Mathematics. — "*An involution of pairs of points and an involution of pairs of rays in space.*" By Dr. C. H. VAN OS. (Communicated by Prof. JAN DE VRIES.)

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§ 1. *Introduction.* By several authors involutions have been treated, consisting of groups of points in the plane or in space. On the contrary involutions, consisting of groups of *straight lines,* do not seem to have been considered. In the following such an involution will be investigated. This involution is derived with the help of an involution of pairs of points, which is itself again connected with a certain *bilinear congruence of twisted cubics.*

The congruence in question $[\varrho^3]$ is formed by all the curves ϱ^3 which pass through two given points A_1 and A_2, and have three given straight lines a_3, a_4 and a_5 as bisecants. These curves are the moveable intersections of the quadratic surfaces out of two given pencils (ϱ^2_{34}) and (ϱ^2_{45}). The base-curve of the pencil (ϱ^2_{34}) consists of the lines a_3 and a_4 and the common transversals b_{134} and b_{234} which we can draw through the points A_1 and A_2 to these straight lines; that of the pencil (ϱ^2_{45}) consists of the lines a_4 and a_5 and their common transversals b_{145} and b_{245} passing through A_1 and A_2. [1])

Through a point P passes one ϱ^3 of the congruence; if we associate to P the point P', which on the curve ϱ^3 is harmonically separated from P by the points A_1 and A_2, we get an involution of *pairs of points* (P, P').

A straight line t is chord of one ϱ^3; let P and Q be its supporting points. Through the involution just found there are associated to the points P and Q two points P' and Q'. If we now associate the line t' connecting the points P' and Q', to t, we get an involution of *pairs of rays* (t, t').

§ 2. *Degenerate* ϱ^3 *of the involution.* We shall show that the

[1]) This congruence $[\rho^3]$ has been investigated by M. STUYVAERT (*Étude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre* Dissertation inaugurale Gand 1902) and by J. DE VRIES (*Bilineaire congruenties van kubische ruimtekrommen.* Proefschrift, Utrecht 1917).

congruence $[\varrho^3]$ contains *seven* systems of ∞^1 curves ϱ^3, each of which is degenerated into a conic k^2 and a straight line d.

In the first place the conic k^2 can pass through the points A_1 and A_2 and therefore lie in a plane π through the straight line $A_1 A_2$. Such a plane intersects the lines a_3, a_4 and a_5 in three points A_3, A_4 and A_5, which together with the points A_1 and A_2 define one conic k^2. The ruled surface ψ^2 formed by the common transversals of the lines a_3, a_4 and a_5, intersects this conic k^2 besides in the points A_3, A_4 and A_5 in one more point D; the transversal d passing through D. forms with k^2 a degenerate ϱ^3. The surface ψ^2 is intersected by the line $A_1 A_2$ in two points B_1 and B_2; the generatrices b_1 and b_2 of ψ^2 passing through these points, form each with the line $A_1 A_2$ a degenerate k^2 of the system just considered. The transversal d, which completes the degenerate k^2, formed by the lines $A_1 A_2$ and b_1 to a ϱ^3, is apparently no other than the line b_2. The three lines b_1, $A_1 A_2$ and b_2 form therefore together a degenerate ϱ^3.

It has just appeared that to every conic k^2 there belongs a definite transversal d; is the reverse also the case? In order to examine this we remark that the line $A_1 A_2$ is twice a component of a degenerate k^2, and is therefore *nodal line* of the surface formed by these conics k^2. A plane π through $A_1 A_2$ intersects this surface along the nodal line and along a conic k^2; it is therefore of order four. A transversal d intersects this surface besides in the lines a_3, a_4 and a_5 in one point D and so forms together with *one* conic k^2 a degenerate ϱ^3.

§ 3. In order to get a second series of degenerate ϱ^3, we draw the transversal b_{134} mentioned in § 1 and bring through the point A_2 and the line a_5 a plane α_{25}. This plane intersects the transversal b_{134} in a point D_1, the lines a_3 and a_4 in two points C_3 and C_4. The points A_2, D_1, C_3 and C_4 determine a pencil of conics each of which forms with the line b_{134} a degenerate ϱ^3.

