Mathematics. — "*Graphical determination of the moments of transition of an elastically supported, statically undeterminate beam.*" [1]) I. By C. B. BIEZENO. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of November 24, 1917).

1. Let a rectangular prismatic beam be charged by forces which cut its axis at right angles and which are parallel to one of the two other principal axes of its centre of gravity.

Its support, which is thought to be elastical, be applied in a number of points of support $A, B, C \ldots$ at the same level in such a way that the reactions of support $R_A, R_B, R_C \ldots$

1. are parallel to the lines of action of the charging forces.

2. are proportional to the local descents $y_A, y_B, y_C \ldots$ of the axis of the beam, so that $\alpha R_A = y_A, \beta R_B = y_B, \gamma R_C = y_C \ldots$.

It is required to define graphically the moments of transition in the beam.

2. In order gradually to conquer the difficulties which arise during the solution of the problem, the case of the beam on three, four and five points of support will successively be dealt with and that on the supposition, that the fieldlengths of the beam as well as the coefficients of stiffness of the elastic supports are equal. This restricting supposition can be introduced, because it does not influence the general construction, as will appear later.

When the case of the beam on five points of support has been treated, the general problem, which finds its analytical interpretation in the so called "theorem of five moments", must have been solved at the same time.

3. In fig. 1 for the beam ABC, supposed to be charged in the middle of each of its fields by a force of 1 ton, the lines

[1]) In the following treatise the reader is supposed to be thoroughly acquainted with the construction of the elastic link-polygon, which we owe to O. MOHR. (See for this construction: O MOHR, Abhandlungen aus dem Gebiete der technischen Mechanik, 2e Auflage S. 367; J. KLOPPER, Leerboek der toegepaste Mechanica, Deel III, p. 160).

($l_A, l_I, l_{II}, l_{III}, l_B\, l_{IV}, l_V, l_{VI}, l_C$)[1] have been drawn, along which the "forces" are acting, which would play a part in the construction of the elastic link-polygon, if the beam lay on *fixed* points of support.

