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Mathematics. — *"On a formula of* SYLVESTER". By Prof. W. KAPTEYN.

(Communicated at the meeting of November 29, 1919).

In his paper "On the partition of numbers" Quart. Journ. of Math. I (1857) p. 141—152, SYLVESTER has given a general formula for the number of solutions in integers (zero included) of the equation

$$a_1\, x_1 + a_2\, x_2 + \ldots\ldots + a_r\, x_r = n. \quad \ldots\ldots \quad (1)$$

where n and a are given integers.

Applying this formula, which is given without proof, to a particular example, I found a fractional number. Of course this result is absurd. I therefore tried to construct a proof and found, as will be shown hereafter, that SYLVESTER's formula wants a slight correction.

If the fraction

$$\frac{1}{\Phi(x)} = \frac{1}{(1-x^{a_1})(1-x^{a_2})\,.\,.\,(1-x^{a_r})}. \quad \ldots\ldots \quad (2)$$

is developed in ascending powers of x, it is evident that the coefficient of x^n gives exactly the number of solutions in integers of the equation (1). We therefore proceed to reduce this fraction to its partial fractions and to develop every one of these in ascending powers of x. The denominator being a compound quantity, the first thing wanted is to determine its different factors.

Let $\underline{1-x^m} = 0$ denote the equation containing all the prime roots of the equation $1-x^m = 0$, then we have

$$1-x^m = \prod_{i=1}^{k} \underline{1-x^{d_i}}. \quad \ldots\ldots\ldots \quad (3)$$

where $d_1, d_2, \ldots d_k$ $(d_1 = 1, d_k = m)$ represent the different divisors of m.

To prove this theorem let $m = p^\alpha q^\beta \ldots t^\lambda$, $p, q, \ldots t$ being prime numbers, then the divisors of m are the several terms of the continued product

$$\left(1 + \sum_1^\alpha p^a\right)\left(1 + \sum_1^\beta q^b\right)\,.\,.\,\left(1 + \sum_1^\lambda t^l\right).$$

One of these being

$$d = p^a\, q^b \ldots t$$

we know that the number of prime roots of the corresponding equation $1-x^d = 0$ is

$$p^a q^b \dots t^l \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\dots\left(1-\frac{1}{t}\right)$$

The number of the prime roots corresponding to all the different divisors of m is therefore

$$\left[1+\left(1-\frac{1}{p}\right)\sum_1^{\alpha} p^a\right]\left[1+\left(1-\frac{1}{q}\right)\sum_1^{\beta} q^b\right]\dots$$
$$\dots\left[1+\left(1-\frac{1}{t}\right)\sum_1^{\lambda} t^l\right]=p^\alpha q^\beta \dots t^\lambda = m.$$

Now these prime roots being all different, they must satisfy an equation of degree m, which, because every one of these roots is also a root of $1-x^m=0$, must coincide with $1-x^m=0$.

To illustrate this theorem, put $m=20=2^2 \cdot 5$, then the divisors are

$$1,\quad 2,\quad 4,\quad 5,\quad 10,\quad 20$$

and the factors corresponding to the prime roots

$$1-x,\ 1+x,\ 1+x^2,\ 1+x+x^2+x^4,\ 1-x^2-x^3+x^4,\ 1-x^2+x^4-x^6+x^8$$

or

$$\underline{1-x},\quad \underline{1-x^2},\quad \underline{1-x^4},\quad \underline{1-x^5},\quad \underline{1-x^{10}},\quad 1-\underline{x^{20}}.$$

The continued product of these factors is evidently $1-x^{20}$, or

$$1-x^{20}=\prod_{i=1}^{6}(1-x^{d_i}).$$

Developing in the same way the several factors of $\Phi(x)$, we may write

$$\Phi(x)=\underline{(1-x^{\alpha_1})}^{r_1}\cdot\underline{(1-x^{\alpha_2})}^{r_2}\dots\underline{(1-x^{\alpha_m})}^{r_m}\quad\dots\quad(4)$$

where the quantities α_i, ranged according to ascending magnitude, represent the different divisors of $a_1, a_2 \dots a_r$, and r_i the numbers of the divisors α_i. We may remark here that $\alpha_1=1$ and $r_1=r$.