As we can take one of the transversals b_{135}, b_{145}, b_{234}, b_{235}, b_{245}, instead of the transversal b_{134}, we get in all *six* pencils of conics degenerated in this way.

Each of the corresponding pencils of conics contains three pairs of lines; for the pencil lying in the plane α_{25} they are the pairs $(A_2 D_1,\ C_3 C_4)$ $(A_2 C_3,\ D_1 C_4)$ and $(A_2 C_4,\ D_1 C_3)$. Each of these pairs forms with the transversal b_{134} a ϱ^3, which has degenerated into *three straight lines*.

Lying in the plane α_{25} the line $A_2 C_3$ intersects the line a_5 and is therefore the transversal b_{235}; in the same way the line $A_2 C_4$ is the same as the transversal b_{345}. The curve $(b_{134}, A_2 D_1, C_3 C_4)$ belongs evidently only to the pencil of degenerate ϱ^3 which contain the line b_{134} as component; the curves $(b_{134}, b_{235}, D_1 C_4)$ and $(b_{134}, b_{245}, D_1 C_3)$ belong each to *two* pencils of degenerate ϱ^3. There are therefore six curves of the first and as many of the second kind. Hence together with the curve $(b_1, A_1 A_2, b_2)$ the congruence $[\varrho^3]$ contains *thirteen* ϱ^3 which have degenerated into three straight lines.

§ 4. *Singular points and bisecants of the congruence.* The points of the three lines a_3, a_4 and a_5 are *singular points* of the congruence.

Let us consider for instance a point A_3 of the line a_3. The surface φ^2_{45} through A_3 intersects an arbitrary surface φ^2_{34} along a curve ϱ^3, which passes through the point A_3. Through A_3 passes therefore a pencil of curves ϱ^3; all these curves pass also through the second point of intersection of the surface φ^2_{45} with the line a_3.

Also the points of the transversals b_{ikl} are *singular points*; for each of these transversals is component of a pencil of degenerate ϱ^3.

The straight lines through the points A_1 and A_2 are *singular bisecants*; for through any point of such a straight line there passes one ϱ^3 and as this passes also through the points A_1 and A_2, it has that straight line as bisecant.

In the second place the straight lines in the planes α_{13}, α_{14}, α_{15}, α_{23}, α_{24}, α_{25} brought through the points A_1 and A_2 and the lines a_3, a_4 and a_5 are *singular bisecants.* For each of these planes contains a pencil of conics k^2, each of which is a component of a degenerate ϱ^3, and a straight line in such a plāne is bisecant of all these conics.

In the third place the generatrices g_{34} of the surfaces φ^2_{34}, which cross the lines a_3 and a_4, are *singular bisecants* of the congruence. Such a line g_{34} is intersected by the surfaces φ^2_{45} in the pairs of points of a quadratic involution and the two points of such a pair are every time the supporting points of a curve ϱ^3. As the surfaces φ^2_{34} pass through the lines a_3, a_4, b_{134} and b_{234}, the lines g_{34} are the transversals of the lines b_{134} and b_{234}.

In the same way the transversals g_{45} of the lines b_{145} and b_{245} and the transversals g_{35} of the lines b_{135} and b_{235} are singular bisecants of the congruence.

The *singular bisecants* form therefore two sheaves, six fields and three bilinear congruences.

§ 5. *Pairs of points on a degenerate* ϱ^3. We now pass on to the consideration of the involution (P, P') and examine first what becomes of this correspondence, if the points P and P' lie on a degenerate ϱ^3.

With a view to this we remark, that the four harmonical points P, A_1, P', A_2 of a curve ϱ^3 from each of its chords s are projected by four harmonical planes. This must remain the case, if we let the ϱ^3 degenerate into a conic k^2 and a straight line d.

In the degeneration considered in § 2, the points A_1 and A_2 lie both on the conic k^2. The following two cases are now possible:

1. The point P lies also on the conic k^2. If we take as chord s a common secant of the conic k^2 and the line d, we see that also the point P' lies on k^2 and is harmonically separated from P by A_1 and A_2.