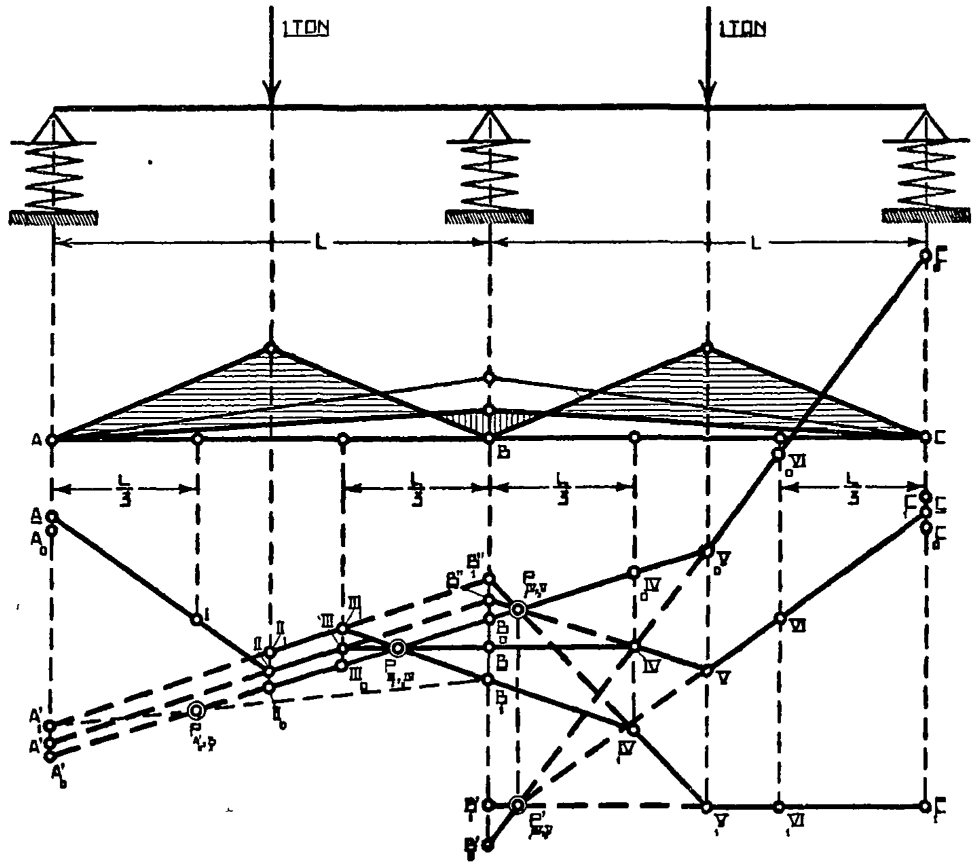

Fig. 1.

If now the descents $A\underline{A}$, $B\underline{B}$ and $C\underline{C}$ of the points of support A, B, C were known, it would be possible to construct the elastic link-polygon of the beam, because together with the point $\underline{A}$ also the point $\underline{A}'$ is fixed, which lies a known distance a under $\underline{A}$ and is the starting point of the construction of Mohr[2]).

The situation of $\underline{A}$, hence also that of $\underline{A}'$, in reality being unknown, we shall for the present try to find a solution of the problem by assigning to the moment of transition M_B a certain value, say x metre-ton. For in this way the reaction and therefore also the descents

[1]) By l_A, l_B, l_C the verticals passing through the points A, B, C are indicated; by l_I, l_{II} etc. the vertical lines on which the angles lie of the elastic link-polygons that will be drawn later on.

[2]) In the figure this point $\underline{A}'$ is by mistake indicated by A'.

AA_x, $\underset{x}{B}B$ and $C{}_xC$ of the points of support A, B and C become known[1]).

Now the point A_x', belonging to A_x, is also fixed, so that the side $\text{II}_x\,\text{III}_x$ can at once be drawn in the right direction. For together with l_B this side must produce a point of intersection $\underset{x}{B}''$, the situation of which is known, because the sides II_x, III_x and III_x, ${}_x\text{IV}$ must cut from l_B a segment of definite length, representing the statical moment relative to B of the "force" acting along l_{III}. With the side $\text{II}_x\,\text{III}_x$ not only the side III_x, ${}_x\text{IV}$, but also the side ${}_x\text{IV}, {}_x\text{V}$ is determined, because the latter, on account of the equality of the fieldlengths AB and BC, must give a point of intersection with $\text{II}_x\,\text{III}_x$ on l_B.

Finally the side ${}_x\text{V}, {}_x\text{VI}, {}_xC$ can be drawn too, as this together with ${}_x\text{IV}, {}_x\text{V}$ must also cut a segment of known length from l_B.

If now the supposition, made with regard to M_B, had been right, the point of intersection ${}_x\overline{C}$ of ${}_x\text{V}, {}_x\text{VI}$ with l_C would coincide with the point ${}_xC$, which apparently does not happen.

The construction might however be repeated with judiciously chosen values of M_B, till the points ${}_x\overline{C}$ and ${}_xC$ quite or nearly coincide.

But these tentative attempts to find the real value of M_B are made superfluous by the construction to be given in the following paragraphs, from which the coincidence of the points ${}_x\overline{C}$ and ${}_xC$ ensues directly and exactly.

4. Closely following the line of thought developed in § 3 let us in the first place assign the value zero to the moment of transition M_B.

In this case the beams AB and BC can be considered as two beams supported at their extremities, of which the reactions of support can be determined directly. The descents of the points A, B and C are also known; if μ represents the coefficient of stiffness of the springs they are:

$$AA_0 = \mu \cdot \tfrac{1}{2}\,,\ \underset{0}{B}B = \mu \cdot 2 \cdot \tfrac{1}{2}\,,\ C{}_0C = \mu \cdot \tfrac{1}{2}.$$

Through the point A'_0, which lies the known distance a, mentioned before, below A_0, the beam $\text{II}_0, \text{III}_0$ must now be drawn, which however, the "force" along l_{III} being zero, must act along the side III_0, ${}_0\text{IV}$, which itself passes through $\underset{0}{B}$.

