Astronomy. — *"Theory of Jupiter's Satellites. II. The variations."* By Prof. W. de Sitter.

(Communicated in the meeting of June 28, 1919).

We still restrict ourselves to the non-periodic part $[R_i]$ of the perturbative function, as in the determination of the intermediary orbit.[1] The quantities h_i, k_i, ν_i, ω_i, which were zero in the intermediary orbit, are now determined by the equations (23),[2] of which the solution is given by (24). For the determination of the five values of β_q^2 we have the determinant (28). Then c'_{iq} and c''_{iq} are determined from (27) and c_{iq} and c'''_{iq} from the first and last of (25). The expressions for the coefficients a_{ij}, a'_{ij}, b_{ij}, b'_{ij}, d_{ij}, d'_{ij}, e_{ij}, e'_{ij} are given in Vol. XII, Part I, of the Annals of the Leiden Observatory, page 31. Then we have

$$A_{ij} = \Sigma_l (a'_{il} a_{lj} + b'_{il} d_{lj}),$$
$$B_{ij} = \Sigma_l (a'_{il} b_{lj} + b'_{il} e_{lj}),$$
$$C_{ij} = \Sigma_l (d'_{il} a_{lj} + e'_{il} d_{lj}),$$
$$D_{ij} = \Sigma_l (d'_{il} b_{lj} + e'_{il} e_{lj}).$$

The details of the computation of the quantities a_{ij}, a'_{ij}, etc. and A_{ij}, B_{ij}, etc. will be published in the Annals of the Observatory at Leiden. Here we shall only give the results. The determinant (28) is: (see formula A, next page).

The coefficients are given in units of the *eighth* decimal place. Denoting the columns by roman, and the rows by arabic numerals, we now perform the following operations:

add $4 \,.\, (V) + 2 \,.\, (VI)$ to (VII)
,, $3 \,.\, (V) - \mathrm{e}_1 \,.\, (I) - \mathrm{e}_2 (II) - \mathrm{e}_3 (III) - \mathrm{e}_4 (IV)$,, (VI)
,, $-2 \,.\, (7)$,, (6)
,, $2 \,.\, (7) - 3 \,.\, (6)$,, (5)
,, $\mathrm{e}_i \, [(6) - 2 \,.\, (7)]$,, (i) $(i = 1 \ldots 4)$.

The determinant then becomes: (see formula B, next page).

[1]) See *Outlines of a new theory of Jupiter's Satellites*, These Proceedings, Vol. XX, p 1289—1308 and *Theory of Jupiter's Satellites. I. The intermediary orbit*, These Proceedings, Vol XXI, p. 1156—1163.

[2]) "*Outlines*" p 1301 The definition of h_i and k_i is slightly different here, in consequence of the introduction of e_i instead of η_i. We now have

$$e_i \cos g_i = \mathrm{e}_i + h_i$$
$$e_i \sin g_i = k_i.$$

A

+ 29100.69 − β^2	+ 21 69	− 46 73	− 1.42	+ 118.18	− 232.81	− 7.12	0
+ 170.79	+ 22765.45 − β^2	+ 102.71	+ 4.77	+ 32.87	+ 114.95	− 361.39	0
− 14.41	+ 37.16	+ 21367.36 − β^2	− 25.51	+ 0.21	+ 12.38	− 25.62	0
+ 0.36	+ 1.69	− 34.51	+ 21087.28 − β^2	+ 0.00	− 0.01	+ 0.01	0
+ 82273	+ 20156	− 10	+ 0	+ 530 06 − β^2	− 1058.08	− 4 08	0
− 97192	+ 44489	+ 20187	− 0	− 618.14	+ 1876.26 − β^2	− 1279.98	0
+ 8	− 16723	− 4933	+ 0	− 0.38	− 158.07	+ 317.68 − β^2	0
+ 0	+ 1	+ 0	− 0	+ 0.00	+ 0.00	− 0.01	− β^2

B

+ 28694.59 − β^2	+ 347.27	+ 78 82	− 1.42	+ 115.60	0	0	0
− 737.84	+ 23493 93 − β^2	+ 383.62	+ 4.76	+ 27.10	0	0	0
− 72.36	+ 83.62	+ 21385.27 − β^2	− 25 51	− 0.15	0	0	0
+ 0.34	+ 1.70	− 34.51	+ 21087.28 − β^2	+ 0.00	0	0	0
+ 373866	− 146757	− 70437	+ 2	+ 2383.70 − β^2	0	0	0
+ 97209	+ 77935	+ 30053	− 1	− 617.37	− β^2	0	0
+ 8	− 16723	− 4933	+ 0	− 0.38	0	− β^2	0
+ 0	+ 1	+ 0	− 0	+ 0 00	0	0	− β^2