2. The point P lies on the line d. If we take the chord s in the same way, we see that the point P' lies on k^2 and by A_1 and A_2 is harmonically separated from the point of intersection D of the two components k^2 and d.

To the point D', which is harmonically separated from D, *all the points* of the line d are therefore associated; for the rest there belongs to every point P of the degenerate ϱ^3 one definite other point P'.

In the degeneration considered in § 3, the point A_1 lies on the line d, the point A_2 on the conic k^2 (or inversely). Two cases are again possible:

1. The point P lies on the conic k^2. If we again take as chord s a secant of k^2 and d, we see that the point P' lies also on the conic k^2 and is harmonically separated from P by the points A_2 and D.

2. The point P lies on the line d. If we take as chord s a straight line in the plane of the conic k^2, we see that the point P lies also on the line d and is harmonically separated from P by the points A_1 and D.

To each point of this degenerate ϱ^3 belongs consequently a definite other point. If P coincides with D, the point P' does the same.

If the ϱ^3 is degenerated into three straight lines, considerations of the same kind hold good.

§ 6. *Singular points of the involution* (P, P'). On every non degenerate ϱ^3 the points A_1 and A_2 are associated to themselves; it appears from the preceding § that this is also the case for the degenerate ϱ^3. These points are therefore *not* singular points of the involution. On the contrary the points of the lines a_3, a_4 and a_5 *are* singular points. Let us consider e.g. a point A_3 of the line a_3. In order to find the point A_3' associated to A_3 on a curve ϱ^3 passing through A_3, we must bring through the bisecant a_3 of this curve ϱ^3 a plane which by the planes α_{13} and α_{23} is harmonically separated from the plane which *touches* the curve ϱ^3 in the point A_3 and passes through the line a_3; this plane intersects the curve ϱ^3 in the point A'_3 in question.

As the plane (A'_3, a_3) passes through the line a_3, it is a tangent plane of the surface φ^2_{34} which contains the considered curve ϱ^3. Now the tangent planes of a ruled surface in the points of a generatrix are projectively associated to the points of contact; the point of contact B_3 of the plane (A_3, a) is therefore harmonically separated from the point A_3 by the points of contact of the planes α_{13} and α_{23}. As these two planes pass through the lines b_{134} and b_{234}, their points of contact B_{13} and B_{23} are the intersections of these transversals with the line a_3.

If the surface $\varphi_{34}{}^2$ describes the pencil ($\varphi_{34}{}^2$), the plane (A'_3 a_3), which touches the surface φ^2_{34} in the point B_3, describes a pencil which is projectively associated to the pencil (φ^2_{34}). The figure produced by these projective pencils is a surface of the third order. To these planes of contact belong the planes α_{13} and α_{23} each of which is at the same time part of a degenerate surface φ^2_{34}; consequently these planes belong to the product and the rest is a *plane*.

The figure produced by two projective pencils passes through the base-curves of these pencils. The plane just found contains therefore the line a_4 as this is the case with neither of the planes α_{13} and α_{23}. It must also pass through the point B_3 as the intersection of a curve φ^2_{34} with its tangent plane in the point B consists besides of the line a_3, of a generatrix through the point B_3.

The locus of the points A'_3 which are associated to the point A_3 on the different curves ϱ^3 laid through the point A_3, belongs to the intersection of the plane (B_3, a_4) with the surface φ^2_{45}, on which all these curves ϱ^3 are situated and which passes through A_3. This intersection consists besides of the line a_4 of a *straight line* λ; this is the locus in question.

This line λ passes through the point of intersection of the plane (B_3, a_4) with the line a_5. Evidently this point of intersection is projectively associated to the point B_3, therefore also to the point A_3. The same must hold good for the intersection of the line λ with the line a_4. If the point A_3 describes the line a_3, the intersections of the line λ with the lines a_4 and a_5 describe two projective sequences of points. Consequently the line λ describes a *quadratic surface* ω^2, the locus of all the points associated to the points of the line a_3.

To each of the lines a_4 and a_5 belongs a similar surface ω^2.