[1]) An index placed under a letter denotes the value of the moment of transition belonging to the point of support indicated by the letter.

An index placed to the right or to the left of a figure or letter denotes the value of the moment of transition in the first point of support to the right or to the left.

The sides $II_0\,III_0$, III_0, ${}_0IV$ and ${}_0IV\,{}_0V$ coincide on the line $A^{0\prime}\underset{0}{B}$.

By finally measuring the known distance $\underset{0}{B}\underset{0}{B'} = b$, starting from $\underset{0}{B}$, it is possible to draw the coinciding sides $\underset{0}{B'}\,{}_0V\,{}_0VI$ and ${}_0VI\,{}_0\overline{C}$.

While the supposition $M_B = 0$ on the one hand causes a descent $C\,{}_0C$ of the point C, it leads on the other hand via the construction of the elastic link-polygon $A_0'\,I_0\,II_0\,III_0\,\underset{0}{B}\,{}_0IV\,{}_0V\,{}_0VI\,{}_0\overline{C}$ to an ascent $C\,{}_0\overline{C}$ of this point.

Let in the second place the value of one metreton be given to the moment of transition M_B[1]).

In this case the situations of the points of support are again known. The supposition $M_B = 1$ metreton namely gives rise, if the fieldlengths AB and BC in metres are indicated by L, to extra reactions of magnitudes: $-\frac{1}{L}, 2\frac{1}{L}$ and $-\frac{1}{L}$ ton, to which correspond the extra descents $-\mu\frac{1}{L}$, $2\mu\frac{1}{L} - \mu\frac{1}{L}$, which can be drawn on the scale once introduced.

In the way indicated in § 3 there arises now a link-polygon $A_1'\,I_1\,II_1\,III_1\,\underset{1}{B}\,{}_1IV\,{}_1V\,{}_1VI\,{}_1\overline{C}$.

While as a result of the introduction of the moment of transition of one metreton the point ${}_0C$ has moved upward over the distance ${}_0C\,{}_1C$, the endpoint ${}_0\overline{C}$ of the elastic link-polygon $A_0'\,I_0\,II_0\,III_0\,\underset{0}{B}\,{}_0IV\,{}_0V\,{}_0VI\,{}_0\overline{C}$ has descended over the distance ${}_0\overline{C}\,{}_1\overline{C}$.

It will now be shown that on the introduction of a moment of transition of x metreton two points ${}_xC$ and ${}_x\overline{C}$ arise, the situations of which are defined by the equations:

$$({}_0C\,{}_xC) = x\,.\,({}_0C\,{}_1C),$$
$$({}_0\overline{C}\,{}_x\overline{C}) = x\,.\,({}_0\overline{C}\,{}_1\overline{C});$$

in other words it will be proved that the two series of points ${}_xC$ and ${}_x\overline{C}$ are similar.

5. If above the point of support B a moment of transition of x

[1]) The moment of bending, appearing in a cross-section of the beam, is called positive, when the right part of the beam exerts a dextro-rotatory couple on the left part.

metreton is introduced, the point A'_0 descends an amount $A_0' A_x' = -x\frac{\mu}{L}$, the point $\underset{0}{B}$ an amount $\underset{0}{B}\underset{x}{B} = x \cdot 2\frac{\mu}{L}$.

$\frac{A'_0 A'_x}{\underset{0}{B}\underset{x}{B}} = 2$ being constant, the series of points A'_x and $\underset{x}{B}$ are similar.

The lines $(A'_x \underset{x}{B})$ connecting their corresponding points, pass therefore through the fixed point $P_{A'_x \underset{x}{B}}$, which divides the distance of the lines l_A and l_B into parts which are to each other as $-1:2$.

The point $\underset{x}{B''}$, which belongs to the point $\underset{x}{B}$, lies at a distance $x \cdot \underset{1}{B''}\underset{1}{B}$ from this point, since the "force" falling along l_{III}, hence also the moment relative to B derived from this "force", increases linearly with the moment M_B.

