Mathematics. — "*On* LAMBERT's *series*". By Prof. J. C. KLUYVER.

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Transforming LAMBERT's series

$$L(z) = \sum_{n=1}^{n=\infty}{}' \frac{z^n}{1-z^n} = \sum_{n=1}^{n=\infty}{}' t(n)\, z^n$$

SCHLÖMILCH [1]) deduced the asymptotic expansion

$$L(z) = \frac{C - \log\log\frac{1}{z}}{\log\frac{1}{z}} + \tfrac{1}{4} - \frac{B_1^{\,2}}{2\,.\,2!}\left(\log\frac{1}{z}\right) - \frac{B_2^{\,2}}{4\,.\,4!}\left(\log\frac{1}{z}\right)^3 - \ldots\ldots$$

suitable to calculate the value of $L(z)$ for real values of z approaching $+1$. WIGERT [2]), slightly changing the formula, obtained a somewhat more general expansion, which can be utilised, when z, tending to $+1$, takes complex values, and LANDAU [3]) has simplified the proof of WIGERT's result. HANSEN [4]) has shown that the circle $|z| = 1$ is a natural limit of $L(z)$ and in his lectures LANDAU established the same result in a simple and direct way. LANDAU's proof is given in a paper by KNOPP [5]), who in that paper, and also in his dissertation [6]), discussed series of the more general type

$$N(z) = \sum_{n-1}^{n=\infty} b_n \frac{z^n}{1-z^n}.$$

Assuming the coefficients b_n to fulfill certain restricting conditions, he could establish several cases in which the continuation of the function $N(z)$ beyond the circle of convergence is impossible. In the present paper I propose to deduce a new asymptotic expansion

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1) Ueber die Lambertsche Reihe. Zeitschr. f. Math. u. Phys., Bd 6, 1861, p. 407.

2) Sur la série de Lambert et son application à la théorie des nombres. Acta Math., XLI, 1918, p. 197.

3) Ueber die Wigertsche asymptotische Funktionalgleichung für die Lambertsche Reihe. Archiv der Math. u. Phys., III. Reihe, XXVII, 1918, p. 141.

4) Démonstration de l'impossibilité du prolongement analytique de la série de Lambert et des séries analogues. Kong. Danske Vidensk. Selskabs Forth., 1907, p. 3.

5) Ueber Lambertsche Reihen. Journal f. d. reine u. ang. Math., Bd. **142**, 1913, p. 283.

6) Grenzwerte von Reihen bei der Annäherung an die **Konvergenzgrenze.** Berlin, 1907.

applying to the supposition that z, moving along a radius of the circle $|z| = 1$, approaches a given rational point on the circumference. Moreover the investigation will serve to add some results to those of KNOPP concerning the function $N(z)$.

1. Supposing p and q to be integers prime to each other, $e^{\frac{2\pi i p}{q}} = \theta^p$ is a rational point of order q on the circle $|z| = 1$, and taking $0 < x < 1$ we have

$$N(x\theta^p) = \sum_{n=1}^{n=\infty} b_{nq} \frac{x^{nq}}{1 - x^{nq}} + \sum_{h=1}^{h=q-1} \sum_{n=0}^{n=\infty} b_{nq+h} \frac{x^{nq+h}\,\theta^{hp}}{1 - x^{nq+h}\,\theta^{hp}}.$$

Now obviously we have

$$\frac{q x^{nq^2}}{1 - x^{nq^2}} = \frac{x^{nq}}{1 - x^{nq}} + \sum_{h=1}^{h=q-1} \frac{x^{nq}\,\theta^{hp}}{1 - x^{nq}\,\theta^{hp}}$$

and

$$\sum_{h=1}^{h=q-1} \frac{\theta^{hp}}{1 - \theta^{hp}} = -\tfrac{1}{2}(q-1),$$

hence we get

$$N(x\theta^p) - q \sum_{n=1}^{n=\infty} b_{nq} \frac{x^{nq^2}}{1 - x^{nq^2}} + \frac{b_0}{2}(q-1) = \sum_{h=1}^{h=q-1} U_h(x,p), \quad . \quad . \quad (1)$$

where the coefficient b_0 is arbitrarily chosen and where

$$U_h(x,p) = \sum_{n=0}^{n=\infty} \left\{ b_{nq+h} \frac{x^{nq+h}\,\theta^{hp}}{1 - x^{nq+h}\,\theta^{hp}} - b_{nq} \frac{x^{nq}\,\theta^{hp}}{1 - x^{nq}\,\theta^{hp}} \right\} \quad . \quad . \quad (2)$$

