Mathematics. — "*On a remarkable functional relation in the theory of coefficientfunctions*". By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

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1. Let $\varphi(t)$ be a function having no singular points without the circle (1,1), i. e. the circle with centre $t=1$, and radius 1. Let $\varphi(\infty)$ be zero and the order g of $\varphi(t)$ on the circumference of the circle (1,1) be different from $+\infty$. Then in the series

$$\varphi(t)=\sum_{0}^{\infty}{}_n \frac{g_n}{t^{n+1}}, \quad \ldots \ldots \ldots \quad (1)$$

the *characteristic* $k=\overline{\lim\limits_{n=\infty}}\left[\log|g_n| : \log n\right]$ of the coefficients g_n is also different from $+\infty$, in virtue of the known relation $k=g-1$. If $g<0$, then the integral

$$\omega(x)=\frac{1}{2\pi i}\int_{(1,1)}\varphi(t)\, t^{x-1}\, dt \quad \ldots \ldots \ldots \quad (2)$$

taken along the circumference of the circle (1,1) *exists* for $R(x)>0$, because in that case the series (1) converges along that circumference; the value of t^{x-1} in it is so defined that the argument of t lies continually between $-\frac{\pi}{2}+\delta$ and $\frac{\pi}{2}-\delta$, δ being a positive quantity with zero limit. The function $\omega(x)$ is called *coefficientfunction* by PINCHERLE[1]), owing to the relation $g_n=\omega(n+1)$. Conversely $\varphi(t)$ is called the *generating* function of $\omega(x)$. PINCHERLE considers the relation between these functions especially from the point of view of the *functional calculus*. If we write

$$\omega(x)=I\left[\varphi(t)\right] \quad \ldots \ldots \ldots \quad (3)$$

I is an *additive functional operation*, which satisfies a certain number of simple functional relations; these relations may be used in order to define the coefficient-function in those cases in which the integral (2) does not exist. Thus we find easily

$$I\left[t\,\varphi(t)\right]=\omega(x+1), \text{ and } I\,\varphi'(t)=-(x-1)\,\omega(x-1). \quad . . . \quad (4)$$

and by combining these two equations

$$I\left[\varphi'(t)\right]=-(x-1)\, I\left[t^{-1}\varphi(t)\right],$$

1) *Sur les Fonctions Déterminantes*, Ann. de l'Ec. Norm. (22) 1905, (Ch. IV).

which, by iteration, passes into

$$I\,[\varphi^{(r)}(t)] = (-1)^r \frac{\Gamma(x)}{\Gamma(x-r)}\, I\,[t^{-r}\,\varphi(t)]$$

It is easy to see that the latter equality is also valid for *negative* integral values of r. It is, however, remarkable that the same equality holds for not-integral values of r. This property I have made use of in the investigation of a function represented[1]) by a binomial series $\Sigma\, c_n \binom{x-1}{n}$, the most typical series in which a coefficientfunction can be expanded. The object of this note is to give a proof of the general validity of the equality in question.

2. We substitute $-\alpha$ for r and replace $\varphi(t)$ by the expression $\varphi(t):(t-1)^{\alpha}$, in order to have always to deal with functions which are regular for $t=\infty$. In accordance with RIEMANN's[2]) definition of the derivative of negative order $-\alpha$ of a function we assume as such the following one

$$(-1)^{\alpha} D^{-\alpha} \frac{\varphi(t)}{(t-1)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} \frac{(u-t)^{\alpha-1}\,\varphi(u)}{(u-1)^{\alpha}}\, du \quad . \; . \; . \quad (5)$$