If, for instance

$$x_1+2x_2+5x_3+10x_4+20x_5=n$$

is the given equation, we have

$$\Phi(x)=(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})$$

where

$$\begin{aligned}
1-x &= \underline{1-x}\\
1-x^2 &= \underline{1-x}\cdot\underline{1-x^2}\\
1-x^5 &= \underline{1-x}\cdot\underline{1-x^5}\\
1-x^{10} &= \underline{1-x}\cdot\underline{1-x^2}\cdot\underline{1-x^5}\cdot\underline{1-x^{10}}\\
1-x^{20} &= \underline{1-x}\cdot\underline{1-x^2}\cdot\underline{1-x^4}\cdot\underline{1-x^5}\cdot\underline{1-x^{10}}\cdot\underline{1-x^{20}}
\end{aligned}$$

hence

$$\Phi(x)=\underline{(1-x)}^5\ \underline{(1-x^2)}^3\ \underline{(1-x^4)}\ \underline{(1-x^5)}^3\ \underline{(1-x^{10})}^2\ \underline{(1-x^{20})}$$

Proceeding now to determine the partial fractions of $\frac{1}{\Phi(x)}$, we know by CAUCHY's formula that

$$\frac{1}{\Phi(x)} = \mathcal{E} \frac{1}{((\Phi(z)))} \frac{1}{x-z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the double parentheses denote that the residues must be taken for all the roots of $\Phi(z)=0$, viz. for all the roots of the equations

$$\underline{1-z}^{\alpha_1}=0, \quad \underline{1-z}^{\alpha_2}=0, \ldots \underline{1-z}^{\alpha_m}=0.$$

By developing the factor

$$\frac{1}{x-z} = -\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\ldots+\frac{x^n}{z^n}+\ldots\right)$$

we get immediately for the required coefficient of x^n

$$P_n = -\mathcal{E} \frac{1}{z^{n+1}} \frac{1}{((\Phi(z)))} = \sum_{i=1}^{m} W_{\alpha_i}$$

where

$$W_{\alpha_i} = -\mathcal{E} \frac{1}{z^{n+1} (\underline{1-z^{\alpha_1}})^{r_1} (\underline{1-z^{\alpha_2}})^{r_2} \ldots ((\underline{1-z}^{\alpha_i}))^{r_i} \ldots (\underline{1-z}^{\alpha_m})^{r_m}}$$

or, restoring the original form of $\Phi(z)$

$$W_{\alpha_i} = -\mathcal{E} \frac{1}{z^{n+1} (1-z^{a_1})(1-z^{a_2}) \ldots (1-z^{a_r})} \frac{\underline{1-z}^{\alpha_i}}{((\underline{1-z}^{\alpha_i}))}$$

where the residue is to be taken for all the roots of the equation

$$\underline{1-z}^{\alpha_i} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Representing one of these roots by ρ and putting

$$z = \rho e^{-t}$$

the preceding value of W_{α_i} takes this form

$$W_{\alpha_i} = \Sigma \mathcal{E} \frac{\rho^{-n} e^{nt}}{(1-\rho^{a_1} e^{-a_1 t})(1-\rho^{a_2} e^{-a_2 t}) \ldots (1-\rho^{a_r} e^{-a_r t})} \frac{t}{((t))} \quad . \quad . \quad (7)$$

wherein the summation must be extended to all the roots ρ of the equation (6).

The term W_1 may be further developed, for the corresponding equation (6)

$$\underline{1-z} = 0 \quad \text{or} \quad 1-z = 0$$

shows that the only root is $\rho = 1$. Therefore W_1 reduces to

$$W_1 = \mathcal{E} \frac{e^{nt}}{(1-e^{-a_1 t})(1-e^{-a_2 t}) \ldots (1-e^{-a_r t})} \frac{t}{((t))}$$

This residue is the coefficient of $\frac{1}{t}$ in

$$X_1 = e^{nt} \quad lg\,(1 - e^{-a_1 t}) \quad lg\,(1 - e^{-a_2 t}) \ldots - lg\,(1 - e^{-a_r t})$$

Now

$$lg\,(1 - e^{-z}) = lg\, z - \tfrac{1}{2} z + \frac{B_1}{2\,!}\frac{z_2}{2} - \frac{B_2}{4\,!}\frac{z^4}{4} + \frac{B_3}{6\,!}\frac{z^6}{6} - \ldots \quad (a)$$

as may be shown by integrating

$$\frac{1}{e^z - 1} \cdot \frac{1}{z} = -\tfrac{1}{2} + \frac{B_1}{2\,!} z \quad \frac{B_2}{4\,!} z^3 + \frac{B_3}{6\,!} z^5 - \ldots \qquad mod\, z < 2\pi$$

where

$$B_1 = \tfrac{1}{6}, \qquad B_2 = \tfrac{1}{30}, \qquad B_3 = \tfrac{1}{42} \ldots$$

between the limits o and z.