As $\underset{x}{B}$ has descended over a distance $x \cdot \underset{0}{B}\underset{1}{B}$ relative to $\underset{0}{B}$, the point $\underset{x}{B''}$ lies $x.\left\{\underset{1}{B}\underset{1}{B''} - \mu\frac{2}{L}\right\}$ above $\underset{0}{B}$.

The ratio $\frac{A'_x A'_0}{B''_x B_0}$ being constant, also the series of points A'_x and $\underset{x}{B''}$ are similar, so that also the lines $A'_x \underset{x}{B''}$ pass through one point $P_{A'_x \underset{x}{B''}}$, not indicated in the diagram.

The three angles of the variable triangle $A'_x \, III_x \underset{x}{B}$, (of which $A'_1 \, III_1 \underset{1}{B}$ gives one position) move in three straight lines l_A, l_{III} and l_B passing through one point, while two sides rotate round fixed points. Hence also the third side must rotate round a fixed point lying on the line connecting the centres of rotation of the two other sides.

If we further fix our attention on the variable triangle $III_x \underset{x}{B''}{}_x IV$, it appears that also the angles of this triangle move in three straight lines (l_{III}, l_B and l_{IV}) passing through one point, while two sides, viz. $III_x \, {}_x IV$ and $III_x \underset{x}{B''}$, rotate round fixed points.

The third side rotates therefore also round a fixed point $P_{{}_x IV_x V}$ on $A'_0 \underset{0}{B}$.

But then the side ${}_x V \, {}_x VI$ too has a fixed centre of rotation $P'_{{}_x IV_x V}$. For the sides ${}_x IV \, {}_x V$ and ${}_x V \, {}_x VI$ cut from the line l_B, hence also from the vertical through $P_{{}_x IV_x V}$, a segment of constant length. As the point of intersection of the sides ${}_xIV\,{}_xV$ with this straight line is a fixed point, the point of intersection of the sides ${}_xV\,{}_xVI$ with the same straight line must also be invariable.

Consequently all the sides of the link-polygon $A_x'\,I_x\,II_x\,III_x\,B_x\,{}_xIV\,{}_xV\,{}_xVI\,{}_x\overline{C}$ rotate round a fixed point.

The series of points ${}_x\overline{C}$ is therefore similar to the series of points A'_x. But also the series of points ${}_xC$ is similar to this latter series. For this reason also the series of points ${}_x\overline{C}$ and ${}_xC$ are similar.

6. The double point $\underline{C}$ of these series at finite distance gives the real situation of the third point of support C of the beam, as it can on the one hand be considered as the point $\overline{C}$, through which the beam must pass on introduction of the moment of transition M_B belonging to $\underline{C}$ by reason of the construction of the elastic link-polygon, and on the other hand may be considered as the point C, which is found by the direct determination of the descents in consequence of the given charge and the moment of transition just mentioned.

When once this point $\underline{C}$ has been determined by the help of the proportion;

$$\frac{{}_0\overline{C}\,\underline{C}}{{}_0\overline{C}\,{}_1\overline{C}} = \frac{{}_0C\,\underline{C}}{{}_0C\,{}_1C}$$

the required link-polygon can be drawn completely, as $\underline{C}$ VI V must pass through $P'_{{}_xIV\,{}_xV}$, V IV through $P_{{}_xIV\,{}_xV}$, IV III through $P_{III_x\,{}_xIV}$, III II A' through the point of intersection $\underline{B}''$ of V IV and l_B and finally II I through the point A (lying at a distance a above A').

The magnitude of the required moment of transition M_B is determined by the segment $\underline{BB''}$.

7. Although in the preceding paragraphs the beam on three elastic supporting points has been fully discussed, we shall before proceeding to the beam on four points of support, make mention of one more theorem bearing upon the situations, considered in a horizontal sense, of the centres of rotation $P_{II_x\,III_x}$, $P_{A'_xB_x}$, $P_{III_x\,{}_xIV}$, $P_{{}_xIV\,{}_xV}$, $P'_{{}_xIV\,{}_xV}$.