In this we take as path of integration the half-line beginning at $u=t$, whose prolonged part passes through $u=1$; and we assign the same arguments, lying between $-\pi+\delta$ and $+\pi-\delta$, to $u-1$ and $u-t$. Then the so-defined derivative is also regular without the circle $(1,1)$ and zero for $t=\infty$; and by the substitution $u-1=(t-1):s$ we find after a slight reduction the expansion

$$(-1)^{\alpha} D^{-\alpha} \frac{\varphi(t)}{(t-1)^{\alpha}} = \sum_{0}^{\infty}{}_n \frac{\Gamma(n+1)g_n}{\Gamma(n+1+\alpha)(t-1)^{n+1}}, \quad . \; . \; . \quad (6)$$

which, therefore, is related in a simple manner to the expansion for $\varphi(t)$ itself. The order of this derivative, as will appear immediately from this series, is α less than that of $\varphi(t)$, and therefore negative together with the latter order. On this supposition we may apply the operation I to it in the form (1), so that by this the *existence* of the first member of the equation to be proved, viz.

$$I\left[(-1)^{\alpha} D^{-\alpha} \frac{\varphi(t)}{(t-1)^{\alpha}}\right] = \frac{\Gamma(x)}{\Gamma(x+\alpha)}\, I\left[\frac{t^{\alpha}\varphi(t)}{(t-1)^{\alpha}}\right] \quad . \; . \; . \quad (7)$$

[1]) These Proceedings Vol. XXII, N⁰. 1. Nieuw Archief v. Wisk. XIII, 2e stuk (1920).

[2]) See BOREL, *Lecons sur les séries à termes positifs*, p. 74. The constant a occurring there is taken equal to ∞ here, in connection with the regularity of $\varphi(t)$ for $t=\infty$.

has been laid down. Passing to the second member we write

$$\frac{\Gamma(x)}{\Gamma(x+\alpha)}=\frac{1}{\Gamma(\alpha)}\int\limits_0^1 s^{x-1}(1-s)^{\alpha-1}\,ds,$$

where the argument of s and of $1-s$ is zero. Further

$$I\left[\frac{t^{\alpha}\varphi(t)}{(t-1)^{\alpha}}\right]=\frac{1}{2\pi i}\int\limits_{(1,1)}\frac{t^{\alpha+x-1}\varphi(t)}{(t-1)^{\alpha}}\,dt,$$

so that the second member in question is equal to

$$\frac{1}{2\pi i\Gamma(\alpha)}\int\limits_{(1,1)}\frac{t^{\alpha+x-1}\varphi(t)}{(t-1)^{\alpha}}\left[\int\limits_0^1 s^{x-1}(1-s)^{\alpha-1}\,ds\right]dt.$$

If we substitute in the second integral $s=u:t$, this expression passes into

$$\frac{1}{2\pi i\Gamma(\alpha)}\int\limits_{(1,1)}\frac{\varphi(t)}{(t-1)^{\alpha}}\left[\int\limits_0^t u^{x-1}(t-u)^{\alpha-1}\,du\right]dt.$$

Since the argument of s was zero, the argument of u is equal to that of t; thus the variable under the sign of integration goes along the *straight* line from $u=0$ to $u=t$ in the u-plane. But it may go as well from $u=-i\delta$, on the circumference of the circle (1, 1), along that circumference in the positive direction to the point $u=t$. On this supposition we consider the system of two integrations, to be performed in succession, as a double integral. Then in the corresponding aggregate (u, t) a definite value u_1 of u has to be associated with all those values of t lying, in the t-plane, on the circumference of the circle (1, 1) between $t=u_1$ and the end-point $t=+i\delta$ of that circle. Hence the double integral may be replaced by the pair of two successive integrations denoted in the expression

$$\frac{1}{2\pi i\Gamma(\alpha)}\int\limits_{(1,1)}u^{x-1}\left[\int\limits_u^0\frac{(t-u)^{\alpha-1}\varphi(t)}{(t-1)^{\alpha}}\,dt\right]du,$$

where the integration according to t has to be performed in the *positive* direction from $t=u$ to $t=+i\delta$ [1]). On account of the properties of $\varphi(t)$ the latter integration may be replaced by an integration

[1]) The here given argument is strong, in so far it is based upon *known* truths, if the functions under consideration are *finite* in the whole domain of integration. This is the case for $R(x)>1$ and $R(\alpha)>1$, but, since both *endforms* are *analytic* functions of x and of α for $R(x)>0$ and $R(\alpha)>0$, they must also be equal for the latter values.

