & D_{22} & D_{33} & D_{12} & D_{23} & D_{31} \\ E_{11} & E_{22} & E_{33} & E_{12} & E_{23} & E_{31} \end{Vmatrix} = 0.$$

According to a well-known rule we find for this

$$(5^2-4^2+3^2-2^2+1^2)(n-2)^2.$$

There are therefore $15(n-2)^2$ curves c_n with a *triple point* S[1]). In such a point the nodal curves have the same tangents d,d'. Any straight line passing through S is to be considered as a cuspidal tangent c.

The null-system therefore has $15(n-2)^2$ *singular points.*

9. I now take three points P, Q, R, arbitrarily, and consider (cf. § 7) the curves $(D)_{PQ}$ and $(D)_{PR}$. To begin with they have the point P in common; for there is a δ^n, which has P as a node, and PQ as tangent and a δ^n, for which one of the tangents lies along PR.

Those curves have further in common the $(5n-9)$ points D, for which QR is one of the tangents d. Another group of common points consists of the singular points S.

Let U be one of the still remaining intersections. There is in that case a δ^n with tangents UP and UQ, and also a δ^n with tangents UP and UR. From this it ensues that all δ^n with node U have the straight line UP as tangent, consequently belong to a pencil in which the tangents d,d' form a parabolic involution.

The double rays of this involution have then coincided in UP, and U is cusp for only *one cuspidal* c^n. If such a point is called *unicuspidal point*, it follows from $(5n-8)^2-1-(5n-9)-15(n-2)^2$ that $(10n^2-25n+12)$ unicuspidal curves send their tangent through P. *The cuspidal tangents of the unicuspidal points envelop a curve of class* $(10n^2-25n+12)$.

10. In any point C of the straight line a I draw the two null-

1) If $n=3$, and the system has 5 base-points, the 15 triple points are easy to indicate. One of them e.g. is the intersection of B_1B_2 with B_3B_4.

rays c,c' (cuspidal tangents), and consider the correspondence between the points L,L', which c,c' determine on the straight line l.

If c is made to revolve round L, the null-points of c describe a curve of order $(5n-8)$, which has a node in L (cf. § 7). To a point L therefore belong $(5n-8)$ points C and $(5n-8)$ points L'. The point $a\,l$ represents two coincidencies $L \equiv L'$. The remaining coincidencies arise from cuspidal tangents u of unicuspidal points U. *The locus of the unicuspidal points is therefore a curve of order* $2(5n-9)$.

This may be confirmed in the following way. If C describes the straight line p, the null-rays c,c' envelop a curve of order $(5n-8)$ which has p as $(5n-10)$-fold tangent. It therefore has, not counting the points of contact, $(5n-8)(5n-9)-(5n-10)(5n-9)$, consequently $2(5n-9)$ points in common with p. In each of these points the null-rays c and c' have coincided.

11. The system $S^{(4)}$ produces in a still different way a *null-system*. Any point F is *flecnodal point* for five curves φ^n. In order to find this out we have only to consider the curve that arises if we make every δ^n that has F as node, to intersect its tangents d,d': This C^{n+2} namely, has in F a quintuple point [1]).

I now associate to each point F as *null-point* the five *null-rays* f, which are inflectional tangents for the five flecnodal curves φ^n.

Any point D of the straight line a is node for a δ^n, which touches the ray PD in D. I now determine the order of the locus of the groups of $(n-3)$ points E which each of the curves δ^n has moreover in common with PD. If E lies in P, δ^n belongs to a complex $S^{(3)}$. According to § 2 there are on a $(4n-5)$ nodes of curves δ^n of $S^{(3)}$ which send their tangent d through P. So P is $(4n-5)$-fold point of the curve (E) and the latter consequently of order $(5n-8)$. In each of its intersections F with a a curve φ^n has a flecnodal point, the inflectional tangent of which passes through P.

From this it ensues that the locus of the null-points F of the rays f out of a point P (*null-curve* of P) is a curve of order $(5n-8)$. As it must have a *quintuple point* in P, an arbitrary straight line f therefore contains $(5n-13)$ null-points. [2])

[1]) In a point S (§ 8) the c^n with triple point replaces three of the curves φ^n; for the other two the inflectional tangent lies along one of the two fixed tangents d, d'.

For a unicuspidal point (§ 9) one of the curves φ^n has its inflectional tangent along the fixed tangent d.

[2]) For $n=3$ is $5n-13=2$. Each Φ^3 is then the combination of a straight line f and a φ^2. Each straight line f belongs in $S^{(4)}$ to a figure (f,φ^2); its intersections with φ^2 are the two null-points F.

12. In the null-system (C,c) P has a null curve of order $(5n-8)$ with node P (§ 7). Of its intersections with the null curve with respect to the system (F,f), 10 lie in P. They also have the unicuspidal points U in common, for which the tangent u passes through P. In each of the remaining $(5n-8)^2-10-(10n^2-25n+12)$ intersections G, a cuspidal curve has with its tangent g four points in common. From this it ensues that *the four-point cuspidal tangents envelop a curve of class* $(15n^2-55n+42)$.

If n is equal to *three*, the curves γ^3 with four-point tangents are replaced by conics, each with one of its tangents. The null-system (F,f) then has the characteristic numbers 5 and 2; the null-curve $(P)^7$ of P is of class 22, consequently sends 12 tangents f through P, and each of these straight lines forms with the conic touching it a γ^3 with four-point tangent. In conformity with this, the form $15n^2-55n+42$ produces for $n=3$ the number 12.