§ 7. The points of the transversals b_{145} etc. are *not* singular points of the involution. For from the construction given in § 5 it follows that to every point of such a transversal a definite other point of the same transversal is associated, no matter of which degenerate ϱ^3 we consider the transversal to be a component.

From § 5 follows further that on each degenerate ϱ^3 of the first series there lies *one* singular point D'. We shall determine the locus of these singular points.

It appeared in § 2 that to this series belongs a ϱ^3 consisting of the straight line A_1A_2 and the two transversals b_1 and b_2 of the four lines $A_1 A_2$, a_3, a_4 and a_5. As we can combine each of the two transversals with the line A_1A_2 to a degenerate conic k^2, there lie on this conic *two* singular points D'_1 and D'_2.

A plane π through the line A_1A_2 contains *one* conic k^2 and consequently it intersects the locus in question, besides in the points D'_1 and D'_2, in one more point D'; the locus is therefore a *twisted cubic* σ^3.

A point D' is associated to a straight line d, which intersects the three lines a_3, a_4 and a_5. To the points associated to D' belong therefore three points which lie on the three lines mentioned; consequently the point D' must lie on the three surfaces ω^2 found in the preceding §. All these three surfaces pass therefore through the curve σ^3.

§ 8. If a point P describes a straight line l, the point P' associated to P describes a curve (l). As the line l has two points in common with each of the three surfaces ω^2, the curve (l) has two points in common with each of the three lines a_3, a_4 and a_5. A surface $\varphi_{34}{}^2$ intersects the line l in two points and contains both the points of the curve (l) associated to this line, so that in all this surface $\varphi_{34}{}^2$ has *six* points in common with the curve (l). For this reason (l) is a *twisted cubic*.

In general the line l and the curve $(l)^3$ have *no* points in common, for as a rule no two associated points of the involution (P, P') lie on l; for the rest this involution has only a finite number of coincidences, viz. the points A_1 and A_2 and the points D, found in § 5, in which the transversals b_{134} etc. intersect the corresponding planes α_{25} etc. As a rule therefore the line l does not contain any coincidences either.

If a point P describes a plane V, the point P' associated to P, describes a surface (V). In order to find the order of this surface, we draw in the plane V a straight line l. The curve $(l)^3$ associated to this line l, intersects the plane V in three points, each of which is associated to a point of l. The line l intersects therefore the locus of the pairs of points (P, P') lying in the plane V, in *three* points; consequently this locus is a curve of order three. The plane V containing as a rule no coincidences, this curve is the complete intersection of V with the surface (V), which for this reason is a surface of order *three*.

The surface $(V)^3$ contains the lines a_3, a_4 and a_5, for each point of one of these lines is associated to a line λ, which cuts the plane V in one point. In the same way the surface $(V)^3$ passes through the curve σ^3.

Let Q be a point of the plane V, l a straight line of V passing through Q. This line contains three points P, associated to a point P' in V. If we connect these points P' with Q and associate these lines of connection to the line l, we get in the pencil of the rays through Q a correspondence (3, 3) with *six* coincidences. These must originate from the rays (P, P') passing through Q and each of these rays furnishes *two* coincidences, as the correspondence (P, P') is involutory. Through Q pass therefore *three* rays $P\,P'$ which lie in plane V and accordingly the lines $P\,P'$ as a rule form a *cubic line complex.*

§ 9. *Singular straight lines of the involution* (t, t'). We now proceed to the consideration of the involution (t, t') and first investigate its *singular rays.*

The line $A_1\,A_2$ is bisecant of *all* the curves ϱ^3. On an arbitrary curve ϱ^3 each of the two supporting points A_1 and A_2 coincides with its associated point; in this case the line $A_1\,A_2$ is associated to *itself.*

According to § 7 the line $A_1\,A_2$ is a component of one degenerate ϱ^3 and as such contains two singular points D_1' and D_2'; to these points correspond all points of the two transversals b_1 and b_2 of the lines $A_1\,A_2$, a_3, a_4 and a_5. If we consider the points D_1' and D_2' as supporting points, there is associated to the line $A_1\,A_2$ a *bilinear congruence of rays* which has the lines b_1 and b_2 as directrices. If we consider one of the points D_1' and D_2' and one arbitrary other point of the line $A_1\,A_2$ as supporting points, we find that to the line $A_1\,A_2$ there are moreover associated two *fields of rays* lying in the planes which connect the lines b_1 and b_2 with the line $A_1\,A_2$.

