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Physics. — "*On expansions in series of algebraic forms with different sets of variables of different degree*" [1]) By Prof. J. A. SCHOUTEN. (Communicated by Prof. CARDINAAL).

(Communicated in the meeting of May 3, 1919).

Notations.

We start from the system S^{l}_{n} [2]) with the covariant and contravariant fundamental units $\mathbf{e}_\lambda$ resp. $\mathbf{e}'_\lambda$, $\lambda = a_1, \dots, a_n$, and the fundamental multiplications $\wedge$ (outer multiplication) $\circ$ (general multiplication) $\frown$ (alternating multiplication and $\smile$ (symmetrical multiplication).

$$\mathbf{e}_\lambda \wedge \mathbf{e}'_\mu = \begin{cases} \varkappa = (-1)^{\frac{n(n-1)}{2}} & \text{for} \quad \lambda = \mu \\ 0 & \text{,,} \quad \lambda \neq \mu \end{cases}$$

$$\left.\begin{array}{l} \mathbf{e}_{\lambda_1} \overset{\wedge}{\dots} \mathbf{e}_{\lambda_i} = \mathbf{e}_{\lambda_1} \overset{\frown}{\dots} \mathbf{e}_{\lambda_i} = \text{covariant quantity of degree } i \text{ (}i\text{-vector)} \\ \mathbf{e}'_{\lambda_1} \overset{\wedge}{\dots} \mathbf{e}'_{\lambda_i} = \mathbf{e}'_{\lambda_1} \overset{\frown}{\dots} \mathbf{e}'_{\lambda_i} = \text{contravariant} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \end{array}\right\} i < n$$

$\mathbf{e}_{a_1} \overset{\frown}{\dots} \mathbf{e}_{a_n} = \mathbf{E} =$ covariant scalar; $\mathbf{e}'_{a_1} \overset{\frown}{\dots} \mathbf{e}'_{a_n} = \mathbf{E}' =$ contravariant scalar

$$\left.\begin{array}{l} \varkappa^n \, \mathbf{e}_{a_2} \dots \overset{\frown}{\mathbf{e}_{a_n}} \mathbf{E}' = \mathbf{e}'_{a_1} \\ \varkappa^n \, \mathbf{e}'_{a_2} \dots \overset{\frown}{\mathbf{e}'_{a_n}} \mathbf{E} = \mathbf{e}_{a_1} \end{array}\right\} \textit{cycl.}$$

By *i-th* (*procurrent*) *transvection* of $\overset{P}{\mathbf{m}} = \mathbf{m}_1 \overset{\circ}{\dots} \mathbf{m}_P = \mathbf{m}_1 \dots \mathbf{m}_P$ and $\overset{Q}{\mathbf{r}'} = \mathbf{r}'_1 \dots \mathbf{r}'_q$ will be understood

$$\overset{P}{\mathbf{m}} \underset{i}{.} \overset{Q}{\mathbf{r}'} = (\mathbf{m}_1 \wedge \mathbf{r}_1') \dots (\mathbf{m}_i \wedge \mathbf{r}_i') \, \mathbf{m}_{i+1} \dots \mathbf{m}_P \, \mathbf{r}'_{i+1} \dots \mathbf{r}'_Q \; ^{3})$$

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[1]) See also: "On expansions in series of co- and contravariant quantities of higher degree etc.", These Proceedings Vol. XXII, p. 251—266, here further cited as E_I, of which paper this communication forms the continuation and an application.

[2]) S^{l}_{n} is found from R^{o}_{n} by omission of all quantities that exist only under the orthogonal group. See for these systems: Over de direkte analyses der lineaire grootheden bij de rotationeele groep enz., Versl. der Kon. Akad. v. Wet. Dl. XXVI, bldz. 567—580; Ueber die Zahlensysteme der rotationalen Gruppe, Nieuw Arch. voor Wisk. Dl. XIII, 1919; Die direkte Analysis der neueren Relativitätstheorie, Verh. der Kon. Akad. v. Wet. Dl. XII N°. 6 (1919) blz. 29.

[3]) The sign $\underset{i}{.}$ instead of $(,)^i$ for the transvections of the theory of invariants was first introduced by E. WAELSCH.

We have therefore, when for the sake of simplicity, is written $.$ for $\underset{1}{.}$:

$$\mathbf{m}.\mathbf{r}' = \mathbf{m}\underset{1}{.}\mathbf{r}' = \mathbf{m}\wedge\mathbf{r}' = \varkappa\left(m_{a_1} r'_{a_1} + \ldots + m_{a_n} r'_{a_n}\right)$$

By *i-th* outer transvection $\underset{i}{\diamond}$ of m affinors $\overset{p}{\mathbf{u}} = \mathbf{u}_1 \ldots \mathbf{u}_p$, $\overset{q}{\mathbf{v}} = \mathbf{v}_1 \ldots \mathbf{v}_q$ etc. will be understood the quantity found from $\overset{p}{\mathbf{u}}\overset{q}{\mathbf{v}}$ by substituting *locally* for the ideal vectors $\mathbf{u}_1, \mathbf{v}_1, \ldots$, and then for $\mathbf{u}_2, \mathbf{v}_2, \ldots$, to $\mathbf{u}_i, \mathbf{v}_i, \ldots$ the ideal factors of their alternating product. When at the same time the other factors are *locally* substituted by the ideal factors of their alternating resp symmetrical product, then the *i-th outer alternating transvection* $\underset{i}{\wedge}$ resp. the *i-th outer symmetrical one* $\underset{i}{\vee}$ is formed.