It has already been pointed out in § 5, that the situation of $P_{A'_xB_x}$ is determined by the ratio $\frac{A'_0\,A'_x}{B_0\,B_x}$, which is independent of the charge of the beam.

The ratio

$$\frac{III_x III_0}{B_xB_0}=\frac{III_1 III_0}{B_1B_0}=\frac{\frac{1}{3}A_1'A_0'+\frac{2}{3}B''_1B_0}{B_1B_0}=\frac{1}{3}\frac{A_1'A_0'}{B_1B_0}+\frac{2}{3}\frac{B''_1B_1-B_1B_0}{B_1B_0}=$$

$$=\frac{1}{3}\frac{A_1'A_0'}{B_1B_0}+\frac{2}{3}\frac{B''_1B_1}{B_1B_0}-\frac{2}{3}$$

by which the situation of $P_{III_x\,{}_xIV}$ is determined, appears to be also independent of the charge of the beam.

But then also the horizontal situations of the other centres of rotation $P_{II_x\,III_x}$, $P_{{}_xIV\,{}_xV}$, $P'_{{}_xIV\,{}_xV}$ are the same for all possible charges of the beam.

For if we consider the two triangles $A_1'B_1\,III_1$ and $\overline{A}_1'\,\overline{B}_1\,\overline{III}_1$ (the latter of which is supposed to bear upon an arbitrary charge differing from the one given), in these two affined figures the points $P_{A'_xB_x}$ and $\overline{P}_{A'_xB_x}$, $P_{III_x\,{}_xIV}$ and $\overline{P}_{III_{xx}IV}$ are homologous points.

From this it can immediately be derived, that also the points $P_{II_x\,III_x}$ and $\overline{P}_{II_x\,III_x}$, $P_{{}_xIV\,{}_xV}$ and $\overline{P}_{{}_xIV\,{}_xV}$ are corresponding points, so that the lines connecting them must pass through the pole of affinity, the point at infinity of the straight lines l.

$P_{II_x\,III_x}$ and $\overline{P}_{II_x\,III_x}$ as well as $\overline{P}_{{}_xIV\,{}_xV}$ and $P_{{}_xIV\,{}_xV}$ lie therefore perpendicularly above each other.

From this follows the theorem referred to in the beginning of this §:

The situation of the centres of rotation $P_{II_x\,III_x}$, $P_{A'_xB_x}$, $P_{III_x\,{}_xIV}$, $P_{{}_xIV\,{}_xV}$, relative to the lines l, is quite independent of the charge of the beam; it is exclusively connected with the stiffness of the beam and that of its supports.

8. Beam on four points of support.

When we have once made ourselves familiar with the line of thought, developed in the preceding paragraphs, it is rational to try and find a solution for the beam on four points of support according to the following program.

1. Cut the beam at the last point of support but one, and construct the situation of the point D in two ways. First by determining the reaction R_D of the beam CD, freely supported at its extremities, and secondly by drawing for the beam $ABCD$ the link-polygon belonging to $M_C=0$. In this way two points

$._0D$ and $._0\overline{D}$,[1]) appear, which do not coincide, unless in reality there is no moment of transition above the point of support C.

2. Construct then in a similar way two points $_1D$ and $_1\overline{D}$ on the supposition that the moment of transition M_C is one metreton.

3. Prove, that the series of points $._yD$ and $_y\overline{D}$ arising on introduction of various moments of transition $M_C = y$ metreton, are similar. Then the double point $\underline{D}$ of these series, to be constructed by the help of $_0D$, $._0\overline{D}$ and $._1D$, $._1\overline{D}$, will indicate the real situation of the last point of support.

4. Starting from this point $\underline{D}$ construct the link-polygon in question $\underline{D}$ IX VIII VII $\underline{C}$ VI V IV $\underline{B}$ III II I $\underline{A}$.

