Mathematics. — "*On a certain point concerning the generating functions of* LAPLACE." By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

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1. The following remarkable proposition of the integral $\int_0^\infty e^{-xr} \varphi(r) dr$,

or of the integral

$$a(x) = \int_0^1 f(t) t^x \, dt, \qquad (1)$$

derived from the former by the substitution $r = -\log t$, has been proved by LERCH[1]):

If the determining function $\alpha(x)$ vanishes for an arithmetical progression of values of x with positive common difference η

$$x = \xi + \mu\eta, \qquad (\mu = 0, 1, 2, \ldots) \qquad (2)$$

then it vanishes for all values of x, and the generating function $f(t)$ also vanishes.

LERCH uses for the proof a theorem of WEIERSTRASS, according to which any function which is continuous in a closed interval can be represented by a uniformly converging series of rational integral functions. Since the theorem, which is also mentioned by PINCHERLE[2]) and by NIELSEN[3]), has a great many interesting consequences, it seems not unuseful to prove it in a manner which is independent of WEIERSTRASS's theorem. The reasoning we give in the next pages makes use of the theorem of FOURIER.

2. The following suppositions are sufficient for the purpose:

1. The function $f(t)$ is *continuous* in the interval of integration, with possible exception as to the value $t = 0$.

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[1]) Acta mathem. 27, 1903.

[2]) "*Sur les fonctions déterminantes*", Ann. de l'Éc. Norm. 22, 1905. PINCHERLE calls $f(t)$ "*fonction génératrice*" and $\alpha(x)$ "*fonction déterminante*", whereas LERCH does the reverse. We have followed the nomenclature of PINCHERLE in the text.

[3]) "*Handbuch der Gammafunktion*", p. 118.

2. The integral (1) *exists* for a certain value $x = c$ of x.

We put

$$g(t) = \int_0^t f(u)\, u^c\, du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then, by 2, $g(t)$ is continuous in the *closed* interval (0,1), and zero for $t = 0$. Further, by 1, $g(t)$ is *differentiable* at all points of that interval, except, possibly, at $t = 0$, and we have

$$g'(t) = f(t)\, t^c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, if $\delta > 0$, we may write

$$\int_\delta^1 f(t)\, t^x\, dt = \int_\delta^1 g'(t)\, t^{x-c}\, dt = \left[g(t)\, t^{x-c}\right]_\delta^1 - (x-c) \int_\delta^1 g(t)\, t^{x-c-1}\, dt$$

If, now, x is a complex number with real part $R(x)$ *greater* than c, the number δ in this equation may be made to approach to zero, and thus we find

$$\int_0^1 f(t)\, t^x\, dt = g(1) - (x-c) \int_0^1 g(t)\, t^{x-c-1}\, dt \quad . \quad . \quad . \quad (5)$$

From this it follows. If the integral (1) exists for a certain value $x = c$ of x, it exists in the whole half-plane defined by $R(x) > R(c)$ [1].

Further it follows from (5) that the integral in the left-hand member represents a *continuous* function of x in any domain S lying wholly in the finite part of the half-plane $R(c) + \delta$, where $(\delta > 0)$. In the same manner as above it is found that the integral

$$\int_0^1 f(t)\, t^x \log t\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

exists for $R(x) > R(c)$ and represents the derivative of $\alpha(x)$ at any point of this half-plane, so that $\alpha(x)$ is also an *analytic* function. These consequences, too, are mentioned by PINCHERLE.

The proof LERCH gives of his theorem equally starts from the equation (5). In the following reasoning, however, we shall use an

[1] This theorem is fundamental in the theory of generating functions. After PINCHERLE different authors have proved it, though often under less general suppositions. The reasoning in the text is due to LERCH. This reasoning is founded upon the *continuity* of $f(t)$, which, presumably, is forgotten by LERCH, when, at the end stating his theorem, he says that $f(t)$ may be as well *discontinuous*. (Of course we do not mean to say that generalization is *impossible*).