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Consider the determinant consisting of the elements of the first five rows and columns. This can be conceived to be the result of the elimination of y_i from the linear equations

$$\Sigma_j F_{ij} y_j - \beta^2 y_i = 0 \qquad (i, j = 1 \,..\, 5)$$

where F_{ij} is the element of the i^{th} row and the j^{th} column

The new unknowns y_i are connected with the original unknowns c'_i and c''_i of the equations (27) by the relations·

$$c_i' = y_i - e_i y_6, \qquad (i = 1 \ldots 4)$$
$$c_1'' = y_5 + 3y_6 + 4y_7,$$
$$c_2'' = y_6 + 2y_7,$$
$$c_3'' = y_7,$$
$$c_4'' = y_8.$$

To each of the five roots β^2_q corresponds a set y_{iq} $(i = 1 \ldots 5)$. Then y_6, y_7 and y_8 are determined by

$$\Sigma_j F_{ij} y_j - \beta^2 y_i = 0. \qquad (j = 1 \ldots 5, \quad i = 6 \,..\, 8)$$

We now determine x_i from

$$F'_{55} = F_{55} + \Sigma_j a_j F_{5j},$$
$$(i, j = 1 \ldots 4)$$
$$F_{i5} + \Sigma_j x_j F_{ij} - x_i F'_{55} = 0,$$

and we put

$$F'_{ij} = F_{ij} - x_i F_{5j}, \qquad (i, j = 1 \ldots 4)$$

and

$$y_i = z_i + x_i y_5. \qquad (i = 1 \ldots 4)$$

Then z_i and β^2 are determined by the equations

$$(\alpha) \qquad \Sigma_j F'_{ij} z_j - \beta^2 z_i = 0, \qquad (i, j = 1 \ldots 4)$$

and y_5 is given by

$$\Sigma_j F_{5j} z_j + (F'_{55} - \beta^2) y_5 = 0. \qquad (j = 1 \,..\, 4)$$

This determines four roots β_q^2. The fifth is given by

$$\beta_5^2 = F'_{55},$$

and the corresponding values of z_i and y_5 are

$$y_{55} = 1 \quad , \quad z_{i5} = 0. \qquad (i = 1 \,..\, 4)$$

To solve the equations (α) we take $z_{qq} = 1$. Since the coefficients of the diagonal F'_{ii} are much larger than the others, we can take

$$\beta^2_q = F'_{qq} + \xi_q$$

$$z_{iq} = \frac{F'_{iq}}{F'_{qq} - F'_{ii}} + \eta_{iq}.$$

Then the quantities ξ_q and η_{iq} are small, and we easily find a set of equations from which they can be conveniently determined.

We find

$$F'_{55} = +\ 1025.13,$$

and the determinant of the four equations (α) is

$$\begin{vmatrix} +30250.34-\beta^2 & -\ 263.42 & -\ 214.29 & -\ 1.41 \\ -\ 235.83 & +23296\ 87-\beta^2 & +\ \cdot 289.04 & +\ 4.76 \\ -\ 71.72 & +\ 83\ 36 & +21385.15-\beta^2 & -\ 25.51 \\ +\ 0.33 & +\ 1.71 & -\ 34.50 & +21087.28-\beta^2 \end{vmatrix}$$

The five roots β_q^2, determined in this way, are (still expressed in units of the 8th decimal place):

$$\beta_1^2 = +\ 30261.15$$
$$\beta_2^2 = +\ 23299.79$$
$$\beta_3^2 = +\ 21374.51$$
$$\beta_4^2 = +\ 21084.19$$
$$\beta_5^2 = +\ 1025.13$$

We now put, for $q = 1 \ldots 4$,

$$\psi_q = \beta_q \tau + \psi_{q0} = \varkappa\tau + \varpi_q + \pi_{q0} = \varkappa\tau + \gamma_q \tau + \varpi_{q0} + \pi_{q0},$$

where

$$\pi_{10} = \pi_{30} = \pi_{40} = 0 \quad , \quad \pi_{20} = 180°.$$

Then $\varpi_q = \gamma_q\tau + \varpi_{qo}$ are the longitudes of the "proper" perijoves, and ψ_q for $q = 1 \ldots 4$ are the arguments of the inequalities of group II, and ψ_5 is the argument of the libration.

The mean motions of these arguments are, in degrees per day [1]):

		El. and Masses.
$\gamma_1 =$	0°148668	0°.14407
$\gamma_2 =$	0 .039842	0 .039593
$\gamma_3 =$	0 .006949	0 .007046
$\gamma_4 =$	0 .001862	0 .001864
$\beta_5 =$	0 .16347	0 .16252

These values are not yet final, since we have neglected:

1. the effect of the periodic part $R_i - [R_i]$ of the perturbative function,

2. the squares and products of the quantities $h_i, k_i, \nu_i, \omega_i$,

3. the inclinations of the orbital planes of the satellites on the plane of Jupiter's equator.

Apart from the corrections which must eventually be applied

[1]) The motions γ_i and β_5 are siderial ones, and do not contain precession. Accordingly the precession has been subtracted from the values of γ_i as given in *El. and Masses.*

later on these three accounts, the values of β_i and γ_i as given here are certainly exact and complete to the last decimal place given.