Affinors and algebraic forms. When the *P-th* transvection of $\overset{P}{\mathbf{m}}$ is formed with a product $\overset{P}{\mathbf{r}'}$ of P different contravariant fundamental elements $\mathbf{r}'_1, \ldots, \mathbf{r}'_P$, we find the form

$$F\overset{P}{\mathbf{m}} = \overset{P}{\mathbf{m}}\underset{P}{.}\mathbf{r}'_1 \ldots \mathbf{r}'_P = (\mathbf{m}_1 . \mathbf{r}'_1) \ldots (\mathbf{m}_P . \mathbf{r}'_P) =$$

$$= \varkappa^P \sum_{\lambda_1 \ldots \lambda_p}^{a_1 \ldots a_n P} m_{\lambda_1 \ldots \lambda_P} r'_{1\lambda_1} \ldots r'_{P\lambda_P}.$$

A special case is that $\mathbf{r}'_1 \ldots \mathbf{r}'_P$ are all not-ideal. Then $F\overset{P}{\mathbf{m}}$ is a form in P sets of n not-ideal variables. Thus the characteristic numbers of a covariant affinor (and therefore also of a contravariant one) may always be considered as the coefficients of such an algebraic form. When the sets $\mathbf{r}'_1, \ldots, \mathbf{r}'_P$ are given and when their order is fixed, $F\overset{P}{\mathbf{m}}$ is singly determined by $\overset{P}{\mathbf{m}}$. When all sets are different, then $\overset{P}{\mathbf{m}}$ is also singly determined by $F\overset{P}{\mathbf{m}}$; in the other case not, as $\overset{P}{\mathbf{m}} + \overset{P}{\mathbf{n}}$, where $\overset{P}{\mathbf{n}}$ is an arbitrary affinor, alternating in two factors that correspond with two equal sets of variables, transvected with $\overset{P}{\mathbf{r}'}$, also forms $F\overset{P}{\mathbf{m}}$.

In the general case $\mathbf{r}'_1, \ldots, \mathbf{r}'_P$ are ideal, $\overset{P}{\mathbf{r}'}$ is however equal to $\overset{p}{\mathbf{x}}{}'_\rho \overset{q}{\mathbf{y}}{}'_\sigma \ldots$, where $p\varrho + q\sigma + \ldots = P$, and where $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}', \ldots$ are not-ideal. Then $F\overset{P}{\mathbf{m}}$ is a form of the degrees $\varrho, \sigma, \ldots$ in sets of variables that may be considered themselves as coefficients of variable forms in $p, q, \ldots$ sets of n covariant variables. Such variables will be called *variables of the degrees* $p, q, \ldots$ When the sets $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}', \ldots$ and the order

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of their ideal factors are given, then $F\overset{P}{\mathbf{m}}$ is singly determined by $\overset{P}{\mathbf{m}}$. For the sake of simplicity we shall choose this order in such way that $\overset{P}{\mathbf{r}}{}' = \overset{p}{\mathbf{x}}{}' \overset{q}{\mathbf{y}}{}' \ldots$ In order to be able to determine also singly $\overset{P}{\mathbf{m}}$, we shall first prove the following theorem:

Principal theorem A. Every algebraic form of the total degree P, homogeneous and of the degrees $\varrho, \sigma, \ldots$, in different sets of variables $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}', \ldots$ each of which may be regarded as coefficients of variable forms $F\overset{p}{\mathbf{x}}{}', F\overset{q}{\mathbf{y}}{}', \ldots$, linear and of the degrees $p, q, \ldots$ in sets of n different covariant variables, can be written as a product of P ideal linear forms. When for the sets of variables $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}' \ldots$ is prescribed, that $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}', \ldots$ are separately annihilated by definitely indicated ordered elementary operators either of the first or of the second kind [1]*, as for the rest the variables being able to obtain all values, then one single definite affinor of degree P belongs to the given form for a definite choice of the order of the sets.*

When the characteristic numbers of the sets are $x'_1, \ldots, x'_\alpha$, $y'_1, \ldots, y'_\beta; \ldots$, α being n^p, β being n^q etc. then every term of F has the shape:

$$\overset{P}{n}_{\varrho_1 \ldots \varrho_\alpha, \sigma_1 \ldots \sigma_\alpha, \ldots} x'^{\varrho_1}_1 \ldots x'^{\varrho_\alpha}_\alpha \, y'^{\sigma_1}_1 \ldots y'^{\sigma_\beta}_\beta .$$

$$\varrho_1 + \cdots + \varrho_\alpha = \varrho$$
$$\sigma_1 + \cdots + \sigma_\beta = \sigma.$$
$$\vdots$$

When

$$\overset{p}{\mathbf{e}}{}_1', \ldots, \overset{p}{\mathbf{e}}{}_\alpha' \;;\; \overset{q}{\mathbf{e}}{}_1', \ldots, \overset{q}{\mathbf{e}}{}_\beta' \;; \ldots$$

are the products of $p, q, \ldots$ of the fundamental units belonging to the characteristic numbers of $\overset{p}{\mathbf{x}}{}', \overset{q}{\mathbf{y}}{}', \ldots$ and

$$\overset{p}{\mathbf{e}}_1, \ldots, \overset{p}{\mathbf{e}}_\alpha \;;\; \overset{q}{\mathbf{e}}_1, \ldots, \overset{q}{\mathbf{e}}_\beta \;; \ldots$$

the products formed in the same way from $\mathbf{e}'_{a_1}, \ldots, \mathbf{e}'_{a_n}$, then the affinor

$$\overset{P}{\mathbf{n}} = \varkappa^P \Sigma \overset{P}{n}_{\varrho_1 \ldots \varrho_\alpha, \varrho_1 \ldots \sigma_\alpha, \ldots} \overset{p}{\mathbf{e}}{}_1^{\varrho_1} \ldots \overset{p}{\mathbf{e}}{}_\alpha^{\varrho_\alpha} \overset{q}{\mathbf{e}}{}_1^{\sigma_1} \ldots \overset{q}{\mathbf{e}}{}_\beta^{\sigma_\beta} \ldots$$

may be formed.

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[1]) See E_I p. 262.

As the transvection

$$\mathbf{e}_{i_1} \dots \mathbf{e}_{i_P} \underset{P}{\cdot} \mathbf{e}'_{j_1} \dots \mathbf{e}'_{j_P} \qquad \begin{array}{l} i_1, \dots, i_P = a_1, \dots, a_n \\ j_1, \dots, j_P = a_1, \dots, a_n \end{array}$$

is equal to $\varkappa^P$ for the case that $i_e = j_e, e = 1, \dots, P$ and equal to zero in every other case, we have

$$F = \overset{P}{\mathbf{n}} \underset{P}{\cdot} \overset{p}{\mathbf{x}'} \overset{q}{\mathbf{y}'} \dots$$

When now on one hand $\overset{P}{\mathbf{n}}$ and on the other hand $\overset{p}{\mathbf{x}'}, \overset{q}{\mathbf{y}'}$ are written as the products of ideal fundamental elements, then F is really reduced to a product of P ideal linear forms. In order to derive from $\overset{P}{\mathbf{n}}$ the affinor $\overset{P}{\mathbf{m}}$ that is singly determined by F we first prove the theorem:

Theorem I.