Also the line a_3 is bisecant of all curves ϱ^3. The supporting points E_3, F_3 are each time the two points of intersection of the line a_3 with a surface φ^2_{45}. The points E'_3 and F'_3 associated to these, lie on the generatrice λ and μ of the surface ω^2 corresponding to E_3 and F_3.

Through each pair of points (E_3, F_3) pass ∞^1 curves ϱ^3; the corresponding points E_3' and F_3' describe apparently two projective sequences of points. Moreover the pairs of points (E_3, F_3) form an involution on the line a_3; the pairs of generatrices (λ, μ) form therefore also an involution. Consequently the pairs of points (E_3', F_3') form an involution on the surface ω^2 and the lines connecting associated points of this involution are the rays associated to the line a_3.

We shall first demonstrate that each generatrix v of the surface ω^2 which belongs to the same system with the lines a_4 and a_5, contains one pair of points (E_3', F_3'). With a view to this we remark that two points E_3' and F_3' are situated on the same curve ϱ^3; this curve intersects the surface ω^2 besides in the supporting points of the bisecants a_4 and a_5. The congruence $[\varrho^3]$ being bilinear, each line v belongs as bisecant to one ϱ^3; the corresponding supporting points are the points in question E_3' and F_3'.

Through a point E_3 of the surface ω^2 there pass two rays of the congruence in question, viz. the line connecting E_3' with its associated point F_3', and the line v passing through the point E_3'; consequently the order of this congruence is *two.*

A tangent plane of the surface ω^2 contains one line v and one line λ. The straight line μ, associated to the line λ, cuts this tangent plane in a point F_3' and the line connecting this point with the associated point E_3' is a ray of the congruence in consideration, which together with the line v lies in this tangent plane. For this reason the *class* of the congruence is *two* as well.

Analogous considerations hold good for the lines a_4 and a_5. *Consequently to each of the lines a_3, a_4 and a_5 there corresponds a congruence* (2,2).

§ 10. A straight line l *through the point* A_1 is bisecant of ∞^1 curves ϱ^3. The point A_1 corresponds to itself; the locus of the points P' corresponding to the points P of the line l is according to § 8 a curve $(l)^3$. This passes through the point A_1; for when P gets into A_1, P' coincides with P. The rays associated to the line l project the curve $(l)^3$ from the point A_1 and form therefore a *quadratic cone.*

The same holds good for a straight line through the point A_2. A straight line l *in the plane* α_{23} is bisecant of ∞^1 conics k^2. Let E and F be the points of intersection of the line l with such a conic. The points E' and F', associated to these points E and F, lie according to § 5 also on the conic k^2 and the straight line $E'F'$ is associated to the line l.

The locus of the points E' and F' is a conic k^2, for the line l

has one point in common with the line a_2. To this point corresponds a line λ, so that the curve $(l)^3$ which corresponds to the line l, must degenerate into this line λ and into a conic k^2, the locus of the pairs of points (E', F'). These pairs of points form an involution on the conic k^2; the line $e'F'$ passes therefore through a fixed point, so that to the line l a *plane pencil* of the plane α_{23} is associated.

The same holds good for a straight line in one of the planes $\alpha_{24}, \alpha_{25}, \alpha_{13}, \alpha_{14}$ and α_{15}.

According to § 4 each transversal g_{34} of the lines b_{134} and b_{234} contains an involution of pairs of points (G,H) which are each time the supporting points of a curve ϱ^3. The associated points G' and H' lie on the curve (l^3), which through the involution (P,P') is associated to the line g_{34}. The pairs of points (G',H') form an involution on this line with two coincidences and the lines $G'H'$ determine a *quadratic ruled surface,* associated to the singular line g_{34}.