The right handside can be transformed, we may write also

$$\left. \begin{aligned} U_h(x,p) = &- (1 - x^h) \sum_{n=0}^{n=\infty} b_{nq+h} \frac{x^{nq}\,\theta^{hp}}{(1 - x^{nq+h}\,\theta^{hp})(1 - x^{nq}\,\theta^{hp})} + \\ &+ \sum_{n=0}^{n=\infty} (b_{nq+h} - b_{nq}) \frac{x^{nq}\,\theta^{hp}}{1 - x^{nq}\,\theta^{hp}}. \end{aligned} \right\} \quad (3)$$

In particular, taking $b_n = 1$ also $b_0 = 1$ we find for the function of LAMBERT

$$L(x\theta^p) - q\,L(x^{q^2}) + \tfrac{1}{2}(q-1) = \sum_{h=1}^{h=q-1} T_h(x,p), \quad . \quad . \quad . \quad (4)$$

where

$$\left. \begin{aligned} T_h(x,p) &= \sum_{n=0}^{n=\infty} \left\{ \frac{x^{nq+h}\,\theta^{hp}}{1 - x^{nq+h}\,\theta^{hp}} - \frac{x^{nq}\,\theta^{hp}}{1 - x^{nq}\,\theta^{hp}} \right\} = \\ &= -(1 - x^h) \sum_{n=0}^{n=\infty} \frac{x^{nq}\,\theta^{hp}}{(1 - x^{nq+h}\,\theta^{hp})(1 - x^{nq}\,\theta^{hp})} \end{aligned} \right\} \quad . \quad . \quad (5)$$

Now we may observe that if $0 \leq x \leq 1$, the moduli of the factors

$(1-x^{nq+h}\,\theta^{hp})$ and $(1-x^{nq}\,\theta^{hp})$ always exceed a fixed value, hence $T_h(x,p)$ remains finite as x tends to unity. Accordingly the difference

$$L(x\,\theta^p)-qL(x^{q^2})$$

is always finite, therefore the point θ^p must be a singular point of $L(z)$ and the continuation of $L(z)$ across the circumference must be regarded impossible.

To examine the behaviour of $L(x\theta^p)$ if $x \to 1$, it will be sufficient to deduce a suitable expansion of the function $T_h(x,p)$.

Putting

$$x=e^{-y} \quad , \quad \theta^p=e^{i\beta},$$

$$\varphi(u)=\frac{1}{e^{u-hi\beta}-1}=-\tfrac{1}{2}+\sum_{k=-\infty}^{k=+\infty}\frac{1}{u-hi\beta+2\pi ik},$$

we have at once

$$T_h(x,p)=\sum_{n=0}^{n=\infty}\{\varphi(nqy+hy)-\varphi(nqy)\},$$

and from this expression it is seen, that the application of a suitable summation-formula will lead to the desired result.