13. In a quintuple infinite system $S^{(5)}$ each point D is node for a net of nodal curves. A straight line d passing through D determines in it a pencil, of which all δ^n touch at d in D. There is consequently *one* cuspidal γ^n, which has a straight line c passing through D as cuspidal tangent. The curves γ^n, with cusp D, form a system with index *two*, for the curves δ^n, passing through any point P, form a pencil, which contains two curves with cusp in D. If every straight line c passing through D is made to intersect with the cuspidal γ^n, which it touches in D, there evidently arises a curve of order $(n+2)$, which has a quintuple point in D. From this it ensues that *five* cuspidal curves have in D a *cusp*, where the cuspidal tangent has a *four-point contact*.

I shall now consider the *null-system* (G,g), in which to a point G are associated the *five* straight lines g, which are four-point cuspidal tangents for cuspidal curves γ^n with cusp G.

14. In each point C of the straight line a I consider the cuspidal curve γ^n, which sends its tangent c through P, and determine the locus of the points E, which γ^n has still in common with PC. If E lies in P, γ^n belongs to a system $S^{(4)}$; in it $(5n-8)$ curves γ^n occur, which have their cusp on a (§ 7). So the curve (E) passes $(5n-8)$ times through P and is of order $(6n-11)$. In each of its intersections G with a, a γ^n has four points in common with PG. The *null-curve* of P is therefore of order $(6n-11)$. As it has a *quintuple point* in P, a straight line g passing through P is *null-ray* for $(6n-16)$ points G.

15. The system $S^{(5)}$ contains ∞^1 curves with a triple point T. If $S^{(5)}$ is represented by

$$\alpha A + \beta B + \gamma C + \delta D + \varepsilon E + \varphi F = 0,$$

the locus of the points T is determined by

$$|A_{kl}\, B_{kl}\, C_{kl}\, D_{kl}\, E_{kl}\, F_{kl}|_6 = 0.$$

It is therefore a curve (T) of order $6(n-2)$ [1]).

A τ^n with triple point T determines with a nodal δ^n which has its node in T, a pencil of nodal δ^n with fixed tangents d, d'. The net of the curves δ^n with node T therefore consists of ∞^1 similar pencils of which the tangents d, d' form an involution. Each of the two nodal rays c_1, c_2 is common cuspidal tangent for a pencil of cuspidal curves and each of these two pencils contains a γ^n with four-point tangent. The five null-rays g of T are therefore represented by the straight lines c_1, c_2, and the three tangents t_1, t_2, t_3 of the curve τ^n. The points T are consequently *not singular.*

16. In a sextuple infinite system $S^{(6)}$ each point T is triple point of a τ^n. To T as *null-point* the three tangents t_1, t_2, t_3 of τ^n are now associated as *null-rays.*

In order to find the second characteristic number of this *null-system,* I consider the curves τ^n, of which the point T lies on the straight line a and I try to find the order of the curve, which contains the groups of $(n-3)$ points E, in which τ is moreover intersected by PT.

If E lies in P, τ^n belongs to an $S^{(5)}$, and T is one of the $6(n-2)$ points which (§ 15) the curve (T) has in common with a. So E is a $(6n-12)$-fold point on the curve (E), which consequently has the order $(7n-15)$.

The *null-curve* of P is therefore of order $(7n-15)$. As it passes three times through P, a straight line t passing through P is tangent for $(7n-18)$ curves τ^n, which have their triple point T on t. A *null-ray,* therefore, has $(7n-18)$ *null points.*

17. The curves (T), which belong to two systems $S^{(5)}$ comprised in $S^{(6)}$, have the $15(n-2)^2$ points T of the system $S^{(4)}$ in common, which forms the "intersection" of the two $S^{(5)}$.

The remaining intersections are *critical points,* viz. each of them is triple point for a pencil of curves τ^n, consequently singular null-point S for (T, t). This null system has consequently $21(n-2)^2$ *singular null-points.*

[1]) If, for $n=3$, the system $S^{(5)}$ has the base points B_1, B_2, B_3, B_4, the curve (T) consists of the straight lines $B_k\, B_l$.

As the triplets of tangents of the curves τ^n of that pencil form an involution, S is triple point with a *cuspidal branch* for *four* curves τ^n. Each singular null-point, therefore, bears *four double null rays.*

18. The null-curves of P and Q have the singular null-points S and the null-points of PQ in common. Each of the remaining intersections T sends a null-ray through P, a second through Q. From $(7n-15)^2-21(n-2)^2-(7n-18)$ it therefore ensues that the null-rays t_2, t_3 will envelop a curve of class $(28n^2-133n+159)$, if t_1 revolves round a point P. The null-rays of P belong each twice to this envelope, each of the remaining tangents, which it sends through P, is evidently double null-ray. The *double null-rays,* therefore, envelop a curve of the class $(28n^2-133n+153)$.

19. In order to find the locus of the points T for which two of the null-rays coincide, I shall consider the curve $(p)_{7n-15}$ enveloped by the null-rays of the points lying on p. It has p as $(7n-18)$-fold tangent, is therefore intersected by p in $(7n-15)(7n-16)-(7n-18)(7n-17)$ points. As for each of these points two null-rays coincide, the points T with *double null-rays* lie on a curve of order $(28n-66)$.

It is at the same time the locus of the triple points that have a cuspidal branch.

For $n=3$ we have a null-system (3,3); the curves τ^3 are three-rays in that case. An arbitrary straight line then forms figures c^3 with the curves of a net of conics. The Jacobian of that net determines the three null-points of the straight line.

If the system $S^{(6)}$ has three base-points, the three null-points of a straight line are produced by the intersection of the sides of a triangle, which has the base-points as vertices. Each base-point is the centrum of a pencil of singular null-rays.