Substituting the values of $lg\,(1 - e^{\alpha_i t})$, we obtain

$$X_1 = \frac{1}{a_1\, a_2 \ldots a_r\, t^r}\, e^{\,vt - \frac{B_1}{1\,!}\frac{s_2\, t^2}{2^2} + \frac{B_2}{3\,!}\frac{s_4\, t^4}{4^2} - \ldots}$$

where

$$v = n + \tfrac{1}{2} s_1 \quad \text{and} \quad s_i = a_1{}^i + a_2{}^i + \ldots + a_r{}^i$$

hence W_1 is the coefficient of t^{r-1} in the product

$$\frac{1}{a_1\, a_2 \ldots a_r}\left(1 + vt + \frac{1}{2\,!} v^2 t^2 + \frac{1}{3\,!} v^3 t^3 + \frac{1}{4\,!} v^4 t^4 + \ldots\right)$$
$$\left(1 - \frac{s_2}{24} t^2 + \frac{s_2{}^2}{1152} t^4 - \ldots\right)$$
$$\left(1 + \frac{s_4}{2880} t^4 \ldots\right)$$

. . . .

Applying the preceding to compute the number of solutions of the equation

$$x_1 + 2\,x_2 + 5\,x_3 + 10\,x_4 + 20\,x_5 = n.$$

we first observe that the different divisors of 1, 2, 5, 10, 20 are

1,	1,	1,	1,	1
	2,	5,	2,	2
			5,	4
			10,	5
				10
				20

thus

$$\Phi\,(x) = \underline{(1-x)^5} \cdot \underline{(1-x^2)^3}\, \underline{(1-x^4)}\, \underline{(1-x^5)^3}\, \underline{(1-x^{10})^2}\, \underline{(1-x^{20})}$$

The number of values α_i being six, we have six different terms W_{α_i}.

In this case, having

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad a_4 = 10, \quad a_5 = 20$$
$$s_1 = 38, \quad s_2 = 530, \quad s_4 = 170642, \quad v = n + 19$$

we obtain

$$W_1 = \frac{1}{48000}\left[v^4 - 265\, v^2 + \frac{72741}{10}\right]$$

or

$$W_1 = \frac{1}{48000}\left[n^4 + 76\, n^3 + 1901\, n^2 + 17366\, n + 41930\frac{1}{10}\right].$$

For W_2 the equation (6)

$$\frac{1 - z^2 = 0}{}$$

or

$$1 + z = 0.$$

shows, that also in this case there is only one root $\varrho = -1$. Therefore W_2 reduces to

$$W_2 = \mathcal{E}\frac{(-1)^n e^{nt}}{(1+e^{-t})(1-e^{-2t})(1+e^{5t})(1-e^{-10t})(1-e^{-20t})}\frac{t}{((t))}$$

or $(-1)^n$ multiplied by the coefficient of $\frac{1}{t}$ in

$$X_2 = e^{nt - lg(1+e^{-t}) - lg(1-e^{-2t}) - lg(1+e^{-5t}) - lg(1-e^{-10t}) - lg(1-e^{-20t})}.$$

Developing the logarithms in this expression by means of the equation (a) and

$$\left.\begin{aligned} lg\,(1+e^{-z}) &= lg\,(1-e^{-2z}) - lg\,(1-e^{-z}) = \\ &= lg\, z - \frac{z}{2} + \frac{3}{2}\cdot\frac{B_1}{2!}z^2 - \frac{15}{16}\cdot\frac{B_2}{4!}z^4 + \ldots \end{aligned}\right\} \cdot (b)$$

we get

$$X_2 = \frac{1}{1600\, t^3}\left(1 + vt + \frac{v^2 t^2}{2!} + \ldots\right)\left(1 - \frac{97}{4}t^2 \ldots\right).$$

Hence

$$W_2 = \frac{(-1)^n}{48000}(15\, v^2 - 727\tfrac{1}{2})$$

or

$$W_2 = \frac{(-1)^n}{48000}(15\, n^2 + 570\, n + 4687\tfrac{1}{2}).$$

For W_4 the equation (6) is

$$\frac{1 - z^4 = 0}{}$$

or

$$1 + z^2 = 0$$

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hence

$$W_4 = \Sigma \mathcal{E} \frac{\rho^{-n} e^{nt}}{(1-\rho e^{-t})(1-\rho^2 e^{-2t})(1-\rho^5 e^{-5t})(1-\rho^{10} e^{-10t})(1-\rho^{20} e^{-20t})} \frac{t}{((t))}$$

where the summation must be extended to both the roots ρ of the equation $1 + z^2 = 0$ Writing $\rho^2 = -1$, we obtain

$$W_4 = \Sigma \mathcal{E} \frac{\rho^{-n} e^{nt}}{(1-\rho e^{-t})(1+e^{-2t})(1-\rho e^{-5t})(1+e^{-10t})(1-e^{-20t})} \frac{t}{((t))}$$

$$= \frac{1}{80} \Sigma \frac{\rho^{-n}}{(1-\rho)^2} = -\frac{1}{160} \Sigma \rho^{-n-1}.$$