When $\overset{P}{\mathbf{q}}$ *is an ordered elementary affinor of the first (second) kind and* $\overset{P}{\mathbf{r}'}$ *a ditto one of the second (first) kind*[1]*, then the P-th transvection of them is zero, when the two elementary operators* $\varepsilon_{ij} \underset{i}{A^{(\alpha)}} \underset{j}{M^{(\alpha)}}, \varepsilon_{lm} \underset{l}{M^{(\beta)}} \underset{m}{A^{(\beta)}}$, *by means of which* $\overset{P}{\mathbf{q}}$ *and* $\overset{P}{\mathbf{r}'}$ ($\overset{P}{\mathbf{r}'}$ *and* $\overset{P}{\mathbf{q}}$) *can arise, are not conjugated,* i.e. when not $l = j$, $m = i$ and $\alpha = \beta$.

As:

$$\overset{P}{\mathbf{q}} = \varepsilon_{ij} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}}\,; \quad \overset{P}{\mathbf{r}'} = \varepsilon_{lm} M_l^{(\beta)} A_m^{(\beta)} \overset{P}{\mathbf{r}'},$$

we have:

$$\begin{aligned} \overset{P}{\mathbf{q}} \underset{P}{\cdot} \overset{P}{\mathbf{r}'} &= \varepsilon_{ij}\, \varepsilon_{lm} \left(A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}}\right)_P M_l^{(\beta)} A_m^{(\beta)} \overset{P}{\mathbf{r}'} = \\ &= \varepsilon_{ij}\, \varepsilon_{lm} \left(M_l^{(\beta)} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}}\right)_P A_m^{(\beta)} \overset{P}{\mathbf{r}'} = \\ &= \varepsilon_{ij}\, \varepsilon_{lm} \left(A_m^{(\beta)} M_l^{(\beta)} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}}\right)_P \overset{P}{\mathbf{r}'}. \end{aligned}$$

Thus the transvection is in fact zero, when not $l = j$ (therefore also $m = i$) and $\alpha = \beta$. The same proof holds m.m. by changing the first into the second kind and vice-versa.

Let now the sums of the ordered elementary operators of the first kind that do not annihilate $\overset{p}{\mathbf{x}'}, \overset{q}{\mathbf{y}'}, \dots$, be ${}_xL, {}_yL, \dots$ and the sums of the conjugate operators ${}^xL, {}^yL, \dots$ etc. In the special case that ${}_xL$ is a sum of elementary operators we evidently have ${}_xL = {}^xL$. From $\overset{P}{\mathbf{n}}$ we first form an affinor $\overset{P}{\mathbf{n}}_1$, by permutating the

[1] See E_I p. 262.

ϱ regions of p factors corresponding with $\overset{p}{\mathbf{x}}{}'^{\varrho}$ and also the σ regions of q factors, corresponding with $\overset{q}{\mathbf{y}}{}'^{\sigma}$ etc. in all possible $\varrho!\,\sigma!\ldots$ ways, adding them and finally dividing by $\varrho!\,\sigma!\ldots$ Then $\overset{P}{\mathbf{n}}_1$ may be written

$$\overset{P}{\mathbf{n}}_1 = \overset{p_\varrho}{\mathbf{n}}_x \overset{q_\sigma}{\mathbf{n}}_y \ldots,$$

where in the way known from symbolism of invariants $\varrho, \sigma, \ldots$ different equivalent quantities $\overset{p}{\mathbf{n}}_x, \overset{q}{\mathbf{n}}_y \ldots$ are to be introduced in order to avoid ambiguities [1]). Then the given form is also obtained by

$$F = \overset{P}{\mathbf{n}}_1 \underset{P}{.} \overset{P}{\mathbf{r}}{}' = \Big(\overset{p}{\mathbf{n}}_x \underset{p}{.} \overset{p}{\mathbf{x}}{}'\Big)^{\varrho} \Big(\overset{q}{\mathbf{n}}_y \underset{q}{.} \overset{q}{\mathbf{y}}\Big)^{\sigma} \ldots$$

According to theorem II we have now

$$F = (\overset{p}{\mathbf{n}}_x \underset{p}{.}\, {}_xL \overset{p}{\mathbf{x}}{}') \; (\overset{q}{\mathbf{n}}_y \underset{q}{.}\, {}_yL \overset{q}{\mathbf{y}}{}') \ldots =$$

$$= ({}^xL \overset{p}{\mathbf{n}}_x \underset{p}{.} \overset{p}{\mathbf{x}}{}') \; ({}^yL \overset{q}{\mathbf{n}}_y \underset{q}{.} \overset{q}{\mathbf{y}}{}') \ldots$$

When we write ${}^xL\overset{p}{\mathbf{n}}_x = \overset{p}{\mathbf{u}}, {}^yL\overset{q}{\mathbf{n}}_y = \overset{q}{\mathbf{v}}$, etc. and $\overset{P}{\mathbf{m}} = \overset{p}{\mathbf{u}} \overset{q}{\mathbf{v}} \ldots$, then we have

$$F = \overset{P}{\mathbf{m}} \underset{P}{.} \overset{P}{\mathbf{r}}{}' = (\overset{p}{\mathbf{u}} \underset{p}{.} \overset{p}{\mathbf{x}}{}') \; (\overset{q}{\mathbf{v}} \underset{q}{\;} \overset{q}{\mathbf{y}}{}') \ldots$$

$\overset{P}{\mathbf{m}}$ is the only affinor of this shape that when transvected with $\overset{P}{\mathbf{r}}{}'$ gives $F\overset{P}{\mathbf{m}}$. In fact every affinor $\overset{P}{\mathbf{m}}_1$ that can be written in the form

$$\overset{P}{\mathbf{m}}_1 = ({}^xL \overset{p}{\mathbf{m}}_x) \; ({}^yL \overset{q}{\mathbf{m}}_y) \ldots$$

and gives zero when transvected with $\overset{P}{\mathbf{r}}{}'$ is identically zero. In fact, as we have supposed that $\overset{p}{\mathbf{x}}{}'$ may obtain all values that are solutions of the equation ${}_xL\overset{p}{\mathbf{x}}{}' = \overset{p}{\mathbf{x}}{}'$; we thus may take for $\overset{p}{\mathbf{x}}{}'$

$$\overset{p}{\mathbf{x}}{}' = {}_xL\, \mathbf{s}'_1 \ldots \mathbf{s}_p',$$

where $\mathbf{s}'_1, \ldots, \mathbf{s}'_p$ are not-ideal *different* sets of variables and the same holds for $\overset{q}{\mathbf{y}}{}'$, etc. Then we have:

$$0 = \overset{P}{\mathbf{m}}_1 \underset{P}{.} \overset{P}{\mathbf{r}}{}' = \{({}^xL \overset{p}{\mathbf{m}}_x) \underset{p}{.}\, {}_xL\, \mathbf{s}_1' \ldots \mathbf{s}_p'\}^{\varrho} \{({}^yL \overset{q}{\mathbf{m}}_y) \underset{q}{.}\, {}_yL\, \mathbf{t}_1' \ldots \mathbf{t}_q'\}^{\sigma} \ldots$$

$$= \{({}^xL \overset{p}{\mathbf{m}}_x) \underset{p}{.} \mathbf{s}_1' \ldots \mathbf{s}_p'\}^{\varrho} \{({}^yL \overset{q}{\mathbf{m}}_y) \underset{q}{.} \mathbf{t}_1' \ldots \mathbf{t}_q'\}^{\sigma}$$

$$= \overset{P}{\mathbf{m}}_1 \underset{P}{.} (\mathbf{s}_1' \ldots \mathbf{s}_p')^{\varrho} (\mathbf{t}_1' \ldots \mathbf{t}_q')^{\sigma} \ldots,$$

[1]) Comp. Die direkte Analysis zur neueren Relativitätstheorie, p. 11, 17.

When now for $\mathbf{s}', \mathbf{t}', \ldots$ are substituted here all possible combinations of $\mathbf{e}'_{\alpha_1}, \ldots, \mathbf{e}'_{\alpha_n}$, we find that each characteristic number of $\overset{p}{\mathbf{m}}$ is zero.

Of the proved property we make use by calling an algebraic form *non special, alternating, symmetrical, locally alternating, symmetrical* or *permutable*, an *elementary form* or an *ordered elementary form of the first or second kind*, when corresponding names are used for the corresponding affinor. By application of an operator $K, A, M, \overset{m}{A}, \overset{a}{M}, \overline{A}, \overline{M}, \overset{m}{\overline{A}}, \overset{a}{\overline{M}}, {}^{k}_{j}I, {}^{k}_{\alpha j}I$, or ${}^{\nu k}_{j}I$ will be understood application of that operator to the corresponding affinor of that form. By means of the corresponding affinor we are now able to reduce a great part of the properties of forms to the formal properties of the operators $K, A, \cdot$ etc., treated in E_I, which simplifies the treatment of forms considerably.

The characteristic numbers occurring in the linear factors are ideal identical with the symbols of Aronhold and Clebsch. When one of the sets e.g. $\overset{p}{\mathbf{x}}'$ is symmetrical, then $\overset{p}{\mathbf{u}}$ is also symmetrical and both may be written as the p-th power of an ideal fundamental element:

$$^{p}\underset{p}{\mathbf{u}} \,.\, ^{p}\underset{p}{\mathbf{x}}' = \mathbf{u}^{p} \,.\, \mathbf{x}'^{p} = (\mathbf{u} \,.\, \mathbf{x}')^{p}.$$

Also in this case the occurring characteristic numbers are symbols of Aronhold and Clebsch. When $\overset{p}{\mathbf{x}}$ is alternating, then $\overset{p}{\mathbf{u}}$ too is alternating and also in this case both may be written as p-th powers:

$$_{p}\underset{p}{\mathbf{u}} \,.\, _{p}\underset{p}{\mathbf{x}}' = \mathbf{u}^{p} \,.\, \mathbf{x}'^{p} = (\mathbf{u} \,.\, \mathbf{x}')^{p}.$$

In this latter case the occurring characteristic numbers are identical with the complex symbols introduced by Waelsch and Weitzenböck, the multiplication of which is anticommutative. When $\overset{p}{\mathbf{x}}'$ and therefore $\overset{p}{\mathbf{u}}$ too is more general, then the notation in the form of powers may be still useful sometimes. Then however, the ideal roots $\mathbf{x}'$ and $\mathbf{u}$ do not determine any longer the isomers of $\overset{p}{\mathbf{x}}'$ and $\overset{p}{\mathbf{u}}$. Both characteristic numbers are ideal numbers of complicated character in the products of which no commutation whatever is allowed any longer. By means of complex symbols Weitzenböck [1]) has proved the first part of the principal theorem A for forms in sets of variables that are all alternating. The above proof is an extension to forms with sets of variables of more general character.

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[1]) Beweis des ersten Fundamentalsatzes der symbolischen Methode. Sitzungsber. der Wiener Akad. 122 (13) 153—168, p. 155 etc.

Polar operators. Let be $p = q \ldots = 1$. As

$$\left(\mathbf{y}' \cdot \frac{d}{d\mathbf{x}'}\right)\mathbf{x}' = \varkappa\, \mathbf{y}',$$

we have

$$\left(\mathbf{y}' \cdot \frac{d}{d\mathbf{x}'}\right)\mathbf{x}'^{\rho}\, \mathbf{y}'^{\sigma} \ldots = \varkappa\, \varrho\, (\mathbf{x}'^{\rho-1} \smile \mathbf{y}')\, \mathbf{y}'^{\sigma} \ldots$$

and

$$\varkappa^{i} \frac{(\varrho - i)!}{\varrho!}\left(\mathbf{y}' \cdot \frac{d}{d\mathbf{x}'}\right)^{i} \overset{P}{\mathbf{m}} \underset{P}{.}\, \mathbf{x}'^{\rho}\, \mathbf{y}'^{\sigma} \ldots = \overset{P}{\mathbf{m}} \underset{P}{.}\, (\mathbf{x}'^{\rho-i} \smile \mathbf{y}'^{i})\, \mathbf{y}'^{\sigma} =$$

$$= \overset{P}{\mathbf{m}} \underset{P}{.}\, \mathbf{x}'^{\rho-i}\, \mathbf{y}'^{\sigma+i} \ldots$$

$\varkappa^{i} \frac{\varrho - i!}{\varrho!}\left(\mathbf{y}' \cdot \frac{d}{d\mathbf{x}'}\right)^{i}$ is therefore the i-th *polar operator* of $\mathbf{x}'$ with respect to $\mathbf{y}'$. By application of this operator the form $F\overset{P}{\mathbf{m}}$ changes into a form with the sets of variables $\mathbf{x}'^{\rho-i}$, $\mathbf{y}'^{\sigma+i}$. The corresponding affinor of this form is no longer $\overset{P}{\mathbf{m}}$, but is derived from $\overset{P}{\mathbf{m}}$ by application of an operator ${}_{\sigma+i}M$, the permutation region of which contains the ideal factors of $\overset{P}{\mathbf{m}}$ corresponding to $\mathbf{y}'^{\sigma+i}$. Application of this operator is therefore equivalent with application of ${}^{\sigma+i}M$ combined with a change of the sets of variables.