I consider the set of trigonometrical series

$$g_1(t)=-\frac{1}{\pi}\sum_1^\infty\frac{\sin 2\pi kt}{k},$$

$$g_2(t)=+\frac{1}{2\pi^2}\sum_1^\infty\frac{\cos 2\pi kt}{k^2},$$

$$g_3(t)=+\frac{1}{2^2\pi^3}\sum_1^\infty\frac{\sin 2\pi kt}{k^3},$$

$$\cdots\cdots\cdots\cdots$$

and the identity

$$0=\int_0^\infty\left\{g'_1\left(t-\frac{h}{q}\right)-g'_1(t)\right\}\varphi(tqy)\,dt \quad \ldots\ldots \quad (6)$$

Integration by parts transforms the indefinite integral into the expression

$$\left\{g_1\left(t-\frac{h}{q}\right)-g_1(t)\right\}\varphi(tqy)+\sum_{k=1}^{k=2m-1}(-1)^k q^k y^k\left\{g_{k+1}\left(t-\frac{h}{q}\right)-g_{k+1}(t)\right\}\varphi^{(k)}(tqy)+$$

$$+q^{2m}y^{2m}\int\left\{g_{2m}\left(t-\frac{h}{q}\right)-g_{2m}(t)\right\}\varphi^{(2m)}(tqy)\,dt$$

and here we have to introduce the limits 0 and ∞. In doing so we must take into account the discontinuities of $g_1\left(t-\frac{h}{q}\right)$ and of

$g_1(t)$. Further, we may notice that $\varphi(tqy)$ and $\varphi^{(k)}(tqy)$ vanish when $t\rightarrow\infty$, and that

$$g_{k+1}\left(-\frac{h}{q}\right)-g_{k+1}(0)=(-1)^{k+1}f_k\left(\frac{h}{q}\right)$$

where f_k denotes the Bernoullian polynomial of order k.

In this way the equation (6) leads to

$$\sum_{n=0}^{n=\infty}\left\{\varphi(nqy+hy)-\varphi(nqy)\right\}=T_h(x,p)=-\sum_{k=0}^{k=2m-1}q^ky^kf_k\left(\frac{h}{q}\right)\varphi^{(k)}(0)+R.$$

In this equation we have

$$\varphi\ \ (0)=\frac{i}{2}\cot\frac{\pi hp}{q}-\tfrac{1}{2},$$

$$\varphi^{(k)}(0)=\left(\frac{i}{2}\right)^{k+1}(D^{(k)}\cot v)_{v=\frac{\pi hp}{q}},$$

and the remainder integral R is given by

$$R=-q^{2m}y^{2m}\int_0^\infty\left\{g_{2m}\left(t-\frac{h}{q}\right)-g_{2m}(t)\right\}\varphi^{(2m)}(tqy)\,dt.$$

Now since

$$|\varphi^{(2m)}(tqy)|<2m!\sum_{k=-\infty}^{k=+\infty}\frac{1}{\{t^2q^2y^2+(h\beta-2\pi k)^2\}^{m+\frac{1}{2}}}<$$

$$<2m!\sum_{k=-\infty}^{k=+\infty}\frac{1}{t^2q^2y^2+(h\beta-2\pi k)^2}\ \frac{1}{|h\beta-\beta\pi k|^{2m-1}},$$

we infer that

$$|R|<2|g_{2m}(0)|\ 2m!\,q^{2m-1}y^{2m-1}\cdot\frac{\pi}{2}\sum_{k=-\infty}^{k=+\infty}\frac{1}{(h\beta-2\pi k)^{2m}},$$

or that

$$|R|<2m\pi\ |g_{2m}(0)|\ q^{2m-1}y^{2m-1}\ |\varphi^{(2m-1)}(0)|.$$

Thus it is shown that $|R|$ is less than a finite multiple of the modulus of the last term in the sum that precedes, and we have found the asymptotic expansion

$$T_h(x,p)=\frac{h}{2q}-\sum_{k=0}^{k=2m-2}\left(\frac{i}{2}\right)^{k+1}q^k\left(\log\frac{1}{x}\right)^kf_k\left(\frac{h}{q}\right)(D^{(k)}\cot v)_{v=\frac{\pi hp}{q}}+$$

$$+Kq^{2m-1}\left(\log\frac{1}{x}\right)^{2m-1},$$

where the value of K is finite and independent of x.