Denoting by Σ_p the sum of the p^{th} powers of the roots, and observing that $\rho^4 = 1$, we know

$$\Sigma_0 = 2, \qquad \Sigma_1 = 0, \qquad \Sigma_2 = -2, \qquad \Sigma_3 = 0,$$

and generally, k being an integer number

$$\Sigma_{4k} = 2, \qquad \Sigma_{4k+1} = 0, \qquad \Sigma_{4k+2} = -2, \qquad \Sigma_{4k+3} = 0.$$

Therefore

$$W_4 = -\frac{1}{160} \Sigma_{3n-1}$$

which gives different values for different values of n.

According to

$$n = 4p, \quad 4p+1, \quad 4p+2, \quad 4p+3$$

we get respectively the four values

$$W_4 = \frac{600}{48000} (0, \ 1, \ 0, \ -1)$$

For W_5 the equation (6) gives

$$\underline{1 - x^5} = 0$$

or

$$1 + x + x^2 + x^3 + x^4 = 0$$

thus

$$W_5 = \Sigma \mathcal{E} \frac{\rho^{-n} e^{nt}}{(1-\rho e^{-t})(1-\rho^2 e^{-2t})(1-e^{-5t})(1-e^{-10t})(1-e^{-20t})} \frac{t}{((t))}.$$

Putting

$$X_5 = e^{nt - lg(1-\rho e^{-t}) - lg(1-\rho^2 e^{-2t}) - lg(1-e^{-5t}) - lg(1-e^{-10t}) - lg(1-e^{-20t})}$$

and reducing by means of the equations (a) and (b)

$$X_5 = \frac{e^{\left(n - \frac{\rho}{1-\rho} - \frac{2\rho^2}{1-\rho^2} + \frac{35}{2}\right)t + \left(\frac{\rho}{2(1-\rho)^2} + \frac{4\rho^2}{2(1-\rho^2)^2} - \frac{525}{24}\right)t^2}}{1000\,(1-\rho)\,(1-\rho^2)\,t^3}$$

we get

$$W_5 = \Sigma \frac{\rho^{-n}}{2000\,(1-\rho)\,(1-\rho^2)} \left[n^2 + \left(35 - \frac{2\rho + 6\rho^2}{1-\rho^2}\right) n + \right.$$

$$\left. + \frac{525}{2} - \frac{34\rho + 98\rho^2 - 42\rho^3 - 114\rho^4}{(1-\rho^2)^2} \right]$$

or

$$W_5 = \frac{n^2 + 35n + \frac{525}{2}}{2000} \Sigma \frac{\varrho^{-n}}{(1-\varrho)(1-\varrho^2)}$$
$$- \frac{n}{2000} \Sigma \frac{\varrho^{-n}(2\varrho + 6\varrho^2)}{(1-\varrho)(1-\varrho^2)^2}$$
$$- \frac{1}{2000} \Sigma \frac{\varrho^{-n}(34\varrho + 98\varrho^2 - 42\varrho^3 - 114\varrho^4)}{(1-\varrho)(1-\varrho^2)^3}.$$

With the same notations as in the preceding case, we obtain

$$\Sigma_0 = 4, \quad \Sigma_1 = -1, \quad \Sigma_2 = -1, \quad \Sigma_3 = -1, \quad \Sigma_4 = -1,$$

and generally

$$\Sigma_{5k} = 4, \quad \Sigma_{5k+1} = -1, \quad \Sigma_{5k+2} = -1, \quad \Sigma_{5k+3} = -1, \quad \Sigma_{5k+4} = -1.$$