The values added for comparison are those of my theory of 1908 [1]), reduced to the masses adopted here. This theory is SOUILLART's with some errors corrected. The computation of the motion of the argument of the libration, however, was carried one order further than was done by SOUILLART.

The values of the coefficients c_{iq}, c'_{iq}, c''_{iq} and c'''_{iq} are:

q	1	2	3	4	5
c_{1q}	+ 0.96868	+ 0.02754	+ 0.02479	+ 0.00230	— 0.00054
c_{2q}	— .04429	+ .93485	— .17327	— .01584	+ .00032
c_{3q}	— 00686	+ .03313	+ .98970	+ .08804	+ .00004
c_{4q}	+ .00006	+ .00012	— .12098	+ 1.00000	.00000
c'_{1q}	+ 0.96038	+ 0.05057	+ 0.02483	+ 0.00229	— 0 00287
c'_{2q}	— .01944	+ .97713	— .15246	— 01400	+ .00155
c'_{3q}	— .00637	+ .04010	+ .99952	+ .08891	+ 00018
c'_{4q}	+ .00006	+ .00012	— .12098	+ 1.00000	.00000
c''_{1q}	+ 2.7649	+ 0.9617	— 0.07900	— 0.00718	+ 0.15668
c''_{2q}	— 3.3850	+ 1.8535	+ .57340	+ .05032	— .26677
c''_{3q}	+ 0.0300	— 0.7324	— 11736	— .01022	+ .02150
c''_{4q}	.0000	.0000	+ .00001	.00000	.00000
c'''_{1q}	+ 0.01195	+ 0.00363	— 0.00028	— 0.00002	+ 0.00012
c'''_{2q}	— .02934	+ .01436	+ .00405	+ 00035	— .00043
c'''_{3q}	+ .00052	— .01113	— .00149	— .00013	+ .00007
c'''_{4q}	0	0	0	0	0

If we neglect the squares and products of ε_q, (as well as products $e_i \varepsilon_q$), and if we put

$$\tau_{iq} = \pm \tfrac{1}{2}(c_{iq} + c'_{iq}),$$

where the lower sign must be taken if either i or q, but *not both* of them, are 2, then the effect of the variations on the radius-vector and the longitude is:

$$\frac{\delta r_i}{a_i} = -\frac{3}{2}\Sigma_q\, c'''_{iq}\,\varepsilon_q \cos \psi_q - \Sigma_q\, \tau_{iq}\,\varepsilon_q \cos(\lambda_i - \varpi_q) -$$

$$- \tfrac{1}{2}\, \Sigma_q\,(c_{iq} - c'_{iq})\,\varepsilon_q \cos(l_i + \psi_q)$$

$$\delta w_i = \Sigma_q\, c''_{iq}\,\varepsilon_q \sin \psi_q + 2\,\Sigma_q\, \tau_{iq}\,\varepsilon_q \sin(\lambda_i - \varpi_q) + \Sigma_q\,(c_{iq} - c'_{iq})\,\varepsilon_q \sin(l_i + \psi_q),$$

where we have put

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[1]) *On the Masses and Elements of Jupiter's satellites and the mass of the System*, these Proceedings, Vol. X, pp. 653 and 710.

$$\lambda_\iota = \lambda_{00} + \pi_{\iota 0} + (c_\iota - \varkappa)\,\tau,$$
$$l_\iota = c_\iota\,\tau.$$

The first term for $q = 1 \ldots 4$ gives the inequalities of group II, and for $q = 5$ it represents the libration. The second term for $q = 1 \ldots 4$ represents the equations of the centre.

We have

q	1	2	3	4
τ_{1q}	+ 0.96453	− 0.03905	+ 0.02481	+ 0.00229
τ_{2q}	+ .03186	+ .95599	+ .16287	+ 01492
τ_{3q}	− .00661	− .03662	+ .99461	+ .08847
τ_{4q}	+ .00006	− .00012	− .12098	+ 1.00000

For $q = 5$ this term is better written in the form

$$-\tfrac{1}{2}\,(c_{\iota 5} + c'_{\iota 5})\,\varepsilon_5 \cos\,(l_\iota - \psi_5)$$
$$+\quad (c_{\iota 5} + c'_{\iota 5})\,\varepsilon_5 \sin\,(l_\iota - \psi_5).$$

It then represents, like the third term for all values of q, small periodic inequalities, whose periods differ little from that of the equations of the centres.

It should be pointed out that the theory here given (intermediary orbit and variations) covers the same ground as SOUILLART's, with the exception of the small periodic perturbations and the terms of very long period arising from the action of the sun, Saturn, etc. SOUILLART does not give any term of the perturbative function, nor any interaction of two terms leading to a term of higher order, which is not taken into account here too; and he omits many terms which are included here. The above theory is *certainly complete* up to the *numerical* limit of accuracy which was fixed beforehand. This can certainly not be said of SOUILLART's theory, though it generally gives many more decimals. The new theory has proved eminently suitable for numerical computation.