Putting

$$A_k=\sum_{h=1}^{h=q-1}f_k\left(\frac{h}{q}\right)(D^{(k)}\cot v)_{v=\frac{\pi hp}{q}},$$

we have finally

$$\left.\begin{array}{l} L(x\theta^p)-qL(x^{q^2}) = -\tfrac{1}{4}(q-1) - \sum\limits_{k=0}^{k=2m-2} A_k \left(\dfrac{i}{2}\right)^{k+1} q^k \left(\log\dfrac{1}{x}\right)^k + \\ + K_1 \cdot q^{2m-1}\left(\log\dfrac{1}{x}\right)^{2m-1}, \end{array}\right\} \quad (7)$$

and again K_1 is finite and independent of x.

From (7) we conclude that, if $x \to 1$, the function $L(x\theta^p)$ tends to infinity in the same manner as

$$qL(x^{q^2}) \backsim \frac{C - \log\log\dfrac{1}{x} - 2\log q}{q\log\dfrac{1}{x}} + \tfrac{1}{4}q,$$

and that

$$\lim_{x\to 1}\{L(x\theta^p) - qL(x^{q^2})\} = -\tfrac{1}{4}(q-1) - \frac{i}{2q}\sum_{h=1}^{h=q-1} h\cot\frac{\pi hp}{q}. \quad (8)$$

Thus we have the rather remarkable result that it is only the real part of $L(x\theta^p)$ that increases indefinitely in a manner quite independent of p

2. If we only wish to shew that, when $x \to 1$, the function $L(x\theta^k)$ cannot remain finite for all non zero values of k, an elementary discussion of the sum $\sum\limits_{k=0}^{k=q-1} L(x\theta^k)$ suffices to obtain this result.

We have at once

$$\sum_{k=0}^{k=q-1} L(x\theta^k) = q\sum_{n=1}^{n=\infty} t(nq)\, x^{nq},$$

and, denoting by D the greatest common measure of n and q, we may substitute

$$t(nq) = \sum_{d/D} \mu(d)\, t\left(\frac{n}{d}\right) t\left(\frac{q}{d}\right),$$

thus obtaining

$$\sum_{k=0}^{k=q-1} L(x\theta^k) = q\sum_{d/q}\mu(d)\, t\left(\frac{q}{d}\right) L(x^{qd}).$$

Hence, making x tend to 1, we get

$$\lim_{x\to 1}\frac{1}{L(x)}\sum_{k=0}^{k=q-1} L(x\theta^k) = \sum_{d/q}\frac{\varphi(d)}{d},$$

where $\varphi(d)$ denotes the number of integers less than d and prime to d.

Supposing q to be equal to the product $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, we have

$$\sum_{d/q} \frac{\varphi(d)}{d} = \prod_{h=1}^{h=s} \left(1 + \alpha_h \frac{p_h - 1}{p_h}\right) > 1,$$

and therefore, if $x \to 1$, for at least one value of k other than zero $| L(x\theta^k) |$ must tend to infinity.

3. It will be readily seen that the method used in obtaining the asymptotic expansion of $L(z)$ can be applied also to the series

$$N(z) = \sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n},$$

if only the coefficient b_n is a simple analytical function of the index n. If we choose, for instance,

$$b_n = -\frac{1}{n},$$

we have

$$M(z) = -\sum_{n=1}^{n=\infty} \frac{1}{n} \cdot \frac{z^n}{1-z^n} = \sum_{n=1}^{n=\infty} \log(1-z^n) = \log \prod_1^{\infty} (1-z^n)$$

and we can put

$$M(x\,\theta^p) - M(x^{q^2}) = \log q + \sum_{h=1}^{h=q-1} V_h(x, p),$$

where

$$V_h(x, p) = \sum_{n=0}^{n=\infty} \{\log(1 - x^{nq+h}\theta^{hp}) - \log(1 - x^{nq}\theta^{hp})\}.$$

Operating as before we shall find

$$M(x\,\theta^p) - M(x^{q^2}) = \log q - \frac{1}{q} \sum_{h=1}^{h=q-1} \left\{ h \log\left| 2 \sin \frac{\pi h p}{q} \right| + \pi i h g_1\left(\frac{hp}{q}\right) \right\} -$$

$$- \frac{1}{24}(q^2 - 1) \log \frac{1}{x} - \sum_{k=1}^{k=2m-1} A'_k \left(\frac{i}{2}\right)^k q^k \left(\log \frac{1}{x}\right)^k + K_2\, q^{2m} \left(\log \frac{1}{x}\right)^{2m}$$

where

$$A'_k = \sum_{h=1}^{h=q-1} f_k\left(\frac{h}{q}\right) \left(D^{(k-1)} \cot v\right)_{v = \frac{\pi h p}{q}},$$

and K_2 has a finite value independent of x.