from $t = u$ to $t = \infty$, and one from $t = \infty$ to $t = 0$. The latter gives an amount which is *independent* of u, and this amount gives zero for the *final* integration. Therefore, after changing the letters u and t, we may write for the preceding expression

$$\frac{1}{2\pi i}\int\limits_{(1,1)} t^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}\int\limits_{t}^{\infty}\frac{(u-t)^{\alpha-1}\varphi(u)}{(u-1)^{\alpha}}\,du\right]dt,$$

and this, if we take (2) and (5) into account, is just equal to the first member of (7). The latter equation thus has been proved in case $g < 0$.

If $g \geqq 0$ and, to begin with, $0 \leqq g < 1$, PINCHERLE defines the coefficientfunction of $\varphi(t)$ by means of an auxiliary function

$$\varphi_1(t) = -D^{-1}\frac{\varphi(t)}{t-1} = \sum_{0}^{\infty}{}_n \frac{g_n}{(n+1)(t-1)^{n+1}} \quad . \quad . \quad . \quad (8)$$

The order of $\varphi_1(t)$ is lower by unity than that of $\varphi(t)$ and thus negative, so that $\omega_1(x) = I\varphi_1(t)$ is defined by (2). By (4) we have

$$I\varphi(t) = -I(t-1)\varphi'_1(t) = xI\varphi_1(t) - \theta^{-1}xI\varphi_1(t), \quad . \quad . \quad (9)$$

if θ be the operation defined by $\theta f(x) = f(x+1)$. If we denote by $\varphi_\alpha(t)$ the result of the operation $(\alpha) = (-1)^\alpha D^{-\alpha}\frac{1}{(t-1)^\alpha}$ applied to $\varphi(t)$, and by $\varphi_{1\alpha}(t)$, the result of the operation $(1) = -D^{-1}\frac{1}{t-1}$, applied to $\varphi_\alpha(t)$, we derive from the preceding equation

$$I\varphi_\alpha = xI\varphi_{1\alpha} - \theta^{-1}[xI\varphi_{1\alpha}] \; . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now, the operation (α) is commutative, as will appear in the simplest manner from the expansion (6). Hence $\varphi_{1\alpha} = \varphi_{\alpha 1}$, if by the latter expression the result is denoted, which is obtained, if first the operation (1) and then (α) is applied. But $\varphi_1(t)$ is a function for which the equality (7) has already been proved; if this is taken into account, we may infer from (10), using the identity $\Gamma(y+1) = y\,\Gamma(y)$,

$$I\varphi_a = \frac{\Gamma(x)}{\Gamma(x+\alpha)}\left[xI\frac{t^\alpha\varphi_1(t)}{(t-1)^\alpha} - (x+\alpha-1)\,I\frac{t^{\alpha-1}\varphi_1(t)}{(t-1)^\alpha}\right]. \quad . \quad (11)$$

Using again the relation (9) we may write

$$x\,I\frac{t^\alpha\varphi_1}{(t-1)^\alpha} - (x-1)\,I\frac{t^{\alpha-1}\varphi_1}{(t-1)^\alpha} = -I\left[(t-1)\,D\frac{t^\alpha\varphi_1}{(t-1)^\alpha}\right] \quad . \quad (12)$$

Further, in connection with (8)

$$-(t-1)\,D\frac{t^{\alpha}\varphi_1}{(t-1)^{\alpha}}=\frac{t^{\alpha}\varphi}{(t-1)^{\alpha}}+\frac{\alpha t^{\alpha-1}\varphi_1}{(t-1)^{\alpha}} \quad . \; . \; . \; . \quad (13)$$

By means of (12) and (13) the equation (11) finally passes into the required identity (7).

The above argument may further be used to prove the same identity for $h \overline{\lessdot} g < h+1$, if it has already been established for $g < h$, h being a positive integral number. The *general* validity has therefore been proved.