In the same way there corresponds to each of the lines g_{45} and g_{35} a *quadratic ruled surface.*

The straight lines which are associated to all the lines g_{34}, form together a line complex, the order of which we shall determine later on.

§ 11. It appeared in § 5 that on each degenerate ϱ^3 of the first system lies one singular point D' which is associated to all the points of the line d. A bisecant l of this ϱ^3 through the point D' corresponds therefore to a *plane pencil* which projects the line d from the point which is associated to the second supporting point of the bisecant.

These bisecants l form two plane pencils, which both have the point D' as base point; the first lies in the plane of the conic k^2, the second projects the line d from the point D'.

The plane of the conic k^2 passing through the line A_1A_2, the bisecants l of the first kind are the common secants of the line A_1A_2 and of the locus σ^3 of the points D'. As A_1A_2 and σ^3 have two points D'_1 and D'_2 in common, their common secants form *a congruence* (1,3).

A plane V intersects the curve σ^3 in three points; through each of these points passes one bisecant l of the second kind lying in the plane V; these bisecants form consequently a *congruence* of class *three.*

From a point P the curve σ^3 is projected by a cubic cone K^3. The planes which project the corresponding lines d from P, envelop a quadratic cone of which the tangent planes are projectively associated to the generatrices of the cone K^3; it happens *five* times that such a plane passes through the corresponding straight line, so that this line is a bisecant l of the second kind passing through P. Hence the *order* of the congruence formed by these bisecants is *five*.

To each ray l of one of the congruences (1, 3) and (5, 3) corresponds a plane pencil of straight lines l' which project a line d from a point of the corresponding conic k^2. For the lines l of the second kind this point coincides with D', so that the congruence (5, 3) is transformed *into itself*; for those of the first kind it is an arbitrary point of the conic k^2.

A plane V intersects the conics k^2 in the points of a curve c^4 that has a node in the intersection of the plane V with the line $A\,A_2$, and the lines d in the points of a conic c^2. Between the points of the curves c^4 and c^2 there evidently exists a correspondence (1, 2). The three points of intersection of these curves lying outside the intersections of the plane V with the lines a_3, a_4 and a_5 and with the two transversels b_1 and b_2 of the four lines A_1A_2, a_3, a_4 and a_5, are points D, hence coincidences of this correspondence. The lines connecting associated points of this correspondence, in other words the rays l' lying in the plane V, envelop therefore a curve of class five.

The rays l' corresponding to the rays l of the congruence (1, 3) *form consequently a line complex of order five.*

The degenerate curves ϱ^3 of the second series, found in § 3, do not contain any singular points.

§ 12. *Coincidences.* A line A produces a coincidence if its supporting points P and Q coincide with their associated points P' and Q'.

The involution (P, P') has a finite number of coincidences, viz. the points A_1, A_2 and the six points D found in § 5, in which the transversals b_{134} etc. cut the corresponding planes α_{25} etc. The line A_1A_2 and the lines connecting the points A_1 and A_2 with the points D are therefore rays of coincidence.

Let us further consider a line l through the intersection D_1 of the line b_{134} with the plane α_{25}. This line is bisecant of a degenerate ϱ^3 formed by the line b_{134} and a conic k^2 in the plane α_{25}; in the point D_1 this conic touches the plane brought through the lines l and b_{134}. For if we cause the two supporting points of a bisecant PQ of which the supporting point P lies on the line b_{135}, the supporting point Q on the conic k^2, to approach D_1, we get such a straight line l. The point P' associated to P lies on the line b_{134} and is harmonically separated from P by the points A_1 and D_1; it approaches therefore also to D_1 and in such a way that lim. $PD : P'D_1 = -1$. In the same way the point Q on the conic k^2 approaches to the point D_1. From this it is easily seen, that in the limit the line $P'Q'$ coincides with PQ so that the line l is a *ray of coincidence.*

Consequently the straight lines through these six points D are also rays of coincidence.

A line t is also a ray of coincidence, if P' coincides with Q and Q' with P, so that the supporting points P and Q are associated to each other in the involution (P, P'). According to § 8 these rays form a *cubic complex.*

§ 13. When a straight line t describes a *plane pencil,* the associated ray t' describes a *ruled surface* R, of which we shall determine the order.