Now for EULER's product $\prod_1^{\infty}(1-x^n)$ we have, when $0 \leqq x < 1$),

$$M(x) = +\frac{1}{24} \log \frac{1}{x} + \tfrac{1}{2} \log 2\pi - \tfrac{1}{2} \log \log \frac{1}{x} - \frac{\pi^2}{6 \log \frac{1}{x}} + M\left(e^{-\frac{4\pi^2}{\log \frac{1}{x}}}\right),$$

hence equation (9) shews completely the behaviour of $M(x\,\theta^p)$, when $x \to 1$.

In fact, we may write

$$\lim_{x\to 1}\{M(x\theta^p)-M(x^{q^2})\} = \log q - \frac{1}{q}\sum_{h=1}^{h=q-1}\left\{h\log\left|2\sin\frac{\pi hp}{q}\right| + \pi i h\, g_1\left(\frac{hp}{q}\right)\right\} \quad (10)$$

and again we may notice that only the real part of $M(x\,\theta^p)$ becomes infinite, the imaginary part tending to a finite limit.

Taking, for instance, $p=1$, $q=2$, we shall find

$$\lim_{x\to 1}\{M(-x) - M(x^4)\} = \tfrac{1}{2}\log 2,$$

or

$$\lim_{x\to 1}\frac{\prod_1^\infty (1+x^{2n-1})}{\prod_1^\infty (1+x^{2n})} = \sqrt{2},$$

a known result in the theory of the ϑ-functions.

4. Finally, I will state that the discussion of the fundamental equations (1), (2) and (3) furnishes the proof that the function

$$N(z) = \sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n}$$

cannot be continued beyond the circle $|z|=1$ in each of the following cases:

I. $A > b_n > B > 0$.

In this case we shall have

$$\frac{A}{q} > \lim_{x\to 1}\frac{1-x}{\log\frac{1}{1-x}} N(x\,\theta^p) > \frac{B}{q}.$$

II. $\lim_{n\to\infty} b_n = A \neq 0$.

Now it will be seen that

$$\lim_{x\to 1}\frac{1-x}{\log\frac{1}{1-x}} N(x\,\theta^p) = \frac{A}{q}.$$

III. $\lim_{n\to\infty}\frac{b_n}{n^s} = A \neq 0, \quad s > 0.$

In this case the equation holds

$$\lim_{x\to 1}(1-x)^{1+s} N(x\theta^p) = \frac{A}{q^{1+s}}\,\Gamma(1+s)\,\zeta(1+s).$$

IV. $\lim_{n \to \infty} b_n = 0$ and either $b_n > 0$, or the series $\sum_{n=1}^{n=\infty} \frac{b_{nq}}{nq}$ converges for an unlimited number of integers q and the sum of this series is different from zero. In the latter supposition we have

$$\lim_{x \to 1} (1-x)\, N(x\,\theta^p) = \sum_{n=1}^{n=\infty} \frac{b_{nq}}{nq}. \text{ [1]}$$

V. $b_n \geqq 0$ and moreover

$$\lim_{n \to \infty} \frac{1}{n} \left[b_{q+h} + b_{2q+h} + \ldots + b_{nq+h}\right] = 0$$
$$(h = 0, 1, 2, \ldots, q-1)$$

for an unlimited number of integers q.

[1]) By a totally different method FRANCK deduced this formula in his paper: Sur la théorie des séries. Math Annalen, Bd. 52, 1899, p. 529.