Capelli's *operators* $H^{(s)}$. Let again be $p = q = \ldots 1$ and let us call the sets of variables $F\overset{P}{\mathbf{m}}$ $\mathbf{x}'_1, \ldots, \mathbf{x}'_m$ and the corresponding exponents $\varrho_1, \ldots, \varrho$, so that $\varrho_1 + \ldots + \varrho_m = P$, then the differential operator $H^{(s)}$ introduced by Capelli is written in our notation:

$$H^{(s)} = \varkappa^{s}\, s!\, \Sigma\, (\mathbf{x}_{i_1} \frown \ldots \frown \mathbf{x}_{i_s}) \underset{s}{.} \left(\frac{\partial}{\partial \mathbf{x}_{i_1}'} \frown \ldots \frown \frac{\partial}{\partial \mathbf{x}_{i_s}'}\right),$$

where the summation has to be extended over all $\binom{m}{s}$ combinations of s of the numbers $1, \ldots, m$. By application of $H^{(s)}$ to $F\overset{P}{\mathbf{m}}$ we find:

$$H^{(s)} \overset{P}{\mathbf{m}} \underset{P}{.}\, \overset{P}{\mathbf{r}'} = H^{(s)} (\mathbf{u}_1 \cdot \mathbf{x}_1')^{\rho_1} \ldots (\mathbf{u}_m \cdot \mathbf{x}'_m)^{\rho_m} = \varkappa^{s}\, s!\, \Sigma\, (\mathbf{x}'_{i_1} \frown \ldots \frown \mathbf{x}'_{i_s}) \underset{s}{.}$$

$$\underset{s}{.} \left\{ \frac{\partial}{\partial \mathbf{x}_{i_1}'} (\mathbf{u}_{i_1} \cdot \mathbf{x}_{i_1}')^{\rho_{i_1}} \ldots \frac{\partial}{\partial \mathbf{x}_{i_s}'} (\mathbf{u}_{i_s} \cdot \mathbf{x}_{i_s}')^{\rho_{i_s}} \right\} (\mathbf{u}_{j_1} \cdot \mathbf{x}_{j_1}')^{\rho_{j_1}} \ldots (\mathbf{u}_{j_{m-s}} \cdot \mathbf{x}'_{j_{m-s}})^{\rho_{j_{m-s}}}$$

where $j_1 \ldots, j_{m-s}$ are the indices of $1, \ldots, m$ that do not belong to $i_1, \ldots, i_s$. The summation has to be extended over all $\binom{m}{s}$ possible combinations.

As

$$\frac{\partial}{\partial \mathbf{x}_i'}(\mathbf{u}_i \,.\, \mathbf{x}_i') = \left(\frac{\partial}{\partial \mathbf{x}'}\,\mathbf{x}_i'\right) \underset{1}{.}\, \mathbf{u}_i = \varkappa \mathbf{u}_i,$$

we have

$$H_s\, F \overset{P}{\mathbf{m}} = s!\, \varrho_{i_1} \ldots \varrho_{i_s} \Sigma (\mathbf{x}'_{i_1} \frown \ldots \mathbf{x}'_{i_s}) \underset{s}{.} (\mathbf{u}_{i_1} \frown \ldots \mathbf{u}_{i_s})(\mathbf{u}_{i_1} \underset{1}{.} \mathbf{x}'_{i_1}{}')^{\rho_{i_1}-1} \ldots$$

$$(\mathbf{u}_{i_s} \,.\, \mathbf{x}'_{i_s})^{\rho_{i_s}-1} (\mathbf{u}_{j_1} \,.\, \mathbf{x}'_{j_1})^{\rho_{j_1}} \ldots (\mathbf{u}_{j_{m-s}} \mathbf{x} \,.\, '_{j_{m-s}})^{\rho_{j_{m-s}}}.$$

As further

$${}_sA \overset{P}{\mathbf{m}} = \mathbf{u}_1^{\rho_1} \ldots \mathbf{u}_{i_1-1}^{\rho_{i_1-1}} \mathbf{u}_{i_1}^{\rho_{i_1}-1} \mathbf{v}_1\, \mathbf{u}_{i_1+1}^{\rho_{i_1+1}} \ldots \mathbf{u}_m^{\rho_m},$$

where

$$\mathbf{v}_1 \ldots \mathbf{v}_s = \mathbf{u}_{i_1} \frown \ldots \mathbf{u}_{i_s},$$

when the permutation region of ${}_sA$ contains of $\mathbf{u}_{i_1}^{\rho_{i_1}}, \ldots, \mathbf{u}_{i_s}^{\rho_{i_s}}$ just the last factor of each, we have also:

$${}_s\overline{A} \overset{P}{\mathbf{m}} = \frac{\varrho_{i_1} \cdots \varrho_{i_s}}{\binom{P}{s}} \Sigma_s A \overset{P}{\mathbf{m}}$$

where the summation has to be extended over all $\binom{m}{s}$ essentially different permutation regions. This infers:

$$H^{(s)} F \overset{P}{\mathbf{m}} = \binom{P}{s} s!\, {}_s\overline{A} F \,.\overset{P}{\mathbf{m}}$$

viz. the CAPELLIAN operator $H^{(s)}$ is identical with the operator $\binom{P}{s} s!\, {}_s\overline{A}$. The linear independency of the operators $H^{(s)}$ discovered by CAPELLI and their commutativity mutually and with other operators composed of polar operators, is therefore a special case of the linear independency of the operators $\overline{A}$, proved in E_1, and their commutativity mutually and with all operators $\overline{M}$, A, M, K and P. As the operators $\overline{A}$, $\overline{M}$, $\overset{m}{\overline{A}}$, $\overset{a}{\overline{M}}$, ${}^i_jI$, K may all be written as sums of multiples of products of operators ${}_s\overline{A}$, $s = 1, \ldots, n$ and the identical operator I, these operators have for a form the significance of definite differential operators. When the sets of variables are of higher degree, these operators have the significance of differential operators of more complicate character. The different kinds of forms mentioned on p. 273 may thus be distinguished by means of the definite differential operators by which they can be obtained and the other differential operators by which they are annihilated.