Each ray is bisecant of one curve ϱ^3; the locus of the supporting points is a curve c; this has a node in the base point B of the plane pencil, for on the two rays t connecting B with the two other points of intersection of the ϱ^3 passing through B one of the two supporting points gets into B. Hence the curve c is of order four.

The curve c^4 has one point in common with each of the three lines a_3, a_4 and a_5; for if a ray t intersects one of these lines, one of the two supporting points gets into the point of intersection.

Through the involution (P,P') a curve $(l)^3$ is associated to a line l, hence to a curve of order four, in general one of order twelve. The curve ϱ^4 has one point in common with each of the straight lines a_3, a_4 and a_5 and to each of these points a line λ is associated, so that moreover a curve ϱ^9 is associated to the curve ϱ^4.

The pairs of supporting points form on the curve c^4 an involution with *six* coincidences; these are the points of contact of the six tangents which can be drawn from the node B at the curve c^4. The pairs of points of the curve ϱ^9, associated to them, form therefore also an involution with six coincidences. The lines connecting associated points of this involution form consequently a ruled surface of order *six,* which is the surface R.

We can also determine the order of R by trying to find the number of points of intersection of this surface with the line a_3. With a view to this we remark that to the line a_3 a surface ω^2 is associated, so that whenever one of the supporting points of a ray t lies on this surface ω^2, one of the supporting points of the associated ray t' lies on the line a_3. The surface ω^2 passes through the lines a_4 and a_5; the curve c^4 intersects this surface besides in the points it has in common with the lines a_4 and a_5, in *six* more points, so that the plane pencil in consideration contains *six* rays t of which one of the supporting points lies on the surface ω^2; consequently there are *six* rays t' intersecting the line a_3.

In the third place we can determine the order of R by trying to find the number of intersections with the line A_1A_2. For this purpose we remark that a ray t' intersecting the line A_1A_2, if it is not a singular ray, must be bisecant of a conic k^2. The two supporting points are associated to two points of the same conic, so that also the associated ray t intersects the line A_1A_2. The plane pencil contains one ray t intersecting the line A_1A_2; the associated ray t' rests also on the line A_1A_2.

According to § 11 there is a complex of order five consisting of rays t associated to singular rays t' which form a congruence (1,3) and each of which intersects the line A_1A_2. The plane pencil contains 5 rays of this complex, hence the surface R five rays t' of the (1,3).

In all the line A_1A_2 is intersected by six rays t', so that the surface R is of order *six*.

§ 14. We can now also determine the order of the line complex associated to the congruence of the singular rays g_{34} found in § 10.

A singular ray t', intersecting the line b_{134}, is bisecant of a degenerate ϱ^3 consisting of the line b_{134} and a conic k^2 of the plane α_{25}, passing through the point of intersection of this plane with the line t'. The supporting points of the associated ray t lie also on the line b_{134} and on the conic k^2. Now the plane pencil considered in the preceding § contains one ray t, which intersects the line b_{134}; hence there is one ray t', which intersects the line b_{134}.

The other five generatrices of the ruled surface R^6 intersecting the line b_{134} must be *singular* rays, therefore lines g_{34}. The plane pencil contains five rays associated to rays g_{34}; consequently these rays form a complex of order *five*.

§ 15. To a sheaf of rays corresponds a congruence $[t']$. In order to determine order and class of this congruence $[t']$, we take the base point B of the sheaf on the line A_1A_2.

It has been found already in § 13 that to a ray t intersecting the line A_1A_2 a ray t' is associated also intersecting A_1A_2. We shall now show that the rays t and t' intersect the line A_1A_2 in *the same* point.

Let k^2 be the conic which has the line t as bisecant, P and Q the corresponding supporting points, P' and Q' the points associated

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to these. Through a linear transformation of the plane π of the conic k^2 we can transform the points A_1 and A_2 into the circle points at infinity. If S be an arbitrary point of the conic k^2, the straight lines SP and SP' will be harmonically separated by SA_1 and SA_2, hence they will be perpendicular to each other after the transformation, so that PP' is a diameter of the circle k^2, the same as the line QQ'. The chords PQ and $P'Q'$ are therefore parallel and consequently intersect on the line A_1A_2.

