Citation:

J. de Vries, Involutions in a field of circles, in:
KNAW, Proceedings, 22 I, 1919-1920, Amsterdam, 1919, pp. 379-382

Mathematics. — "*Involutions in a field of circles*". By Prof. JAN DE VRIES.

(Communicated in the meeting of September 27, 1919).

1. In a plane are given three systems of coaxial circles (α), (β), (γ) in each of which the circles are arranged in the pairs α_1, α_2 etc. of an involution. Let δ_1 be the circle which intersects the circles $\alpha_1, \beta_1, \gamma_1$ orthogonally, δ_2 the orthogonal circle of the corresponding circles $\alpha_2, \beta_2, \gamma_2$, then δ_1 and δ_2 are conjugated in an *involutory correspondence* in the field of circles.

Since α_1 coincides twice with α_2, β_1 twice with β_2 and γ_1 twice with γ_2, the involution (δ_1, δ_2) has *eight coincidences*.

In general an arbitrary circle δ_1 is intersected orthogonally by one circle α only. However, when δ_1 belongs to the system (α') of coaxial circles orthogonally intersecting the circles of (α) then α_1, and α_2 also, is an arbitrary circle from (α), whilst β_2 and γ_2 are perfectly defined. In this case every circle δ_2 intersecting β_2 and γ_2 orthogonally corresponds with δ_1.

Hence the *orthogonal systems* (α'), (β'), (γ') of (α), (β), (γ) consist of *singular circles*, i.e. of circles which in the involution are conjugated each to an infinite number of circles.

There is still another way in which δ_1 may be singular. On a circle α the systems (β) and (γ) determine two involutions; since these have one pair in common, on α are to be found the two points of intersection of a circle β with a circle γ. Hence every circle α (or β, or γ) belongs to a triplet $\alpha_2, \beta_2, \gamma_2$, belonging to one system of circles and for which the orthogonal circle accordingly becomes indefinite. The circle δ_1 which intersects the corresponding circles $\alpha_1, \beta_1, \gamma_1$ orthogonally is therefore *singular* and conjugated to every circle of a certain system of coaxial circles.

2. A further investigation of the involution (δ_1, δ_2) becomes comparatively simple, when we make use of a representation of the circles of the field on the points of space, to which Dr. K. W. WALSTRA has attracted attention in 1917 [1]).

In order to obtain this representation we take the plane of our circles as the plane of coordinates $z = 0$. A circle we then represent

[1]) These Proceedings XIX, p. 1130.

by the point on its axis with coordinate z equal to the power of the origin O with respect to the circle

All circles with radius zero are represented by the points of a paraboloid of revolution **G** (limiting surface) and the images of two orthogonal circles are harmonically separated by **G**. Two reciprocal polar lines are the images of two systems of coaxial circles orthogonal to each other.

The systems $(\alpha), (\beta), (\gamma)$ are represented by three involutions (A_1, A_2),. (B_1, B_2), (C_1, C_2) situated on three straight lines a, b, c. The image D_1 of the circle σ_1, which intersects $\alpha_1, \beta_1, \gamma_1$ orthogonally is the pole of the plane $A_1 B_1 C_1$. So we have now to consider an involution (D_1, D_2) of the points of space, which involution is characterized by the property that the polar planes Δ_1 and Δ_2 of D_1 and D_2 meet the given lines a, b, c in the pairs (A_1, A_2), (B_1, B_2), (C_1, C_2) of three given involutions.

3. It is now easy to find the *singular* elements of the involution of circles again. In the first place we observe that A_1 becomes indefinite as soon as Δ_1 passes through a; for Δ_2 now any plane may be chosen which contains the points B_2 and C_2, hence for D_2 any point of the polar line a'_2 of the straight line $a_2 \equiv B_2 C_2$. If Δ_1 is made to revolve about a, then D_1 moves along the polar line a' of a, and a_2 describes a ruled quadric. The line a'_2 also describes a ruled quadric $(a'_2)^2$ of which the polar lines b' and c' of b and c are directrices. It is obvious that to every point of a' a definite straight line of $(a'_2)^2$ is conjugated. Similarly to the *singular lines* b', c' correspond the ruled quadrics $(b'_2)^2, (c'_2)^2$.

Secondly D_2 becomes indefinite as soon as A_2, B_2 and C_2 are collinear and therefore situated on a transversal s of a, b, c. When s is made to coincide successively with the generators of the ruled quadric having a, b, c for directrices, then A_1, B_1 and C_1 describe three projective ranges, so that Δ_1 osculates a twisted cubic σ^3, of which the lines a', b' and c' are bisecants. To every point $S \equiv D_1$ of this *singular curve* σ^3 evidently is correlated a line s' *viz.* the polar line of the corresponding line s. The lines s' form a ruled quadric $(s')^2$ with the directrices a', b', c'.