The operator Ω. For $m = n$:

$$n!\,\varkappa^n\, \mathrm{E}'_{n};\left(\frac{d}{d\mathbf{x}_1'} \frown \cdot \frac{d}{d\mathbf{x}_n'}\right)$$

is the well-known operator Ω (Ω-Prozesz). According to the preceding we have:

$$\Omega = \frac{\mathrm{E}'}{(\mathbf{x}_1' \frown .. \mathbf{x}_n')} H^{(n)}$$

and the application of Ω is therefore equivalent to the application of $n!\binom{P}{n}\,{}_n\overline{A}$ combined with a division by the determinant of the sets of variables. We may therefore also say that a non reducible form in n sets of n variables is a form that is annihilated by Ω[1]. A form in n sets of n variables containing a factor $\dfrac{\mathbf{x}'_1 \frown . \mathbf{x}'_n}{\mathrm{E}'}$ can never be non-reducible. In fact, the corresponding affinor $\overset{P}{\mathbf{m}}$ possesses a linear covariant of degree $P-n$. Such a form is therefore not annihilated by the operator Ω.[2])

Expansion in series of a form in sets of two variables.

Let $F\overset{P}{\mathbf{m}}$ be a form in m sets of 2 variables and $\overset{P}{\mathbf{m}}$ the corresponding affinor. As $n = 2$, the expansion in series of $\overset{P}{\mathbf{m}}$ with regard to elementary affinors is identical with that with regard to non-reducible covariants[3]). When we apply this expansion, we find for $F\overset{P}{\mathbf{m}}$ the expansion:

$$F\overset{P}{\mathbf{m}} = \sum_{\alpha}^{0,\,,m'} \frac{\binom{P}{2\alpha}^2 \binom{2\alpha}{\alpha}^2}{\binom{P-\alpha+1}{\alpha}^2}\, {}_{\alpha\,2}\overset{\frac{m}{-}}{A}\, \underset{P}{\overset{P}{\mathbf{m}}} \,.\, \mathbf{x}'^{\rho}\, \mathbf{y}'^{\sigma} \ldots;\quad m' = \begin{cases} \dfrac{m}{2} & \text{for } m \text{ even} \\[2ex] \dfrac{m-1}{2} & \text{,, ,, odd,} \end{cases}$$

where each term is a sum of products of one single non-reducible form with a certain number of determinants of the form $x'_a\, y'_b - x'_b\, y'_a$, or shortly $(x'\,y')$ as written usually (Klammerfactoren), that is characteristic of that term. That such an expansion is possible and singly determined, has first been proved by GORDAN. For the special case that there are only two sets of variables an application of permutation laws gives

[1]) Comp. E_I p. 264.

[2]) Comp. p. 279.

[3]) Comp. E_I p. 265.

$$ {}_{\alpha.2}\overset{m}{A}\, \mathbf{u}^{\rho}\, \mathbf{v}^{\sigma} = \frac{\binom{\rho}{\alpha}\binom{\sigma}{\alpha}}{\binom{P}{2\alpha}\binom{2\alpha}{\alpha}}\, 2^{\alpha}\, (\mathbf{u}^{\rho} \underset{\alpha}{\vee} \mathbf{v}^{\sigma}), $$

so that for this special case the expansion in series becomes:

$$ F\overset{P}{\mathbf{m}} = \sum_{\alpha}^{0,\,..,m'} \frac{\binom{\rho}{\alpha}\binom{\sigma}{\alpha}}{\binom{P-\alpha+1}{\alpha}}\, 2^{\alpha} \binom{P}{2\alpha}\binom{2\alpha}{\alpha} (\mathbf{u}^{\rho} \underset{\alpha}{\vee} \mathbf{v}^{\sigma}) \underset{P}{.}\, \mathbf{x}'^{\rho}\, \mathbf{y}'^{\sigma}. $$

This expansion of $F\mathbf{u}^{\rho}\mathbf{v}^{\sigma}$ remains applicable for $n > 2$, because, there being only two sets of variables, only alternations of the form ${}_{\alpha.2}A$ give not identically zero. This is the so-called second expansion in series of GORDAN [1]).

The terms of the expansion with regard to non-reducible covariants may now be further decomposed in different ways. First each operator ${}_{\alpha.2}\overset{m}{A}$ can be decomposed into simple mixed alternations. Then an expansion of $F\overset{P}{\mathbf{m}}$ is obtained with regard to locally alternating forms for which in each term the power of a determinant of the variables is *the same* as the in the same term occurring power of the determinant of the ideal factors of $\overset{P}{\mathbf{m}}$ corresponding to those sets of variables. That such an expansion is possible and singly determined has been first proved by A. REISSINGER [2]).

Secondly each elementary affinor may be decomposed in ordered elementary affinors of the first kind. With this decomposition corresponds an expansion of the form:

$$ F\overset{P}{\mathbf{m}} = \sum_{\alpha}^{0,\,..,m'} \sum_{\lambda}^{1,\,..,\delta_{\alpha}} 2^{\alpha} \frac{P-2\alpha+1}{P-\alpha+1}\, {}_{\alpha.2}\overset{m}{A}{}^{(\lambda)}\, {}_{P-\alpha,\alpha}\overset{\alpha}{M}{}^{(\lambda)}\, \overset{P}{\mathbf{m}} \underset{P}{.}\, \mathbf{x}'^{\rho}\, \mathbf{y}' \ldots {}^{3)} $$

The factors of the form $(x'.y')$ occurring in each term satisfy the condition that they belong to the permutation regions of a definite *ordered* alternation ${}_{\alpha 2}A$ acting on $\mathbf{x}'^{\sigma}\, \mathbf{y}'^{\rho}$ and characteristic of that

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[1]) STUDY, Methoden zur Theorie der ternären Formen § 3 and § 4. The so-called first expansion in series of GORDAN corresponds to an expansion in series of a *mixed* affinor and is not discussed here.

[2]) Ausgezeichnete Form der Polaren-Entwicklung eines symbolischen Produktes. Progr. Realsch. Kempten 1906 – 07.

[3]) See *EI* p. 263.

term That such an expansion is possible and singly determined has been first proved by W. Godt[1]).