To an arbitrary ray t through the point B corresponds, therefore a ray through the same point, so that to the congruence $[t']$ there belongs in the first place the sheaf itself.

To the line A_1A_2 corresponds a *bilinear congruence of rays*, also belonging to the congruence $[t']$, besides *two fields of rays*.

Through the point B passes a cubic cone of singular rays of the congruence (1,3) considered in § 11. To each of these rays corresponds a plane pencil which projects a line d from a point Q' of the corresponding conic k^2. The point Q' is associated to the second point of intersection Q of the ray with the conic k^2.

The cubic cone mentioned has the line A_1A_2 as nodal generatrix. The two generatrices coinciding with A_1A_2 belong to the two degenerate conics k^2 consisting of the line A_1A_2 and of one of the two lines b_1, b_2; hence the two leaves of the cone K^3, which pass through the line A_1A_2, touch at the planes of these degenerate conics consequently they also touch the two leaves both passing through A_1A_2, of the surface of order four, found in § 2, described by the conics k^2; the line A_1A_2 belongs therefore *six times* to the intersection of the cone K^3 with this surface. The rest of the intersection consists of the curve σ^3 projected by the cone K^3 and of the locus τ^3 of the points Q. The cone K^3 has *three* points in common with each of the lines a_3, a_4 and a_5 lying on the quartic surface mentioned; the curve σ^3 having these lines as bisecants, two of these points lie every time on the curve σ^3, while the third must lie on the curve τ^3.

It is further easily found that the curves σ^3 and τ^3 lying on one and the same cubic cone, have *three* points in common. In general through the involution (P,P'), to a cubic curve a curve of order nine is associated. However the curve τ^3 having one point in common with each of the lines a_3, a_4 and a_5, three straight lines λ belong to this associated curve and as it has three points in common with the curve σ^3 and for this reason contains three singular points D', three lines d belong to it. The complete locus of the points Q' is therefore a curve τ_1^3.

The rays in question, associated to the generatrices of the cone

K^4, project the lines d from the corresponding points Q' of the τ_1^4. In the same way as in § 11 we should therefore find that these rays form a congruence (5, 3). But it happens *three times* that the point Q' coincides with the point D and hence lies on the line d; these points are associated to the three points of intersection of the curves σ^3 and τ^3, for in them the point Q coincides with the point D'. In this case *all* rays through the point Q' intersect the line d. Accordingly, from the congruence (5, 3), which we should find in general, three sheaves are split off and we only find a congruence (2, 3).

To the sheaf of rays through a point B of the line A_1A_2 are associated one sheaf, two fields of rays, one bilinear congruence and one congruence (2, 3). In general there corresponds therefore to a sheaf of rays a congruence (4, 6).

§ 16. To a *field of rays* corresponds also a certain congruence. In order to investigate this, we consider the rays lying in a plane π through the line $A_1 A_2$.

A non singular ray of this field is bisecant of a conic k^2 in this plane π, hence associated to another bisecant of this conic. To the congruence in question belongs therefore in the first place the field of rays *itself*. To the line $A_1 A_2$ in the plane π correspond a bilinear congruence of rays and two fields of rays.

To an arbitrary straight line through the point A_1 corresponds a quadratic cone with the point A_1 as vertex. This intersects the plane π along two straight lines. The sheaf of the rays through the point A_1 belongs therefore also to the congruence in question and each of these rays must be counted *twice*, because it is associated to *two* rays of the plane π. The same holds good for the sheaf of the rays through the point A_2.

The plane π intersects the curve σ^3 besides in the points D'_1 and D'_2 in one more point; through this point passes a plane pencil of singular rays of the congruence (1, 3). To each of these rays corresponds a plane pencil, which projects the line d, belonging to the conic k^2, from a point of this conic; hence to the plane pencil mentioned corresponds the congruence of the lines resting on k^2 and d. As these have a point D in common, those lines of intersection form a congruence (1, 2). A *field of rays* is therefore transformed into a congruence (6, 6).