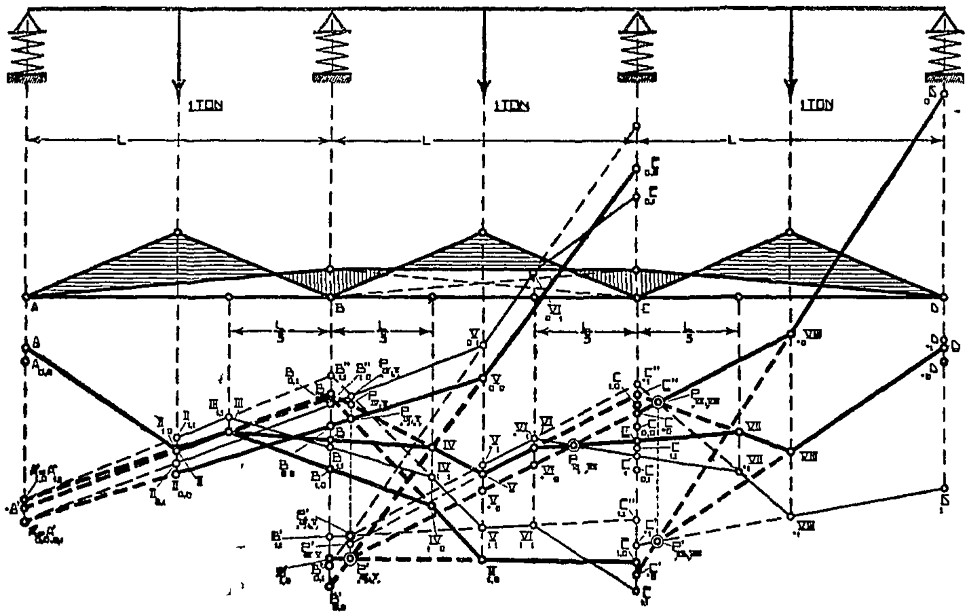

Fig. 2.

9. In fig. 2 the working program, developed to this end, has been put into execution on the supposition that each of the fields AB, BC and CD of the beam is charged in the middle by a force of one ton.

First the construction given in §§ 3—7 has been executed for the beam ABC, which besides by the two forces of one ton on each of the fields AB and BC is supposed to be charged at its extremity C by a force of $\frac{1}{2}$ ton (originating from the charge of the last field CD).

[1]) By the indices 0 and . added to the letters D, is indicated that the moment of transition in C is zero and that the moment of transition in B has the *right* value belonging to the supposition $M_C = 0$.

If we cut this beam at B the whole extra charge acts on the spring under C, so that in the determination of the points A_{00} [1]), B_{00}, C_{00} only the point C_{00} appears to have an extra descent.

Without any difficulty with the help of the link-polygons $A'_{00}\,II_{00}\,B_{00}$, ${}_0V_0\,\overline{C}_{00}$ and $A'_{1,0}\,II_{1,0}$, B_{10}, ${}_0IV_0$, ${}_1V_0$, $\overline{C}_{10}$ the points $\overline{C}_{00}$ and $\overline{C}_{10}$ can then be constructed, which together with the point C_{00} and C_{10} determine the point $C_{.0}$ through which the beam ABC must pass at its extremity C, when besides the given charges it must bear in C a force of $\frac{1}{2}$ ton.

On the supposition $M_C = 0$ the side $P'_{x}IV_{0}\,{}_{x}V_{0}$, $.V_0$, $.VI_0$, $C_{.0}$ can now be prolonged as far as l_{VIII}. After that from $C_{.0}$ a segment $C_{.0}\,C'_{.0} = c_0$ must be drawn in downward direction in order to make it possible to draw the side $C'_{.0}\;{}_{.0}VIII\;{}_{.0}\overline{D}$. In this way, however, the point ${}_0\overline{D}$ is determined.

10. It is far more difficult to find the point $C_{.1}$, through which the beam ABC, considered as a whole, must pass if in C a moment of transition of one metreton is applied. For, when the connection of the beam above B is broken, this moment will, in opposition to the force just applied in C, besides on C also exert its influence on the point B.

The place of the point B_{00} is taken by a point B_{01} which lies $\frac{\mu}{L}$ higher. In the same way the point C_{01} lies a distance $2\frac{\mu}{L}$ below C_{00} because the couple of unity acting on the field BC as well as that, acting on the field CD, gives an extra descent $\frac{\mu}{L}$ to the spring under C.