equation derived from (5) by repeating once more the process which leads to the latter equation. So we put

$$h(t) = \int_0^t g(u)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then, again, $h(t)$ is continuous and differentiable in (0,1) and we have

$$h'(t) = g(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The principal point, however, is that the latter equation is also valid at $t = 0$. Thus the derivative of $h(t)$ is a *limited* function in the closed interval (0,1). Further, observing that

$$\lim_{t=0}\, [h(t) : t] = h'(0) = g(0) = 0,$$

we find on integrating by parts, for $R(x) > R(c)$

$$\int_0^1 g(t)\, t^{x-c-1}\, dt = h(1) - (x-c-1) \int_0^1 h(t)\, t^{x-c-2}\, dt. \quad . \quad . \quad (9)$$

and hence

$$a(x) = g(1) - (x-c)\, h(1) + (x-c)(x-c-1) \int_0^1 h(t)\, t^{x-c-2}\, dt \quad (10)$$

3. The preceding statements are valid independently of any further hypothesis as to the character of $f(t)$. Now, suppose that $a(x)$ becomes zero for the arithmetical progression of values

$$x = \xi + \mu, \qquad (\mu = 0, 1, 2, \ldots). \quad . \quad . \quad . \quad . \quad (11)$$

Choosing the number c in the preceding equations equal to ξ, we find $g(1) = 0$ and the integral in the right-hand member of (5) vanishes for

$$x = \xi + 1 + \mu, \qquad (\mu = 0, 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (12)$$

From this it follows that $h(1) = 0$, and, in connection with the latter result, from (10)

$$\int_0^1 h(t)\, t^{\mu}\, dt = 0, \qquad (\mu = 0, 1, 2, \ldots) \quad . \quad . \quad . \quad (13)$$

Now we saw that the derivative of $h(t)$ is *limited*. According to a well-known proposition $h(t)$ can therefore be expanded in a series of FOURIER. We have

$$h(t) = \sum_0^{\infty}{}_n\, (a_n \cos 2\pi nt + b_n \sin 2\pi nt) \quad . \quad . \quad . \quad . \quad (14)$$

where

$$a_0 = \int_0^1 h(t)\, dt$$

and, for $n = 1, 2, 3, \ldots$

$$a_n = 2\int_0^1 h(t) \cos 2\pi nt\, dt, \qquad b_n = 2\int_0^1 h(t) \sin 2\pi nt\, dt$$

Now the functions $\cos 2\pi nt$ and $\sin 2\pi nt$ are for any value of n expansible in power-series

$$\cos 2\pi nt = \sum_0^\infty{}_\mu A_\mu t^\mu, \qquad \sin 2\pi nt = \sum_0^\infty{}_\mu B_\mu t^\mu,$$

converging uniformly in the interval (0,1). Since $h(t)$ is limited in that interval we may use the following reduction

$$\int_0^1 h(t) \cos 2\pi nt\, dt = \int_0^1 h(t) \sum_0^\infty{}_\mu A_\mu t^\mu\, dt = \sum_0^\infty{}_\mu A_\mu \int_0^1 h(t)\, t^\mu\, dt$$

and in a similar manner we find

$$\int_0^1 h(t) \sin 2\pi nt\, dt = \sum_0^\infty{}_\mu B_\mu \int_0^1 h(t)\, t^\mu\, dt$$

Hence by (13) all coefficients in the expansion of Fourier are zero, and therefore $h(t)$ is identically zero in the interval (0,1). Since, further, $g(t) = h'(t)$, the same thing holds for $g(t)$, and since $f(t)\, t^\xi = g'(t)$ (except at $t = 0$), the generating function $f(t)$ itself is zero in the interval (0,1). This is the second part of Lerch's theorem. Since the first part follows immediately from the second, the theorem has been proved in the particular case that the arithmetical progression of zeros of $\alpha(x)$ has 1 for its common difference.

If this difference is equal to the positive number η and if, therefore, the zeros are given by formula (2), we substitute

$$t^\eta = s, \qquad x = \eta y, \qquad \xi = \eta c$$

by which the integral passes into

$$\frac{1}{\eta}\int_0^1 f\left(s^{\frac{1}{\eta}}\right) s^{\frac{1}{\eta}-1+y}\, ds = \int_0^1 \varphi(s)\, s^y\, ds. \quad . \quad . \quad . \quad . \quad (15)$$

The function $\frac{1}{\eta} f\left(s^{\frac{1}{\eta}}\right) s^{\frac{1}{\eta}-1} = \varphi(s)$ has the properties 1 and 2 of § 2, so that the foregoing arguments may be applied to it. The integral (15) vanishes for the sequence of values (11), hence $\varphi(s)$, and therefore also $f(s)$, identically vanishes in the interval (0,1). The theorem of LERCH has thus been proved completely.