Thirdly it is possible to decompose each elementary affinor into ordered elementary affinors of the second kind. To this corresponds the singly determined expansion:

$$F\overset{P}{\mathbf{m}} = \sum_{i} \varepsilon_{ij} \sum_{j}^{1,\ldots,\delta_{ij}} M_j^{(i)} A_i^{(j)} \overset{P}{\mathbf{m}} \underset{P}{.} \mathbf{x}'^{\rho} \mathbf{y}'^{\sigma} \ldots$$

E. Waelsch[2]) has given another expansion which is also singly determined and which corresponds to an expansion of $\overset{P}{\mathbf{m}}$ in terms of the form

$$\{(\mathbf{u}^{\rho} \underset{\alpha_2}{\vee} \mathbf{v}^{\sigma}) \underset{\alpha_3}{\vee} \mathbf{w}^{\tau}\} \underset{\alpha_4}{\vee} \ldots$$

with coefficients that for a definitely chosen order of $\mathbf{u}^{\sigma}, \mathbf{v}^{\sigma}, \ldots$ are functions of $\alpha_2, \alpha_3, \ldots$ It is remarkable that the number of terms of this expansion for a P-linear form is equal to that of the expansion with regard to ordered elementary affinors of the first kind, e.g. $1 + 5 + 9 + 5 = 20$ for $P = 6$.

Expansion in series of a form in m sets of n variables.

Let

$$F\overset{P}{\mathbf{m}} = \mathbf{u}^{\rho} \mathbf{v}^{\sigma} \ldots \underset{P}{.} \mathbf{x}'^{\rho} \mathbf{y}'^{\sigma} \ldots$$

be a form in the m sets of variables $\mathbf{x}', \mathbf{y}', \ldots$ and $\overset{P}{\mathbf{m}} = \mathbf{u}^{\rho}\mathbf{v}^{\sigma}\ldots$ the corresponding affinor. We can expand $\overset{P}{\mathbf{m}}$ in non-reducible covariants. Each term is then a sum of ordered alternations each consisting of a penetrating general product of a number α of factors $\mathbf{E}$ that is characteristic of that term with a linear homogeneous non-reducible affinor of degree $P-\alpha n$. To this corresponds

[1]) W. Godt deduces this expansion in quite another way and this may be the reason that he has not seen the connexion with the group characters of Frobenius and the possibility of an analogous expansion for $n > 2$. "Ueber die Entwicklung binärer Formen mit mehreren Variabelen", Arch. f. Math. u. Phys. 13 (08) 1—12.

[2]) Ueber Reihenentwicklungen mehrfachbinärer Formen. Sitz. Ber. der Wiener Akad. 113 (04) 1209—1217, Waelsch has used for the first time the expansions in series of the theory of binary invariants to decompose directed quantities in parts covariant under the *orthogonal* group (e.g. the decomposition of the affinor of deformation in scalar, vector and deviator), "Ueber höhere Vectorgröszen der Kristallphysik etc." Wien. Ber. 113 (04) 1107—1119; Extension de l'algèbre vectorielle etc., Comptes Rendus 143 (06) 204—207.

an expansion of the form $F\overset{p}{\mathbf{m}}$ in a number of terms each of which is a sum of products of a non-reducible form with a certain number (characteristic of that term) of determinants formed by n of the sets of variables.

For $m = n$ this expansion has been given first by CAPELLI[1]) and for the general case by J. DERUYTS[2]) and K. PETR[3])[4]). Both CAPELLI and PETR base their proof upon the property mentioned p. 276, that a form in n sets of n variables containing the determinant of the variables as a factor, is not annihilated by Ω. The deduction of CAPELLI, which is most analogous to the above is based upon the theory of the differential operators $H^{(s)}$. DERUYTS uses his theory of the semi-invariants and -covariants and PETR makes use of differential-operators that can be constructed by means of auxiliary variables.

The terms of the expansion in non-reducible covariants may again be decomposed in different ways. Firstly each term of $\overset{p}{\mathbf{m}}$ can be decomposed into general mixed alternations and these again into simple ones. To this corresponds an expansion of $\overset{p}{\mathbf{m}}$ in a sum of terms consisting each of a sum of products of a number of determinants with $s_1, \ldots, s_l$ rows formed from the characteristic numbers of the sets $\mathbf{x}', \mathbf{y}' \ldots$ with one single symmetrical form. All terms are covariants, the sub-terms only then when $s_1 = s_2 = \ldots = n$.

In each sub-term the power of a determinant of the characteristic numbers of the variables is the same as the power of the determinant of the characteristic numbers of the corresponding ideal factors of $\overset{p}{\mathbf{m}}$. Under these conditions the expansion is singly determined and an extension of the one given by REISSINGER for $n = 2$.

Secondly each term of $\overset{p}{\mathbf{m}}$ may be decomposed into ordered alternations of the form ${}_{\gamma n}A$. Then the determinants occurring in each sub-term must belong to the permutation regions of a definite ordered alternation ${}_{\alpha n}A$, characteristic of that sub-term, and acting on $\mathbf{x}'^{\rho}\,\mathbf{y}'^{\sigma}$.

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1) Fondamenti di una teoria generale delle forme algebriche, Mem. dei Linces (82) § 74; Sur les opérations dans la théorie des formes algébriques, Math Ann. 37 (90) 1–37.

2) Essai d'une théorie générale des formes algébriques. Mém de Liège. 2. 17 (92) 4. 1—156; Détermination des fonctions invariantes de formes a plusieurs séries de variables. Mém. couronnés et mém des sav. étr. de Bruxelles 53 (90—93) 2. 1—23.

3) Ueber eine Reihenentwicklung für algebraische Formen, Bull. Intern. de Prague 12 (07) 163—191.

4) The forms called here non-reduceable are called by CAPELLI: "formes impropres" and by DERUYTS: "covariants de formes primaires".