In case the moment of transition in B is supposed to be zero we have therefore to do with the link-polygon A'_{01}, II_{01}, B_{01}, ${}_0V_1$, ${}_0VI_1$, $\overline{C}_{01}$ of which the two last sides now deviate and cut a segment of known length from l_C.

[1]) By a second index, placed to the right or to the left of a letter, the value of the moment of transition in the second point of support to the right or to the left is indicated, etc.

If in B a moment of one segmentton is introduced, the point $\underset{01}{C}$ is serment replaced by the point $\underset{11}{C}$ lying $\frac{\mu}{L}$ higher, while the construction of the elastic link-polygon A'_{11}, II_{11}, III_{11}, $\underset{11}{B}$, ${}_1IV_1$, ${}_1V_1$, ${}_1VI_1$, $\underset{11}{\overline{C}}$ removes the point $\underset{01}{\overline{C}}$ to $\underset{11}{\overline{C}}$.

Instead of the earlier fixed centres of rotation $P_{III_{x,0}\ {}_xIV_0}$, $P_{{}_xIV_0\ {}_xV_0}$, $P'_{{}_xIV_0,\ {}_xV_0}$, other points $P_{III_{x,1}\ {}_xIV_1}$, $P_{{}_xIV_1,\ {}_xV_1}$, $P'_{{}_xIV_1,\ {}_xV_1}$, lying perpendicularly above them, appear; of these points for the present only the last is of importance.

For when the double point $\underset{.1}{C}$ of the series $\underset{x1}{C}$ and $\underset{x1}{\overline{C}}$ is constructed, also the point $\underset{.1}{C''}$ is known, through which the side $\underset{.1}{V}$, $\underset{.1}{VI}$ must pass. But this side must also contain the point $P'_{{}_xIV_1,\ {}_xV_1}$; it is therefore determined.

Consequently also the sides $VI_{.1}$, ${}_{.1}VII$, ${}_{.1}VII$, ${}_{.1}VIII$ and ${}_{.1}VIII$, ${}_{.1}D$ can be drawn, so that now ${}_{.1}D$ is determined.[1])

The construction of the points ${}_{.0}D$ and ${}_{.1}D$ conjugated to the points ${}_{.1}\overline{D}$ and ${}_{01}\overline{D}$ just found, does not present any difficulties.

11. According to the outline given in § 8 we must now investigate whether the series of points ${}_{.y}D$ and ${}_{.y}\overline{D}$, which appear on the introduction of various moments of transition $M_C = y$ metre-ton in the way described above, are similar.

To that purpose we consider in the first place the centres of rotation $P_{II_{x1}\ III_{x1}}$, $P_{III_{x1}\ {}_xIV_1}$, $P_{{}_xIV_1\ {}_xV_1}$ just mentioned, belonging to the moment of transition $M_C = 1$ metre-ton. These centres of rotation

[1]) Strictly speaking the construction of the link-polygon A'_{11}, II_{11}, .. $\underset{11}{B}$.... $\underset{11}{\overline{C}}$, mentioned in this § and drawn in fig. 2 for completeness' sake, is superfluous. For it serves exclusively for the determination of the ratio $\frac{\underset{01}{\overline{C}}\ \underset{11}{\overline{C}}}{\underset{01}{C}\ \underset{11}{C}}$, which only depends on the horizontal situation of the centres of rotation $P_{III_{x,1}\ {}_xIV_1}$, $P_{{}_xIV_1,\ {}_xV_1}$ etc. which corresponds to that of $P_{III_{x,0}\ {}_xVI_0}$, $P_{{}_xIV_0\ {}_xV_0}$ etc.. $\frac{\underset{01}{\overline{C}}\ \underset{11}{\overline{C}}}{\underset{01}{C}\ \underset{11}{C}}$ can therefore be put equal to the ratio $\frac{\underset{00}{\overline{C}}\ \underset{10}{\overline{C}}}{\underset{00}{C}\ \underset{10}{C}}$ already found.

lie perpendicularly above the centres of rotation $P_{II_{x0}\,III_{x0}}$, $P_{III_{x0}\,{}_xIV_0}$, $P_{{}_xIV_0\,{}_xV_0}$ on a straight line through A'_{00}, determined by the point $\underset{01}{B}$, lying $\frac{\mu}{L}$ above $\underset{00}{B}$.