4. The first part of the theorem, that $a(x)$ becomes *identically* zero, if this is the case for an arithmetical progression of x-values, may be proved in a *direct* manner, without first proving the second part; and it is an immediate consequence of the proposition:

A function $a(x)$ defined by an integral of the form (1) *can, under the suppositions* 1 *and* 2 *mentioned at the beginning of* § 2, *be expanded in a binomial series*

$$a(x) = \sum_{0}^{\infty} {}_n c_n \binom{x-\beta}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where β is a number lying in the domain of convergence of the integral.

Suppose, for a moment, this proposition to be true. If, then, $a(x)$ becomes zero for the sequence of values (11), we take $\beta = \xi$ in the equation (16). Substituting for x the values ξ, $\xi+1, \xi+2, \ldots$ in succession, we find that all coefficients c_n of the binomial expansion vanish and therefore *that $a(x)$ vanishes identically.*

The first part of LERCH's theorem is very easily proved in this manner and it would therefore be desirable that we might derive from it the second part in a short manner. But as yet we are not in a position to do this. The above demonstration is, after all, rather short, but besides, on grounds that, with a view to conciseness, we prefer not to state, we do not think it likely that the *identical* vanishing of $a(x)$ is more effective for the purpose than the vanishing for an *arithmetical progression* of values of the argument.

Nevertheless the first part of LERCH's theorem has an interest in itself, because remarkable consequences may be inferred from it. Among these LERCH mentions the truth that simple functions such as

$$\sin kx, \quad \cos kx, \quad \frac{2}{\Gamma(l-kx)}, \quad (k > 0)$$

cannot be the *determining* functions of *generating* functions, in other words that they cannot be represented by integrals of the form (1), neither can products of these functions with others which remain within finite limits in the finite part of a certain halfplane $R(x) > c$.

The proposition concerning the expansion of the integral (1) in a binomial series may be proved in different manners. In the first place integrals of that form belong to the general category of functions of which I showed, in an earlier communication in these

Proceedings (Vol. XXII, N°. 1) that they are expansible in series of the form (16). Consider a domain $R(x) \geq c + \delta$, take a positive number $\delta_1 < \delta$ and substitute $x = c + \delta_1 + y$ in the second integral of the right-hand member of (5), then $R(y) \geq \delta - \delta_1$ and thus positive, so that we have

$$\left| \int_0^1 t^{x-c-1} g(t)\, dt \right| \leq \int_0^1 t^{R(y)} \left| t^{\delta_1 - 1} g(t) \right| dt < \int_0^1 t^{\delta_1 - 1} \left| g(t) \right| dt,$$

where the latter integral exists, since $g(t)$ is a limited function in the interval (0,1). Hence $a(x)$ is in the whole domain considered of the form

$$a(x) = (x-b)\, \mu(x)$$

where $\mu(x)$ is a function remaining within finite limits and b a number lying without the domain. Suchlike functions, however, can always be expanded in series of the form in question.

A second, more direct proof, is obtained by substituting $t = 1-u$ in the same integral as considered before, and using the following reduction

$$(1-u)^{x-c-1} = (1-u)^{\beta-c-1} (1-u)^{x-\beta} = (1-u)^{\beta-c-1} \sum_0^\infty {}_m (-1)^m \binom{x-\beta}{m} u^m$$

where the series for $R(x) > R(\beta)$ converges uniformly in the interval $0 \leq u \leq 1$. Since, for $R(\beta) > R(c)$ the integral

$$\int_0^1 g(1-u)\,(1-u)^{\beta-c-1}\, du$$

converges *absolutely* (on account of the continuity of $g(1-u)$), we may, after performing the substitution in question, integrate term by term, and then we find (replacing again $1-u$ by t in the partial integrals)

$$\int_0^1 g(t)\, t^{x-c-1}\, dt = \sum_0^\infty {}_m (-1)^m \binom{x-\beta}{m} \int_0^1 (1-t)^m t^{\beta-c-1} g(t)\, dt \quad . \qquad (17)$$

This expansion is, therefore, valid for $R(x) > R(\beta) > R(c)$. Since the product of this series with $x-c$ can be transformed into a series of the same form, the required proposition has been proved again [1]).