This expansion is singly determined, the corresponding expansion of $\overset{p}{\mathbf{m}}$ being singly determined [1]) and an extension of that given by GODT for $n = 2$. We have thus found the theorem:

Principal theorem B. Every algebraic form, homogeneous and of the degrees $\varrho, \sigma, \ldots$ *in* m *sets of* n *variables* $\mathbf{x}', \mathbf{y}' \ldots$ *can be expanded in one and only one way in a series of terms, each of which consists of a product of a number* α *of determinants that are formed from the variables of* n *of the sets with a non-reducible form, in such a way that the determinants in each term belong to the permutation regions of a definite ordered alternation* $_{\nu, n}A$*, characteristic of that term and acting on the affinor* $\mathbf{x}'^{\varrho}\, \mathbf{y}'^{\sigma} \ldots$*, the number* α *being characteristic of a definite group of those terms.*

Thirdly we can proceed so far with the division, that $\overset{p}{\mathbf{m}}$ becomes a sum of ordered elementary affinors all of the first or all of the second kind. With this corresponds an expansion of $F\overset{p}{\mathbf{m}}$ in ordered elementary forms of the first resp. of the second kind, which may be characterized in the following way:

Principal theorem C. Every algebraic form, homogeneous and of the degrees $\varrho, \sigma, \ldots$ *in* m *sets of* n *variables* $\mathbf{x}', \mathbf{y}', \ldots$*, can be expanded in one and only one way in a series of ordered forms of the first resp. of the second kind.*

Examples:

The 6-linear form

$$F\overset{6}{\mathbf{m}} = \mathbf{m}_1 \ldots \mathbf{m}_6 \underset{6}{.} \mathbf{x}'_1 \ldots \mathbf{x}_6$$

can be decomposed into 76 ordered elementary forms of the first kind corresponding with the affinors.

1)	$_6A_{11}\ {}^0M^1$	$\overset{6}{\mathbf{m}}$	
2, ..., 6)	$_5A_{10}^{(\lambda)}\ {}^2M_2^{(\lambda)}$	$\overset{6}{\mathbf{m}}$,	$\lambda = 1, \ldots, 5$
7, ..., 15)	$_{4,2}A_9^{(\lambda)}\,{}^{2,2}M^{3(\lambda)}$	$\overset{6}{\mathbf{m}}$,	$\lambda = 1, \ldots, 9$
16, ..., 25)	$_4A_8^{(\lambda)}\ {}^3M_5^{(\lambda)}$	$\overset{6}{\mathbf{m}}$,	$\lambda = 1, \ldots, 10$
26, ..., 30)	$_{2\,3}A_7^{(\lambda)}\,{}^{3\,2}M_4^{(\lambda)}$	$\mathbf{m}$,	$\lambda = 1, \ldots, 5$
31, ..., 46)	$_{3,2}A_6^{(\lambda)}\ {}^{3,2}M_6^{(\lambda)}$	$\overset{6}{\mathbf{m}}$,	$\lambda = 1, \ldots, 16$
47, ..., 56)	$_3A_5^{(\lambda)}\ {}^4M_8^{(\lambda)}$	$\overset{6}{\mathbf{m}}$,	$\lambda = 1, \ldots, 10$

[1]) E_I p. 265

$$
\begin{array}{llll}
57,\ldots,61) & {}_{3\,2}A_4^{(\lambda)}\,{}^{2.3}M_7^{(\lambda)} & \overset{6}{\mathbf{m}}, & \lambda=1,\ldots,5 \\
62,\ldots,70) & {}_{2,2}A_3^{(\lambda)}\,{}^{4,2}M_9^{(\lambda)} & \overset{6}{\mathbf{m}}, & \lambda=1,\ldots,9 \\
71,\ldots,75) & {}_{2}A_2^{(\lambda)}\,{}^{5}M_{10}^{(\lambda)} & \overset{6}{\mathbf{m}}, & \\
76) & {}_{0}A_1\,{}^{6}M_{11} & \overset{6}{\mathbf{m}}, &
\end{array}
$$

for $n=5$, 1 becomes zero, for $n=4:1,\ldots,6$, for $n=3:1,\ldots,25$ and for $n=2$: $1,\ldots,56$. The expansion in elementary forms corresponds to an expansion of $\overset{6}{\mathbf{m}}$ which is found from the preceding one by taking together the horizontal rows 1; $2,\ldots,6$; $7\ldots,15$; etc. From this can be deduced again the expansion in non-reducible covariants. For $n>6:1\ldots,76$, for $n=6\cdot 1$; $2,\ldots,76$; for $n=5:2,\ldots,6$; $7,\ldots,76$; for $n=4.\ 7,\ldots 25$; $26,\ldots,76$; for $n=3:26,\ldots,30$; $31,\ldots,56$; $57,\ldots,76$, and for $n=2\cdot 57,\ldots,61$; $62,\ldots,70$; $71,\ldots,75$; 76. As to the expansion of a form of the sixth degree in a number of sets of variables less than 6 e.g.

$$F\overset{6}{\mathbf{n}}=\underset{6}{\overset{2}{\mathbf{n}}_1\overset{2}{\mathbf{n}}_2\overset{2}{\mathbf{n}}_3}\quad \mathbf{x}'^2_1\,\mathbf{x}'^2_2\,\mathbf{x}'^2_3$$

we may remark, that $\mathbf{x}'^2_1\,\mathbf{x}'^2_2\,\mathbf{x}'^2_3$ can be obtained by a definite simple mixing ${}^{3\,2}M_3^{(\alpha)}$. Hence, in the expansion of $\overset{6}{\mathbf{n}}$ all ordered elementary operators of the first kind vanish, when their first factor is an alternation that is annihilated by ${}^{3.2}M_3^{(\alpha)}$. In the first place thus $1,\ldots,25$. Of 26—30 remains one term, of 31—46 there remain nine terms, among which three different ones, of 47—56 four equal terms, of 57—60 two equal terms, of 61—69 six terms, among which three different ones, of 70—75 four terms, among which two different ones, while 76 remains. In total there remain therefore for $n\leq 3$ twelve terms and for $n=2$ seven terms. This last number gives also the number of terms in the expansion of WAELSCH [1]).

Expansion of a form in m sets of n variables of arbitrary degree.

Principal theorem D. Every algebraic form $F\overset{p}{\mathbf{m}}$, homogeneous and of the degrees $\varrho, \sigma, \ldots$ in m different sets of variables $\overset{p}{\mathbf{x}'}, \overset{q}{\mathbf{y}'},\ldots$ can be expanded in one and only one way in a series of ordered elementary forms of the first resp. of the second kind.

[1]) Comp. p. 278.

The *expansion may* be, *obtained by expanding* $\overset{p}{\mathbf{m}}$ *in ordered* elementary affinors.

Also the expansion in non-reducible covariant forms (principal theorem B) can be obtained for this most general case by expanding $\overset{p}{\mathbf{m}}$ with respect to non-reducible covariants. Then the determinants of n rows that occur in the expansion of the form generally have only an ideal meaning and therefore the non-reducible forms occurring in each term have only an ideal significance too. The terms themselves however keep their non-ideal significance and are found by taking together definite groups of terms from the expansion in elementary forms of the first kind.