4. If D_1 describes the line l, then Δ_1 revolves about the polar line l', so that A_1, B_1 and C_1 describe projective ranges. A_2, B_2 and C_2 then also describe projective ranges; hence Δ_2 osculates a twisted cubic λ^3, of which a', b' and c' are bisecants. Consequently D_1 and D_2 are conjugated in a *cubic correspondence.*

Since l has two points in common with $(s')^2$, λ^3 rests on σ^3 in two points. The rays of space are in this way transformed into the fourfold infinity of twisted cubics, which intersect each of the lines a', b', c' and the curve σ^3 twice.

A plane Φ is transformed into a cubic surface passing through a', b', c' and σ^3. The images of two planes have these four lines and the image λ^3 of their line of intersection in common.

5. A tangent plane of the limiting surface **G** is the image of the circles which pass through a given point. The involution (σ_1, σ_2) therefore (by § 4) has the following property: *A system of coaxial circles is transformed into a class of circles with index three.*

This class contains *six circles with radius zero* and *three straight lines.* The *singular circles* form *three coaxial systems* (§ 1) and a class with *index* three (§ 3).

To each singular circle a system of coaxial circles is conjugated; these systems form four classes.

The image of a system of coaxial circles contains *eight* singular circles.

6. Evidently the representation of the field of circles on the points of space enables us to deduce from each involution in the latter an involution in the field of circles and vice versa.

A particularly simple involution is obtained as follows. On every ray h which meets OZ at right angles the paraboloid **G** determines an involution of conjugated pairs (P, P'). In the field of circles the analogon hereof is the correspondence which conjugates to each other two circles intersecting orthogonally and having the same power with respect to a fixed point O.

The point P', conjugated to P, is the intersection of the ray h with the polar plane π of P. If P lies on OZ, then for h may be taken any perpendicular to OZ passing through P. Since π now is perpendicular to OZ, to P will be conjugated every point of the line of infinity of $z = 0$.

A point of **G** lies in its own polar plane and therefore constitutes a *coincidence* of the correspondence. When P reaches the vertex of **G** or the point at infinity of OZ, then P' is an arbitrary point of $z = 0$ or of $z = \infty$.

If P moves along a line l, then h describes a ruled quadric ϱ^2 and π a pencil of planes projective with ϱ^2, so the locus of P' is a twisted cubic λ^3. The polar line l' of l meets ϱ^2 in two points P'; each plane through l' contains besides these two points still another point P' not lying on l'. Hence l' is a chord of λ^3. So is l, for its points of intersection with **G** are coincidences.

7. To the points P of a plane Ψ correspond the points P' of a cubic surface Ψ^3. Two such surfaces in the first place have the

curve λ^3 in common, which is the image of the line of intersection of the two corresponding planes. In order to obtain a proper insight into the meaning of the figure which they have in common in addition to this, we observe that the involution (P,P') is a particular case of the following correspondence.

Let a quadric surface Φ^2 be given and the pair of polar lines d,d'. Through a point P the straight line t is drawn which meets d and d'; the polar plane π of P defines on t the point P', which we conjugate to P.

The points of intersection of d and Φ^2 we denote by E_1, E_2, those of d' and Φ^2 by E'_1, E'_2. The straight line $E_1\,E'_1$ lies in Φ^2; to each of its points P evidently is conjugated any of its points. To each point of d corresponds every point of d'. Thus all the edges of the tetrahedron $E_1\,E_2\,E'_1\,E'_2$ are singular, so that these six lines are conjugated to their points of transit through a plane Ψ. In addition to the curve λ^3 two surfaces Ψ^3 then have these six singular lines in common.

If Φ^2 now again is replaced by **G**, then d becomes the axis OZ, d' is the line at infinity of $z=0$ and the other four singular lines are to be found in the imaginary lines along which **G** is intersected by $z=0$ and $z=\infty$.

8. If P is caused to move along a line l, which meets OZ, then h describes a system of parallel lines which is projective to the pencil constituted by the polar line of P with respect to the parabola in the plane through l and OZ. The points P' now are situated on a rectangular hyperbola which by the line at infinity of $z=0$ is completed to a λ^3.

By the correspondence of the orthogonal circles, which is alluded to in § 6 a *system of coaxial circles* is again transformed into a class *with index three*. The circles with radius zero are coincidences. The two circles of a pair are real only if they have a negative power with respect to O. When O lies without a circle, then the conjugated circle has an imaginary radius.