[1]) In NIELSEN's book (l.c. p. 125) we find an analogous proof of the proposition in question; this, however, does not part from the integral in the second member of (5), but from the original integral, so that the hypothesis must be made that the latter converges *absolutely* for $\lim t = 0$. The reduction (5) makes this hypothesis superfluous.

A third proof has the advantage of showing that expansion of (1) according to factorials of $x-c$ is possible when the integral only exists for $x=c$, even when the straight line $R(x)=R(c)$ were the limit between the domains of convergence and divergence in the x-plane, and when the integral did not exist at all points of that line. The proof consists in repeating the process which led to the theorem of LERCH an infinite number of times. We write

$$\left.\begin{aligned} g(t) &= \int_0^t u^c f(u)\,du, \qquad g_1(t) \equiv h(t) = \int_0^t g(u)\,du, \\ g_2(t) &= 2\int_0^t g_1(u)\,du, \ldots\, g_n(t) = n\int_0^t g_{n-1}(u)\,du, \ldots \end{aligned}\right\} \quad . \quad . \quad (18)$$

Then formula (10) may be generalized in the following manner:

$$\left.\begin{aligned} a(x) &= g(1) - g_1(1)(x-c) + g_2(1)\binom{x-c}{2} - g_3(1)\binom{x-c}{3} + \ldots \\ &+ (-1)^{n-1} g_{n-1}(1)\binom{x-c}{n-1} + (-1)^n \binom{x-c}{n}\int_0^1 g'_n(t)\, t^{x-c-n}\,dt \ldots \end{aligned}\right\} \quad . \quad (19)$$

The remainder has zero as a limit for $R(x) > R(c)$, for if G is the maximum modulus of the limited function $g(t)$ in the interval (0,1), we have in succession

$$|g_1(t)| < Gt \qquad |g_2(t)| < Gt^2, \ldots \qquad |g_n(t)| < Gt^n, \ldots$$

hence

$$\begin{aligned} \left|\int_0^1 g'_n(t)\, t^{x-c-n}\,dt\right| &= \left| n\int_0^1 g_{n-1}(t)\, t^{x-c-n}\,dt\right| \\ &< n\int_0^1 G t^{R(x-c)-1}\,dt \\ &< \frac{nG}{R(x-c)}, \text{ voor } R(x-c) > 0. \end{aligned}$$

Now $\binom{x-c}{n}$ is for $n=\infty$ equivalent to $n^{-R(x-c)-1}$, and thus the modulus of the remainder in formula (19) is for all n-values less than

$$\frac{H(x-c)}{R(x-c)\, n^{R(x-c)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where H is a certain positive number greater than G. For $R(x) > R(c)$ the remainder has therefore zero as a limit as n increases indefinitely. Moreover the majorant-value (20) shews that on the half-line going from $x = c$ in the direction of the positive part of the real axis, the binomial series converges *uniformly*; for $R(x-c) = x-c$ on this line. PINCHERLE has observed (l.c.) that a similar statement, which is analogous to a known theorem of ABEL on power series, holds for the integral (1), and that it follows from the equation (5), which has been found by means of integration by parts. In the same manner the just mentioned proposition may be proved generally by means of *summation* by parts, both for series of *integral* factorials (the binomial series treated of in this note) and the series of factorials in the more restrictive sense of the word, which proceed according to *inverse* factorials. For the latter series I have shown this in a communication on those series [1]). The expansion of the integral (1) in such a series is, however, as appears from investigations of NIELSEN [2]) and PINCHERLE [3]), only possible under restricting conditions for $f(t)$, viz. if it is an *analytic* function, whose circle of convergence for the point $t = 1$ passes through $t = 0$, and whose order on this circle is different from $+\infty$.

[1]) Proceedings XXII, N⁰. 1.

[2]) Handbuch, p. 244.

[3]) *Sulla sviluppabilità di una funzione in serie di fattorali*, Rendic. d. R. Acc. d. Lincei 1903 (2e Semestre).