According to the values

$$n = 5p, \; 5p+1, \; 5p+2, \; 5p+3, \; 5p+4$$

we find

$$\Sigma \frac{\varrho^{-n}}{(1-\varrho)(1-\varrho^2)} = \frac{1}{5} \Sigma \varrho^{-n}(1-\varrho^3)(1-\varrho^4) = \frac{1}{5} \Sigma \varrho^{4n}(1 + \varrho^2 - \varrho^3 - \varrho^4) =$$
$$= \frac{1}{5}(\Sigma_{4n} + \Sigma_{4n+2} - \Sigma_{4n+3} - \Sigma_{4n+4}) = (1 \;\; 0, \; 1, \; -1, \; -1)$$

$$\Sigma \frac{\varrho^{-n}(2\varrho + 6\varrho^2)}{(1-\varrho)(1-\varrho^2)^2} = \frac{1}{5}(2\,\Sigma_{4n+3} + 4\,\Sigma_{4n+4} - 6\,\Sigma_{4n}) = (-6, \; 0, \; 0, \; 2, \; 4)$$

$$\Sigma \frac{\varrho^{-n}(34\varrho + 98\varrho^2 - 42\varrho^3 - 114\varrho^4)}{(1-\varrho)(1-\varrho^2)^3} = \frac{1}{25}(-538\,\Sigma_{4n} - 8\,\Sigma_{4n+1} +$$
$$+ 32\,\Sigma_{4n+2} + 162\,\Sigma_{4n+3} + 352\,\Sigma_{4n+4}) = \frac{2}{5}(-269, \; -4, \; 16, \; 81, \; 176)$$

and therefore

$$W_5 = \frac{1}{48000}(24n^2 + 840n + 6300)(1, \; 0, \; 1, \; -1, \; -1)$$
$$+ \frac{1}{48000} \cdot 48n(3, \; 0, \; 0, \; -1, \; -2)$$
$$+ \frac{1}{48000}\frac{48}{5}(269, \; 4, \; -16, \; -81, \; -176).$$

In the same way we obtain, according to

$$n = 10p, \quad 10p+1, \quad 10p+2, \ldots \quad 10p+9$$

$$W_{10} = \frac{1}{48000}(120\,n + 1800)(-1, \; 0, \; 1, \; 3, \; 3, \; 1, \; 0, \; -1, \; -3, \; -3)$$
$$+ \frac{1}{48000} \cdot 60(5, \; 16, \; 21, \; 29, \; 19, \; -5, \; -16, \; -21, \; -29, \; -19)$$

and according to

$$n = 20p, \quad 20p + 1, \ldots \quad 20p + 19$$

$$W_{20} = \frac{1200}{48000}(-5, \ -8, \ -5, \ -7, \ -5, \ \ 3, 0, \ \ 3, \ \ 5, \ \ 7,$$

$$5, \ \ 8, \ \ 5, \ \ 7, \ \ 5, \ \ 3, 0, \ -3, \ -5, \ -7).$$

From the preceding formulae we may deduce the several results for $n = 10p + q$ $(q = 0,1 \ldots 19)$

To illustrate this, taking $n = 10p + 7$, we obtain

$$W_1 = \frac{1}{48000}(n^4 + 76n^3 + 1901n^2 + 17366n + 41930\tfrac{1}{10})$$

$$W_2 = \frac{1}{48000}(\qquad - \ 15n^2 - \ 570n - \ 4687\tfrac{1}{2})$$

$$W_4 = \frac{1}{48000}(\qquad - \ 600)$$

$$W_5 = \frac{1}{48000}(\qquad 24n^2 + \ 840n + \ 6300)$$

$$+ \frac{1}{48000}(\qquad - \ 153\tfrac{3}{5})$$

$$W_{10} = \frac{1}{48000}(\qquad - \ 120n - \ 1800)$$

$$+ \frac{1}{48000}(\qquad - \ 1260)$$

$$W_{20} = \frac{1}{48000}(\qquad 3600)$$

and finally

$$\Sigma W_{\alpha_i} = \frac{1}{48000}(n^4 + 76n^3 + 1910n^2 + 17516n + 43329)$$

$$= \tfrac{1}{3}(p+1)(p+2)(10p^2 + 22p + 9).$$

That this value is an integer may be easily seen by writing

$$\Sigma W_{\alpha_i} = \tfrac{1}{3}p(p+1)(p+2)(10p+22) + 3(p+1)(p+2).$$

Comparing the general formula of SYLVESTER

$$W_{\alpha_i} = \Sigma \mathcal{E} \frac{\varrho^n e^{nt}}{(1-\varrho^{a_1} e^{-a_1 t})(1-\varrho^{a_2} e^{-a_2 t}) \ldots (1-\varrho^{a_r} e^{-a_r t})} \frac{t}{((t))}$$

with the formula (7), it is evident that in his formula ϱ^n ought to be replaced by ϱ^{-n}. This makes no difference in the values W_1 and W_2. In the example treated by SYLVESTER

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = n$$

the values of W_3, W_4, W_5 and W_6 however want correction.