As on introduction of the other moments of transition $M_C = y$ meterton there appear points $\underset{0y}{B}$, defined by $\underset{00}{B}, \underset{0y}{B} = y. \underset{00}{B}, \underset{01}{B}$, it is evident, that the centres of rotation mentioned, undergo vertical displacements, which are proportional to these moments.

Especially at the introduction of $M_C = y$ metreton the segment $P'_{{}_xIV_0\,{}_xV_0}, P'_{{}_xIV_y\,{}_xV_y}$ will be equal to y times the segment $P'_{{}_xIV_0\,{}_xV_0}, P'_{{}_xIV_1\,{}_xV_1}$.

On account of the law of superposition, on which the whole problem is founded, the descent $\underset{.0}{C}\,\underset{.1}{C}$ of the. point C will increase in direct ratio to the value y of the moment of transition M_C.

The distance of the point $\underset{.y}{C''}$ to the point $\underset{.0}{C}$ can therefore be put equal to:

$$y\,.\,(\underset{1}{C''}\,\underset{1}{C} - \underset{0}{C}\,\underset{.1}{C}).$$

The lines $(P'_{{}_xIV_y\,{}_xV_y}, \underset{.y}{C''})$ connect therefore corresponding points of two similar series of points; they pass through one point.

As $.VI_y\ .VI_0$ can be linearly expressed in $\underset{.y}{C''}\ \underset{.0}{C}$ and $P'_{{}_xIV_0\,{}_xV_0}$, $P'_{{}_xIV_y\,{}_xV_y}$, the series of points $.VI_y$ is also similar to the series $\underset{y}{C}$, so that the lines $.VI_y$, ${}_yVII$, too have a fixed centre of rotation $P_{\underset{y}{.VI}\ \underset{y}{VII}}$. But then also the sides ${}_{.y}VII\ {}_yVIII$ and ${}_{.y}VIII\ {}_{.y}\overline{D}$ have fixed centres of rotation $P_{{}_yVII\ {}_yVIII}$ and $P'_{{}_yVII\ {}_yVIII}$.

The series of points ${}_{.y}\overline{D}$ is therefore similar to the series $\underset{.y}{C'''}$, $\underset{.y}{C}$, $P'_{{}_xIV_y\ {}_xV_y}$ ... which in their turn are similar to the series ${}_yD$, for which holds good:

$${}_yD\ {}_0D = y \times {}_1D\ {}_0D.$$

Hence the series of points ${}_y\overline{D}$ and ${}_yD$ are also similar. Their double point D at finite distance is the extreme point of the link-polygon in question for the beam on four points of support.

Now that this double point is known, the construction of the whole link-polygon no longer presents any difficulty.

For by the centre of rotation $P'_{y\text{VII}\,y\text{VIII}}$ the side $\underline{D}$ IX VIII is determined, by the centre of rotation $P_{y\text{VII}\,y\text{VIII}}$ the side VIII VII C'', by the centre of rotation $P_{\text{VI}_y\,y\text{VII}}$ the side VII $\underline{C}$ VI.

If we furthermore draw $\underline{C''}$ VI in the first place the side VI V and in the second place the centre of rotation $P'_{\text{IV V}}$, hence also the point $P_{\text{IV V}}$, lying perpendicularly above it, through which V IV must pass, are fixed.

By means of $P_{\text{IV V}}$ we also find the line $A'_{00}\,P_{\text{IV V}}$, on which the centre of rotation of all the other sides must lie.

Now the link-polygon in question $\underline{D}$ IX VIII VII $\underline{C}$ VI V IV $\underline{B}$ III II I $\underline{A}$ can be completed.